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Numerical Solution of Sine Gordon Equations Through Reduced Differential Transform Method

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Abstract

In this paper, we present an effective algorithm namely the reduced differential transform method ($RDTM$) for solving sine-Gordon equations. The results obtained through the proposed method are compared with the results through Adomian decomposition method (ADM), Variation iteration method (VIM), Homotopy perturbation method (HPM) and Homotopy analysis method (HAM). The comparison of $RDTM$ with these methods reveals that $RDTM$ is very effective and easy to apply.

AMS subject classification: 35K15, 35K55.
Keywords: Sine Gordon equation, reduced differential transform, inverse reduced differential transform.

1. Introduction

In this paper, we consider the nonlinear sine-Gordon equation

$$u_{tt} - C^2 u_{xx} + \mathcal{K} sin \mathrm{u} = 0, x \in \mathcal{R}, t > 0 \tag{1.1}$$

Subject to the initial conditions

$$u(x, 0) = f(x), u_t(x, 0) = g(x) \tag{1.2}$$

Where C^2 and κ are constants. This is one of the basic nonlinear evolution equations that describe various important nonlinear phenomena such as the propagation of magnetic flux and stability of fluid motions. It has a wide range of applications in mathematics and Physics e.g. in differential geometry, relativistic field theory, solid state physics and nonlinear optics etc.

In the recent years, many analytical/numerical methods have been developed to find the exact/ approximate solution of sine-Gordon equations. For example, Wazwaz [2] proposed tanh method to obtain the exact solution of sine-Gordon equations. Kaya [3] applied the modified decomposition method to obtain the numerical solution of sine-Gordon equations. Batiha et al. [4] used variation iteration method to find the approximate analytic solution of sine-Gordon equation.Yucel [5] presented the exact solution of sine-Gordon equation by means of homotopy analysis method. Mohyud-Din et al. [6] applied modified variation iteration method to solve the sine-Gordon equations. Biazar et al. [7] proposed the differential transform method to obtain the semi analytical solution of sine-Gordon equations.Xinping shao et al. [1] investigated variation iteration method coupled with adomian decomposition and homotopy perturbation method to obtain the exact solution of sine-Gordon equations. Hasan et al. [17] proposed a new technique to solving nonlinear sine-Gordon equation through homotopy analysis method.

The present work aims to investigate the applicability and effectiveness of reduced differential transform method ($RDTM$) on nonlinear sine-Gordon equations. The reduced DTM was first envisioned by Keskin [10] and successfully employed to many nonlinear partial differential equations. Also, Keskin and Oturnac [8, 9] applied this method to obtain the analytical solutions of generalized KdV equations. $RDTM$ has been widely used by many researchers [11, 12, 13, 14, 15] successfully for different nonlinear physical systems such as higher dimensional Burger equations, Burgers-Huxley equations, Newell-Whitehead-Segel equation, generalized Hirota-Satsuma coupled KdV equation, generalized Drenfled-Sokolov equations and Kaup-kuperschmidt equation.

2. Basic idea of reduced differential transform method

Consider a function $u(x, t)$ of two variables and assume that it can be represented as a product of two single variable functions, i.e., $u(x, t) = f(x)g(t)$. On the basis of the properties of the one dimensional differential transform, the function $u(x, t)$ can be represented as

$$u(x, t) = \sum_{h=0}^{\infty} F(h)x^h \sum_{k=0}^{\infty} G(k)t^k = \sum_{h=0}^{\infty}\sum_{k=0}^{\infty} U(h, k)x^h t^k$$

where $U(h, k) = F(h)G(k)$ is called the spectrum of $u(x, t)$.

The basic definitions and properties of reduced differential transform (RDT) are introduced below.

The reduced differential transform of $u(x, t)$ at $t = 0$ is defined as

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k u(x, t)}{\partial t^k}\right]_{t=0} \tag{2.1}$$

Where $u(x, t)$ is the given function and $U_k(x)$ is the transformed function.

The reduced differential inverse transform of $U_k(x)$ is defined as

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^k \tag{2.2}$$

And from equations 2.1 and 2.2, we have

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{\partial^k u(x,t)}{\partial t^k} \right]_{t=0} t^k \tag{2.3}$$

The fundamental theorems of the RDTM are derived from generalized Taylor series [10] and they are given in the table 1.

Theorem 2.1. If $w(x,t) = u(x,t) + v(x,t)$ then $W_k(x) = U_k(x) + V_k(x)$.

Theorem 2.2. If $w(x,t) = \alpha u(x,t)$ then $W_k(x) = \alpha U_k(x)$.

Theorem 2.3. If $w(x,t) = \alpha \frac{\partial^n u(x,t)}{\partial t^n}$ then $W_k(x) = \alpha \frac{(k+n)!}{k!} U_{k+n}(x)$.

Theorem 2.4. If $w(x,t) = x^m t^n$ then $W_k(x) = x^m \delta(k-n) = \begin{cases} x^m, & \text{if } k = n \\ 0, & \text{if } k \neq n \end{cases}$

Theorem 2.5. If $w(x,t) = \alpha \frac{\partial^n u(x,t)}{\partial x^n}$ then $W_k(x) = \alpha \frac{\partial^n U_k(x)}{\partial x^n}$.

Theorem 2.6. If $w(x,t) = u(x,t)v(x,t)$ then $W_k(x) = \sum_{k_1=0}^{k} U_{k_1}(x) V_{k-k_1}(x)$.

Theorem 2.7. If $w(x,t) = x^m t^n u(x,t)$ then $W_k(x) = x^m U_{k-n}(x)$.

Theorem 2.8. If $w(x,t) = t^n u(x,t)$ then $W_k(x) = U_{k-n}(x)$.

In addition, we introduce the theorem which provides us with a simple way to apply the $RDTM$ to the Sine Gordon equation.

Theorem 2.9. Assume that $F_k(x)$, $G_k(x)$ and $U_k(x)$ are the differential transforms of the functions $f(x,t)$, $g(x,t)$ and $u(x,t)$ respectively, then we have the following:

i) If $f(x,t) = sinau(x,t)$, then

$$F_k(x) = RDT(f(x,t)) = \begin{cases} sin(aU_0), & \text{if } k = 0 \\ a \sum_{k_1=0}^{k-k_1} \left(1 - \frac{k_1}{k}\right) G_{k_1}(x) U_{k-k_1}(x), & \text{if } k \geq 1 \end{cases} \tag{2.4}$$

ii) If $g(x,t) = cosau(x,t)$, then

$$G_k(x) = RDT(g(x,t)) = \begin{cases} cos(aU_0), & \text{if } k = 0 \\ -a\sum_{k_1=0}^{k-k_1}\left(1-\frac{k_1}{k}\right)F_{k_1}(x)U_{k-k_1}(x), & \text{if } k \geq 1 \end{cases} \tag{2.5}$$

where 'a' is a constant.

Proof. According to the fundamental operations of reduced differential transform given in the table, it is obvious that

$$F_0(x) = sinaU_0$$

Using Leibnitz rule of higher order derivatives of the products, we get

$$\begin{aligned} f(x,t) &= sinau(x,t) \\ \frac{\partial^k}{\partial t^k}f(x,t) &= a\frac{\partial^{k-1}}{\partial t^{k-1}}\left(cosau\frac{\partial u}{\partial t}\right) \\ &= a\sum_{k_1=0}^{k-1}\binom{k-1}{k_1}\frac{\partial^{k_1}g(x,t)}{\partial t^{k_1}}\frac{\partial^{k-k_1}u(x,t)}{\partial t^{k-k_1}} \end{aligned}$$

Therefore,

$$\begin{aligned} \left[\frac{\partial^k f(x,t)}{\partial t^k}\right]_{t=0} &= a\sum_{k_1=0}^{k-1}\binom{k-1}{k_1}k!(k-k_1)!G_{k_1}(x)U_{k-k_1}(x) \\ &= a\sum_{k_1=0}^{k-1}(k-1)!(k-k_1)G_{k_1}(x)U_{k-k_1}(x) \end{aligned}$$

and then, from using (defn), for $k = 1, 2, 3, \ldots,$ we get

$$F_k(x) = a\sum_{k_1=0}^{k-1}\left(1-\frac{k_1}{k}\right)G_{k_1}(x)U_{k-k_1}(x)$$

ii) According to the fundamental operations of reduced differential transform given in the table, it is obvious that

$$F_0(x) = cosaU_0$$

Using Leibnitz rule of higher order derivatives of the products, we get

$$\begin{aligned} g(x,t) &= cosau(x,t) \\ \frac{\partial^k}{\partial t^k}f(x,t) &= -a\frac{\partial^{k-1}}{\partial t^{k-1}}\left(sinau\frac{\partial u}{\partial t}\right) \\ &= -a\sum_{k_1=0}^{k-1}\binom{k-1}{k_1}\frac{\partial^{k_1}f(x,t)}{\partial t^{k_1}}\frac{\partial^{k-k_1}u(x,t)}{\partial t^{k-k_1}} \end{aligned}$$

Therefore,

$$\left[\frac{\partial^k f(x,t)}{\partial t^k}\right]_{t=0} = -a\sum_{k_1=0}^{k-1}\binom{k-1}{k_1}k!(k-k_1)!F_{k_1}(x)U_{k-k_1}(x)$$
$$= -a\sum_{k_1=0}^{k-1}(k-1)!(k-k_1)F_{k_1}(x)U_{k-k_1}(x)$$

and then, from using (defn), for $k = 1, 2, 3, \ldots,$ we get

$$G_k(x) = a\sum_{k_1=0}^{k-1}\left(1-\frac{k_1}{k}\right)F_{k_1}(x)U_{k-k_1}(x)$$

■

3. Test Examples

Example 3.1. We consider the Sine Gordon equation 1.1 with $\mathcal{C} = 1$, $\kappa = -1$ and the initial conditions

$$u(x,0) = \frac{\pi}{2} \tag{3.1}$$

$$u_t(x,0) = 0 \tag{3.2}$$

Apply reduced differential transform to the equation 1.1, we obtain the recursive formula:

$$(k+1)(k+2)U_{k+2}(x) = \frac{\partial^2}{\partial x^2}U_k(x) + F_k(x) \tag{3.3}$$

where $F_k(x)$ is given by the equation 2.4 with $a = 1$.

The differential transform of the equations 3.1 and 3.2 are

$$U_0(x) = \frac{\pi}{2} \tag{3.4}$$

$$U_1(x) = 0 \tag{3.5}$$

Using the equations 2.4 and 2.5 in to the equation 3.3 with $k = 0, 1, 2, 3, 4, 5$, we obtain the first six components of $U_k(x)$ are as follows:

$$
\begin{aligned}
2U_2(x) &= \frac{\partial^2}{\partial x^2}U_0(x) + F_0(x)\\
6U_3(x) &= \frac{\partial^2}{\partial x^2}U_1(x) + G_0(x)U_1(x)\\
12U_4(x) &= \frac{\partial^2}{\partial x^2}U_2(x) + G_0(x)U_2(x) - \frac{1}{2}F_0(x)U_1^2(x)\\
20U_5(x) &= \frac{\partial^2}{\partial x^2}U_3(x) + G_0(x)U_3(x) - \frac{2}{3}F_0(x)U_1(x)U_2(x)\\
&\quad -\frac{1}{3}F_0(x)U_2(x)U_1(x) - \frac{1}{6}G_0(x)U_1^3(x)\\
30U_6(x) &= \frac{\partial^2}{\partial x^2}U_4(x) + G_0(x)U_4(x) - \frac{3}{4}F_0(x)U_1(x)U_3(x)\\
&\quad -\frac{1}{2}F_0(x)U_2^2(x) - \frac{1}{4}G_0(x)U_1^2(x)U_2(x)\\
&\quad -\frac{1}{4}F_0(x)U_1(x)U_3(x) - \frac{1}{6}G_0(x)U_1^2(x)U_2(x)\\
&\quad -\frac{1}{12}G_0(x)U_1^2(x)U_2(x) - \frac{1}{24}G_1(x)U_1^3(x)\\
42U_7(x) &= \frac{\partial^2}{\partial x^2}U_5(x) + G_0(x)U_5(x)\\
&\quad -\frac{9}{20}G_0(x)U_1^2(x)U_3(x) - \frac{1}{20}G_0(x)U_3(x)U_2^2(x)\\
&\quad +\frac{1}{30}F_0(x)U_1^3(x)U_2(x) + \frac{1}{60}F_0(x)U_1^3(x) + \frac{1}{120}G_0(x)U_1^5(x)
\end{aligned}
$$

Using the equations 3.4 and 3.5 in the above system of equations, we obtain the values of components of $U_k(x)$ as follows:

$$U_2(x) = \frac{1}{2},\ U_3(x) = 0,\ U_4(x) = 0,\ U_5(x) = 0,\ U_6(x) = -\frac{1}{240},\ U_7(x) = 0, \ldots$$

Substituting all these values in the equation 1.2, we obtain

$$u(x,t) = \frac{\pi}{2} + \frac{t^2}{2} - \frac{t^6}{240} + \ldots \tag{3.6}$$

This result shows an excellent agreement with the one obtained by DTM [7] and $MVIM$ [6].

Example 3.2. We consider the Sine Gordon equation 1.1 with $\mathcal{C} = 0, \kappa = -1$ and the initial conditions

$$u(x,0) = \pi \tag{3.7}$$

$$u_t(x,0) = -2 \tag{3.8}$$

The exact solution of this problem is

$$u(x,t) = 2arcsin(secht) \tag{3.9}$$

Taking differential transform to both sides of equation 1.1, we obtain the following recursive formula:

$$(k+1)(k+2)U_{k+2}(x) = F_k(x) \tag{3.10}$$

where $F_k(x)$ is given by the equation 2.4 with $a = 1$.

The differential transform of the equations 3.7 and 3.8 are

$$U_0(x) = \pi \tag{3.11}$$

$$U_1(x) = -2 \tag{3.12}$$

Similar to the previous problem, we obtain the first six components of $U_k(x)$ are as follows:

$$\begin{aligned}
2U_2(x) &= F_0(x) \\
6U_3(x) &= G_0(x)U_1(x) \\
12U_4(x) &= G_0(x)U_2(x) - \frac{1}{2}F_0(x)U_1^2(x) \\
20U_5(x) &= G_0(x)U_3(x) - \frac{2}{3}F_0(x)U_1(x)U_2(x) \\
&\quad -\frac{1}{3}F_0(x)U_2(x)U_1(x) - \frac{1}{6}G_0(x)U_1^3(x) \\
30U_6(x) &= G_0(x)U_4(x) - \frac{3}{4}F_0(x)U_1(x)U_3(x) \\
&\quad -\frac{1}{2}F_0(x)U_2^2(x) - \frac{1}{4}G_0(x)U_1^2(x)U_2(x) \\
&\quad -\frac{1}{4}F_0(x)U_1(x)U_3(x) - \frac{1}{6}G_0(x)U_1^2(x)U_2(x) \\
&\quad -\frac{1}{12}G_0(x)U_1^2(x)U_2(x) - \frac{1}{24}G_1(x)U_1^3(x) \\
42U_7(x) &= G_0(x)U_5(x) - \frac{9}{20}G_0(x)U_1^2(x)U_3(x) \\
&\quad -\frac{1}{20}G_0(x)U_3(x)U_2^2(x) + \frac{1}{30}F_0(x)U_1^3(x)U_2(x) \\
&\quad +\frac{1}{60}F_0(x)U_1^3(x) + \frac{1}{120}G_0(x)U_1^5(x)
\end{aligned}$$

and so on. Utilizing the equations 3.11 and 3.12 in the above system of equations, we obtain the values of components of $U_k(x)$ as follows:

$$U_2(x) = U_4(x) = U_6(x) = 0$$

and

$$U_3(x) = \frac{1}{3}, U_5(x) = -\frac{1}{12}, U_7(x) = \frac{61}{2520}$$

Finally, substituting all these values in the equation 1.2, we obtain

$$u(x,t) = \pi - 2t + \frac{t^3}{3} - \frac{t^5}{12} + \frac{61}{2520}t^7 - \dots \tag{3.13}$$

Consequently the solution in closed form is

$$u(x,t) = 2arcsin(secht)$$

which is exactly the same as the results obtained by VIM [4], HAM [5] $MVIM$ [6] and DTM [7].

Example 3.3. Consider the nonlinear Sine-Gordon equation 1.1 with $\mathcal{C} = 1, \kappa = -1$ and the initial conditions

$$U_0(x) = \frac{\pi}{2} \tag{3.14}$$

$$U_1(x) = 1 \tag{3.15}$$

Taking differential transform to both sides of equation 1.1, we obtain the following recursive formula:

$$(k+1)(k+2)U_{k+2}(x) = \frac{\partial^2}{\partial x^2}U_k(x) + F_k(x) \tag{3.16}$$

The differential transform of the equations 3.14 and 3.15 are

$$U_0(x) = \pi \tag{3.17}$$

$$U_1(x) = -2 \tag{3.18}$$

Utilizing the recurrence equation 3.16 for $k = 0, 1, 2, 3, 4, 5$, we obtain the first six components of $U_k(x)$ are as follows:

$$
\begin{aligned}
2U_2(x) &= \frac{\partial^2}{\partial x^2}U_0(x) + F_0(x)\\
6U_3(x) &= \frac{\partial^2}{\partial x^2}U_1(x) + G_0(x)U_1(x)\\
12U_4(x) &= \frac{\partial^2}{\partial x^2}U_2(x) + G_0(x)U_2(x) - \frac{1}{2}F_0(x)U_1^2(x)\\
20U_5(x) &= \frac{\partial^2}{\partial x^2}U_3(x) + G_0(x)U_3(x) - \frac{2}{3}F_0(x)U_1(x)U_2(x)\\
&\quad -\frac{1}{3}F_0(x)U_2(x)U_1(x) - \frac{1}{6}G_0(x)U_1^3(x)\\
30U_6(x) &= \frac{\partial^2}{\partial x^2}U_4(x) + G_0(x)U_4(x) - \frac{3}{4}F_0(x)U_1(x)U_3(x)\\
&\quad -\frac{1}{2}F_0(x)U_2^2(x) - \frac{1}{4}G_0(x)U_1^2(x)U_2(x)\\
&\quad -\frac{1}{4}F_0(x)U_1(x)U_3(x) - \frac{1}{6}G_0(x)U_1^2(x)U_2(x)\\
&\quad -\frac{1}{12}G_0(x)U_1^2(x)U_2(x) - \frac{1}{24}G_1(x)U_1^3(x)\\
42U_7(x) &= \frac{\partial^2}{\partial x^2}U_5(x) + G_0(x)U_5(x) - \frac{9}{20}G_0(x)U_1^2(x)U_3(x)\\
&\quad -\frac{1}{20}G_0(x)U_3(x)U_2^2(x)\\
&\quad +\frac{1}{30}F_0(x)U_1^3(x)U_2(x) + \frac{1}{60}F_0(x)U_1^3(x) + \frac{1}{120}G_0(x)U_1^5(x)
\end{aligned}
$$

And so on. Using the transformed initial conditions 3.17 and 3.18 in the above system of equations, we obtain the values of components of $U_k(x)$ as

$U_2(x) = \frac{1}{2}, U_3(x) = 0, U_4(x) = -\frac{1}{24}, U_5(x) = -\frac{1}{40}, U_6(x) = -\frac{1}{360}, U_7(x) = 0, \ldots$

Substituting all these values in the equation 1.2, we obtain

$$
u(x,t) = \frac{\pi}{2} + t + \frac{t^2}{2} - \frac{t^4}{24} - \frac{1}{40}t^5 - \frac{1}{360}t^6 + \frac{1}{336}t^7 + \ldots \tag{3.19}
$$

which is exactly the same as the results obtained by DTM [7] and ADM [16].

4. Conclusion

In this work, the reduced differential transform method is introduced for solving sine-Gordon equations. The results reveal that the proposed method is very effective and a convenient mathematical tool for finding the exact and semi analytic solution of sine-Gordon equations.

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