Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 7 (2017), pp. 3889-3894

http://www.ripublication.com/gjpam.htm

Identities of symmetry for Carlitz's twisted (h, q)-tangent polynomials associated with p-adic integral on $\mathbb{Z}_p$

C. S. Ryoo
Department of Mathematics,
Hannam University, Daejeon 306-791, Korea

Abstract

In this paper, we discover symmetric properties for Carlitz's twisted (h, q)-tangent polynomials.

AMS subject classification: 11B68, 11S40, 11S80.
Keywords: Tangent numbers and polynomials, twisted (h, q)-tangent polynomials, symmetric identities, Carlitz's twisted (h, q)-tangent number and polynomials.

1. Introduction

Many mathematicians have worked some identities of symmetry for q-extension of Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, tangent numbers and polynomials (see [2, 3, 4, 6, 7, 8, 9, 10, 11]). Recently, Y. He derived several identities of symmetry for Carlitz's q-Bernoulli numbers and polynomials in complex field (see [2]). D. Kim *et al.* [3] investigated some identities of symmetry for Carlitz's q-Euler numbers and polynomials in complex field. J. Y. Kang and C. S. Ryoo obtained some identities of symmetry for q-Genocchi polynomials (see [1]). In [5], we obtained some identities of symmetry for Carlitz's twisted q-Euler polynomials associated with p-adic integral on $\mathbb{Z}_p$. Our aim in this paper is to discover special symmetric properties for Carlitz's twisted (h, q)-tangent polynomials. Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ denotes the ring of rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{C}$ denotes the set of complex numbers, and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. Throughout this paper we use the notation:

$$[x]_q = \frac{1-q^x}{1-q} \text{ (cf. [1, 2, 3, 4])} .$$

Hence, $\lim_{q \to 1}[x] = x$ for any x with $|x|_p \leq 1$ in the present p-adic case. Let

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\}.$$

For $g \in UD(\mathbb{Z}_p)$, the p-adic invariant integral on $\mathbb{Z}_p$ is defined by Kim to be

$$I_{-1}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} g(x)(-1)^x, \text{ see [4].} \tag{1.1}$$

First, we introduce the Carlitz's type twisted (h, q)-tangent numbers $T_{n,q,\zeta}^{(h)}$ and polynomials $T_{n,q,\zeta}^{(h)}(x)$ and investigate their properties(see [5]). Let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, $h \in \mathbb{Z}$, and

$$T_p = \cup_{m \geq 1} C_{p^m} = \lim_{m \to \infty} C_{p^m},$$

where $C_{p^m} = \{\zeta | \zeta^{p^m} = 1\}$ is the cyclic group of order p^m. For $\zeta \in T_p$, we denote by $\phi_\zeta : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \longmapsto \zeta^x$. For $q \in \mathbb{C}_p$ with $|1-q|_p < 1$ and $\zeta \in T_p$, the Carlitz's type twisted (h, q)-tangent polynomials $T_{n,q,\zeta}^{(h)}(x)$ are defined by

$$T_{n,q,w}^{(h)}(x) = \int_{\mathbb{Z}_p} w^y q^{hy} [2y + x]_q d\mu_{-1}(y). \tag{1.2}$$

When $x = 0$, $T_{n,q,\zeta}^{(h)}(0) = T_{n,q,\zeta}^{(h)}$ is called the nth Carlitz's twisted (h, q)-tangent numbers.

2. Symmetric identities for Carlitz's twisted (h, q)-tangent numbers and polynomials

Our primary goal of this section is to obtain symmetric identities for Carlitz's twisted (h, q)-tangent numbers $T_{n,q,\zeta}^{(h)}$ and polynomials $T_{n,q,\zeta}^{(h)}(x)$. Since $[x + 2y]_q = [x]_q +$

$q^x[2y]_q$, we see that

$$\begin{aligned} T_{n,q,\zeta}^{(h)}(x) &= \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{xl} T_{l,q,\zeta}^{(h)} \\ &= \left(q^x T_{q,\zeta}^{(h)} + [x]_q\right)^n \\ &= 2\sum_{m=0}^{\infty} (-1)^m \zeta^m q^{hm} [x+2m]_q^n, \end{aligned} \tag{2.1}$$

with the usual convention of replacing $(T_{q,\zeta}^{(h)})^n$ by $T_{n,q,\zeta}^{(h)}$. Let w_1 and w_2 be odd numbers. Then we have

$$\begin{aligned} &\int_{\mathbb{Z}_p} \zeta^{w_1 y} q^{w_1 h y} e^{\left[w_2 x + \frac{2w_2}{w_1} j + 2y\right]_{q^{w_1}} [w_1]_q t} d\mu_{-1}(y) \\ &= \lim_{N\to\infty} \sum_{y=0}^{w_2 p^N - 1} \zeta^{w_1 y} q^{w_1 h y} e^{[w_1 w_2 x + 2w_2 j + 2w_1 y]_q t} (-1)^y \\ &= \lim_{N\to\infty} \sum_{i=0}^{w_2-1} \sum_{y=0}^{p^N-1} \zeta^{w_1(i+w_2 y)} q^{w_1 h(i+w_2 y)} e^{[w_1 w_2 x + 2w_2 j + 2w_1(i+w_2 y)]_q t} (-1)^{i+w_2 y} \end{aligned} \tag{2.2}$$

From (2.2), we can derive the following equation (2.3):

$$\begin{aligned} &\sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2 j} q^{h w_2 j} \int_{\mathbb{Z}_p} \zeta^{w_1 y} q^{w_1 h y} e^{\left[w_2 x + \frac{2w_2}{w_1} j + 2y\right]_{q^{w_1}} [w_1]_q t} d\mu_{-1}(y) \\ &= \lim_{N\to\infty} \sum_{j=0}^{w_1-1} \sum_{i=0}^{w_2-1} \sum_{y=0}^{p^N-1} (-1)^{i+j} \zeta^{w_2 j} \zeta^{w_1 i} \zeta^{w_1 w_2 y} q^{w_2 h j} q^{w_1 h i} q^{w_1 w_2 h y} \\ &\qquad \times e^{[w_1 w_2 x + 2w_2 j + 2w_1 i + 2w_1 w_2 y]_q t} (-1)^y \end{aligned} \tag{2.3}$$

By the same method as (2.3), we obtain

$$\begin{aligned} &\sum_{j=0}^{w_2-1} (-1)^j \zeta^{w_1 j} q^{h w_1 j} \int_{\mathbb{Z}_p} \zeta^{w_2 y} q^{w_2 h y} e^{\left[w_1 x + \frac{2w_1}{w_2} j + 2y\right]_{q^{w_2}} [w_2]_q t} d\mu_{-1}(y) \\ &= \lim_{N\to\infty} \sum_{j=0}^{w_2-1} \sum_{i=0}^{w_1-1} \sum_{y=0}^{p^N-1} (-1)^{i+j} \zeta^{w_1 j} \zeta^{w_2 i} \zeta^{w_1 w_2 y} q^{w_1 h j} q^{w_2 h i} q^{w_1 w_2 h y} \\ &\qquad \times e^{[w_1 w_2 x + 2w_1 j + 2w_2 i + 2w_1 w_2 y]_q t} (-1)^y \end{aligned} \tag{2.4}$$

Therefore, by (2.3) and (2.4), we have the following theorem.

Theorem 2.1. For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, we have

$$\begin{aligned}&\sum_{j=0}^{w_1-1}(-1)^j\zeta^{w_2 j}q^{hw_2 j}\int_{\mathbb{Z}_p}\zeta^{w_1 y}q^{w_1 hy}e^{\left[w_2x+\frac{2w_2}{w_1}j+2y\right]_{q^{w_1}}[w_1]_q t}d\mu_{-1}(y)\\&=\sum_{j=0}^{w_2-1}(-1)^j\zeta^{w_1 j}q^{hw_1 j}\int_{\mathbb{Z}_p}\zeta^{w_2 y}q^{w_2 hy}e^{\left[w_1x+\frac{2w_1}{w_2}j+2y\right]_{q^{w_2}}[w_2]_q t}d\mu_{-1}(y).\end{aligned} \tag{2.5}$$

By substituting Taylor series of e^{xt} into (2.5) and after calculations, we obtain the following corollary.

Corollary 2.2. For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, we have

$$\begin{aligned}&[w_1]_q^n\sum_{j=0}^{w_1-1}(-1)^j\zeta^{w_2 j}q^{hw_2 j}\int_{\mathbb{Z}_p}\zeta^{w_1 y}q^{w_1 hy}\left[w_2x+\frac{2w_2}{w_1}j+2y\right]_{q^{w_1}}^n d\mu_{-1}(y)\\&=[w_2]_q^n\sum_{j=0}^{w_2-1}(-1)^j\zeta^{w_1 j}q^{hw_1 j}\int_{\mathbb{Z}_p}\zeta^{w_2 y}q^{w_2 hy}\left[w_1x+\frac{2w_1}{w_2}j+2y\right]_{q^{w_2}}^n d\mu_{-1}(y).\end{aligned} \tag{2.6}$$

By (1.2) and Corollary 2.2, we have the following theorem.

Theorem 2.3. For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, we have

$$\begin{aligned}&[w_1]_q^n\sum_{j=0}^{w_1-1}(-1)^j\zeta^{w_2 j}q^{hw_2 j}T^{(h)}_{n,q^{w_1},\zeta^{w_1}}\left(w_2x+\frac{2w_2}{w_1}j\right)\\&=[w_2]_q^n\sum_{j=0}^{w_2-1}(-1)^j\zeta^{w_1 j}q^{hw_1 j}T^{(h)}_{n,q^{w_2},\zeta^{w_2}}\left(w_1x+\frac{2w_1}{w_2}j\right).\end{aligned}$$

By (2.6), we can derive the following equation (2.7):

$$\begin{aligned}&\int_{\mathbb{Z}_p}\zeta^{w_1 y}q^{w_1 hy}\left[w_2x+\frac{2w_2}{w_1}j+2y\right]_{q^{w_1}}^n d\mu_{-1}(y)\\&=\sum_{i=0}^{n}\binom{n}{i}\left(\frac{[w_2]_q}{[w_1]_q}\right)^i[2j]_{q^{w_2}}^i q^{w_2(n-i)j}\int_{\mathbb{Z}_p}\zeta^{w_1 y}q^{w_1 hy}[w_2x+2y]_{q^{w_1}}^{n-i}d\mu_{-1}(y)\\&=\sum_{i=0}^{n}\binom{n}{i}\left(\frac{[w_2]_q}{[w_1]_q}\right)^i[2j]_{q^{w_2}}^i q^{w_2(n-i)j}T^{(h)}_{n-i,q^{w_1},\zeta^{w_1}}(w_2x).\end{aligned} \tag{2.7}$$

By (2.7) and Theorem 2.3, we have

$$
\begin{aligned}
&[w_1]_q^n \sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2 j} q^{h w_2 j} \int_{\mathbb{Z}_p} \zeta^{w_1 y} q^{w_1 h y} \left[w_2 x + \frac{2w_2}{w_1} j + 2y \right]_{q^{w_1}}^n d\mu_{-1}(y) \\
&= \sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2 j} q^{h w_2 j} \sum_{i=0}^{n} \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i} [2j]_{q^{w_2}}^i q^{w_2(n-i)j} T_{n-i,q^{w_1},\zeta^{w_1}}^{(h)}(w_2 x) \\
&= \sum_{i=0}^{n} \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i} T_{n-i,q^{w_1},\zeta^{w_1}}^{(h)}(w_2 x) \sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2 j} q^{w_2(n-i+h)j} [2j]_{q^{w_2}}^i \\
&= \sum_{i=0}^{n} \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i} T_{n-i,q^{w_1},\zeta^{w_1}}^{(h)}(w_2 x)\, \mathcal{S}_{n,i}^{(h)}(w_1, \zeta^{w_2}, q^{w_2}),
\end{aligned}
\tag{2.8}
$$

where

$$
\mathcal{S}_{n,i}^{(h)}(w_1, \zeta, q) = \sum_{j=0}^{w_1-1} (-1)^j \zeta^j q^{(n-i+h)j} [2j]_q^i .
$$

is called as the sums of twisted even q-integer powers. By the same method as (2.8), we get

$$
\begin{aligned}
&[w_2]_q^n \sum_{j=0}^{w_2-1} (-1)^j \zeta^{w_1 j} q^{h w_1 j} \int_{\mathbb{Z}_p} \zeta^{w_2 y} q^{w_2 h y} \left[w_1 x + \frac{2w_1}{w_2} j + 2y \right]_{q^{w_2}}^n d\mu_{-1}(y) \\
&= \sum_{i=0}^{n} \binom{n}{i} [w_1]_q^i [w_2]_q^{n-i} T_{n-i,q^{w_2},\zeta^{w_2}}^{(h)}(w_1 x)\, \mathcal{S}_{n,i}^{(h)}(w_2, \zeta^{w_1}, q^{w_1}).
\end{aligned}
\tag{2.9}
$$

By (2.8) and (2.9), we have the following theorem.

Theorem 2.4. For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, we have

$$
\begin{aligned}
&\sum_{i=0}^{n} \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i} \mathcal{S}_{n,i}^{(h)}(w_1, \zeta^{w_2}, q^{w_2}) T_{n-i,q^{w_1},\zeta^{w_1}}^{(h)}(w_2 x) \\
&= \sum_{i=0}^{n} \binom{n}{i} [w_1]_q^i [w_2]_q^{n-i} \mathcal{S}_{n,i}^{(h)}(w_2, \zeta^{w_1}, q^{w_1}) T_{n-i,q^{w_2},\zeta^{w_2}}^{(h)}(w_1 x) .
\end{aligned}
\tag{2.10}
$$

Observe that if $h = 1$, then (2.10) reduces to Theorem 2.4 in [8]. If we take $x = 0$ in Theorem 2.4, we also derive the interesting identity for Carlitz's twisted (h,q)-tangent numbers as follows:

Corollary 2.5. For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, we have

$$\sum_{i=0}^{n}\binom{n}{i}[w_2]_q^i[w_1]_q^{n-i}\mathcal{S}_{n,i}^{(h)}(w_1,\zeta^{w_2},q^{w_2})T_{n-i,q^{w_1},\zeta^{w_1}}^{(h)}$$
$$=\sum_{i=0}^{n}\binom{n}{i}[w_1]_q^i[w_2]_q^{n-i}\mathcal{S}_{n,i}^{(h)}(w_2,\zeta^{w_1},q^{w_1})T_{n-i,q^{w_2},\zeta^{w_2}}^{(h)}.$$

Acknowledgement

This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) (No. 2017R1A2B4006092).

References

[1] J. Y. Kang, C. S. Ryoo, *On Symmetric Property for q-Genocchi Polynomials and Zeta Function*, Int. Journal of Math. Analysis **8(1)** (2014), 9–16.

[2] Yuan He, *Symmetric identities for Carlitz's q-Bernoulli numbers and polynomials*, Advances in Difference Equations **246**(2013), 10 pages.

[3] D. Kim, T. Kim, S.-H. Lee, J.-J. Seo, *Symmetric Identities of the q-Euler Polynomials*, Adv. Studies Theor. Phys., **7(24)** (2014), 1149–1155.

[4] T. Kim, *q-Euler numbers and polynomials associated with p-adic q-integrals*, J. Nonlinear Math. Phys., **14**(2007), 15–27.

[5] C. S. Ryoo, *Carlitz's Type Twisted (h, q)-tangent numbers and polynomials*, Applied Mathematical Sciences, **9(30)** (2015), 1475–1482.

[6] C.S. Ryoo, *A numerical investigation on the zeros of the tangent polynomials*, J. Appl. Math. & Informatics **32** (2014), 315–322.

[7] C.S. Ryoo, *Reflection Symmetries of the q-Genocchi Polynomials*, J. Appl. Math. & Informatics **28** (1010), 1277–1284.

[8] C. S. Ryoo, *Symmetric identities for Carlitz's twisted q-tangent numbers and polynomials associated with fermionic p-adic integral on $\mathbb{Z}_p$*, Far East Journal of Mathematical Sciences **96(6)** (2015), 699–706.

[9] C. S. Ryoo, *A Note on the Reflection Symmetries of the Genocchi polynomials*, J. Appl. Math. & Informatics, **27** (2009), 1397–1404.

[10] H. Shin, J. Zeng, *The q-tangent and q-secant numbers via continued fractions*, European J. Combin. **31**(2010), 1689–1705.

[11] Y. Simsek, *Complete Sum of Products of (h, q)-Extension of Euler Polynomials and Numbers*, Journal of Difference Equations and Applications, **16(11)**(2010), 1331–1348.