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Nonlinear Implicit Caputo Fractional Differential Equations with Integral Boundary Conditions in Banach Space

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Abstract

In this paper, we establish sufficient conditions for the existence of solutions for nonlinear Caputo-type implicit fractional differential equations with integral boundary conditions. The proof of the main results is based on the measure of noncompactness and the Darbo's fixed point theorem. An example is included to show the applicability of our results.

AMS subject classification:
Keywords: Caputo fractional derivative, Implicit fractional differential Equations in Banach space, Measure of noncompactness, Integral boundary, Fixed point.

1. Introduction

Fractional calculus is a generalization of the classical differentiation and integration to arbitrary noninteger order. The idea of fractional calculus has been a subject of interest not only among mathematicians but also among physicists and engineers. They have used it effectively to improve the mathematical modelling of several phenomena occuring in scientific and engineering diciplines such as viscoelasticity, eletrochemistry, electromagnetism, biology, control,diffution process, economics, chaotic theory, variational problems, and so forth; see [3, 7, 13, 14, 16–21] and the refferences therein.

The mathematical models are in fact types of fractional differential system such that the existence of their solutions make these models are more applicable. Hence, the existence theorems as theoretic tools are necessarily for any fractional differential models. Many reseachers focused on the existence of solution for initial and boundary value vproblems of fractional differential equations; see [6, 10, 11, 22, 23], and the references therein.

Recently, considerable attention has been given to the existence of solutions of boundary value problem and boundary conditions for implicit fractional differential equations and integral equations with Caputo fractional derivative. See for example [2, 5, 15], and the references therein.

Motivated by above cited works, the purpose of this paper, is to establish esistence result to the following implicit fractional differential equation with integral boundary conditions:

$$ {}^{c}D^{\alpha}_{0+}y(t) = -f(t, y(t), {}^{c}D^{\alpha}_{0+}y(t)), \quad \text{for each } t \in J = [0, 1],\ 1 < \alpha < 2 \tag{1.1} $$

$$ ay(0) - by^{'}(0) = 0 \tag{1.2} $$

$$ y(1) = \int_0^1 k(s)g(t, y(s))ds + \mu $$

where ${}^{c}D^{\alpha}_{0+}$ is the Caputo fractional derivative, $(E, \|\cdot\|)$ is real Banach space, $f : J \times C([0, 1], E) \times E \to E$, $g \in C(E, E)$, $k \in C([0, 1], E), k \neq 0$, $a, b, \in \mathbb{R}_{+}$, $a+b > 0$ and $\dfrac{a}{a+b} < \alpha - 1$.

The rest of this paper is organized as follows. In section 2, we give some notations and recall some preliminaries about fractional calculus and the Kuratowski's measure of non compactness and auxiliary results. In section 3, based on Darbo's fixed point theorem combined with the technique of measures of noncompactness, the result is discussed. In the last section, we present an example illustrating the presented main results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Let $(E, \|\cdot\|)$ be a Banach space. We denote by $C(J, E)$ the space of E-valued continuous functions on J with the usual supremum norm

$$ \|y\|_{\infty} \sup\{\|y(t)\| : t \in J\} \text{ for every } y \in C(J, E). $$

Also a measurable function $y : J \to E$ is Bochner integrable if and only if $\|y\|$ is Lebesgue measure.

Let $L^1(\mathcal{J}, E)$ denote the Banach space of measurable functions $y : \mathcal{J} \to E$ which are Bochner integrable normed by

$$ \|y\|_{L^1} = \int_1^T \|y(t)\|dt. $$

For properties of the Bochner integrable, see [23].

Definition 2.1. Let $x : (0, \infty) \to R$ be a function and $\alpha > 0$. The Riemann-Liouville fractional integral of orders α of x is defined by

$$I_{0+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} = \int_0^t (t-s)^{\alpha-1} x(s) ds$$

provided that the integral exists. The Caputo fractional derivative of order α of x ix defined by

$${}^C D_{0+}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} = \int_0^t (t-s)^{n-\alpha-1} x^n(s) ds$$

provided that the the right side is point wise defined on $(0, \infty)$, where $n = [\alpha]+1,\ n-1 < \alpha < n$, and Γ denotes the gamma function. If $\alpha = n$, then ${}^C D_{0+}^{\alpha} x(t) = x^{(n)}(t)$.

Lemma 2.2. If $x \in AC^n[0, 1]$, then the Caputo derivative ${}^C D_{0+}^{\alpha} x(t)$ exists almost everywhere on [0,1], where $AC^n[0, 1] = \left\{ x \in C^{n-1}[0, 1] | x^{(n-1)} \text{ is absolutely continuous} \right\}$ and n is the smallest integer greater than or equal to α.

Lemma 2.3. If $x \in C^n[0, 1]$ then $I_{0+}^{\alpha}\, {}^C D_{0+}^{\alpha} x(t) = x(t)+c_0+c_1 t+c_2 t^2+\cdots+c_{n-1} t^{n-1}$, where n is the smallest integer greater than or equal to α.

Moreover, for a given set V of functions $v : J \to E$ let us denote by $V(t) = \{v(t),\ v \in V\},\ t \in J$ and $V(J) = \{v(t),\ v \in V,\ t \in J\}$. Next we give the definition of the concept of measure of noncompactness and some auxiliary result; see for more details [4, 8, 9] and the references therein.

Definition 2.4. Let E be a Banach space and Ω_E the bounded subsets of E. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_E \to [0, \infty]$ defined by $\alpha(B) = \inf \left\{ \epsilon > 0 : B \subseteq \cup_{i=1}^n B_i \text{ and } \operatorname{diam}(B_i) \leq \epsilon \right\}$; here $\operatorname{diam}(B_i) = \sup\{\|x - y\| : x, y \in B_i\}$.

The Kuratowski measure of noncompactness satisfies the following properties.

Lemma 2.5. ([4,8,9]) Let A and B are bounded sets.

(a) $\alpha(B) = 0 \Leftrightarrow \bar{B}$ is compact (B is relatively compact), where $\bar{B}$ denotes the closure of B.

(b) nonsingularity: α is equal to zero on every one element set.

(c) $\alpha(B) = \alpha(\bar{B}) = \alpha(\text{conv } B)$, where conv B is the convex hull of B.

(d) monotonicity: $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$.

(e) algebraic semi-additively: $\alpha(A + B) \leq \alpha(A) + \alpha(B)$, where $A + B = \{x + y : x \in A,\ y \in B\}$.

(f) semi-homogencity: $\alpha(\lambda A) = |\lambda| \alpha(B) : \lambda \in \mathbb{R}\}$ where $\lambda B = \{\lambda x :\ x \in B\}$.

(g) semi-additivity: $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$.

(h) invariance under translation: $\alpha(B + x_0) = \alpha(B)$ for any $x_0 \in E$.

For our purpose we will only need the following fixed point theorem and important lemma.

Theorem 2.6. (Darbo's fixed point theorem) ([12]) Let X be a Banach space and C be bounded, closed, convex and nonempty subset of X. Suppose a continuous mapping $N : C \to C$ is such that for all closed subsets D of C,

$$\alpha(N(D)) \leq k\alpha(D), \tag{2.1}$$

where $0 \leq k \leq 1$. Then N has a fixed point in C.

Remark 2.7. Mappings satisfying the Darbo-condition (2.1) have subsequently been called k-set contractions.

Lemma 2.8. ([13]) If $V \subset C(J, E)$ is a bounded and equicontinuous set, then

i. the function $t \to \alpha(V(t))$ is continuous on J, and $\alpha_c(V) = \sup_{1 \leq t \leq T} \alpha(V(t))$.

ii. $\alpha\left(\int_1^T x(s)ds : x \in V\right) \leq \int_1^T \alpha(V(s))ds$, where $V(s) = \{x(s) : x \in V\}$, $s \in J$.

Theorem 2.9. (Ascoli-Arzela) ([12]) Let $A \subset C(J, E)$, A is relatively compact (i.e. $\bar{A}$ is compact) if:

i. A is uniformly bounded i.e., there exists $M > 0$ such that $\|f(t)\| \leq M$ for every $f \in A$ and $t \in J$.

ii. A is equicontinuous i.e., for every $\epsilon > 0$, there exists $\delta > 0$ such that for each $t, \bar{t} \in J$, $|t - \bar{t}| \leq \delta$ implies $\|f(t) - f(\bar{t})\| \leq \epsilon$, for every $f \in A$.

iii. The set $\{f(t) : f \in A; t \in J\}$ is relatively compact in E.

3. Existence of Solutions

Let us defining what we mean by a solution of problem (1.1)-(1.2).

Definition 3.1. A function $y \in C(J, E)$ is said to be solution of (1.1)-(1.2) if y satisfies the equation ${}^C D^{\alpha}_{0+} y(t) = f(t, y(t), {}^C D^{\alpha}_{0+} y(t))$ on J, and the conditions $ay(0) - by'(0) = 0$ and $y(1) = \int_0^1 k(s)g(t, y(s))ds + \mu$ on [0, 1].

To prove the existence of solutions to (1.1)-(1.2), we need the following auxiliary lemma.

Lemma 3.2. Let $y, z \in C[0, 1]$ the linear fractional boundary value problem (BVP)

$$\left.\begin{array}{l} {}^{C}D_{0+}^{\alpha}y(t) = -h(t),\ 0 < t < 1 \\ ay(0) - by^{'}(0) = 0 \\ y(1) = \int_0^1 k(s)z(s)ds + \mu \end{array}\right\} \tag{3.1}$$

has a unique solution

$$y(t) = \int_0^1 G(t, s)h(s)ds + \frac{at + b}{a + b}\int_0^1 k(s)z(s)ds + \frac{at + b}{a + b}\mu \tag{3.2}$$

where

$$G(t, s) = \frac{1}{\Gamma(\alpha)}\begin{cases} \frac{(at + b)(1 - s)^{\alpha-1}}{a + b} - (t - s)^{\alpha-1}, & 0 \le s \le t \le 1 \\ \frac{(at + b)(1 - s)^{\alpha-1}}{a + b}, & 0 \le t \le s \le 1 \end{cases} \tag{3.3}$$

Proof. From Lemma 2.3, we have

$$\begin{aligned} y(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t - s)^{\alpha-1}h(s)ds + c_0 + c_1 t, \\ y^{'}(t) &= -\frac{1}{\Gamma(\alpha - 1)}\int_0^t (t - s)^{\alpha-2}h(s)ds + c_1 \end{aligned}$$

By the boundary condition in BVP (3.1) we get

$$ac_0 + bc_1 = 0,\ -\frac{1}{\Gamma(\alpha)}\int_0^1 (1 - s)^{\alpha-1}h(s)ds + c_0 + c_1 = \int_0^1 k(s)z(s)ds + \mu$$

Hence

$$\begin{aligned} c_0 &= \frac{b}{a + b}\left[\frac{1}{\Gamma(\alpha)}\int_0^1 (1 - s)^{\alpha-1}h(s)ds + \int_0^1 k(s)z(s)ds + \mu\right], \\ c_1 &= \frac{a}{a + b}\left[\frac{1}{\Gamma(\alpha)}\int_0^1 (1 - s)^{\alpha-1}h(s)ds + \int_0^1 k(s)z(s)ds + \mu\right], \end{aligned}$$

This means that

$$y(t) = \int_0^1 G(t, s)h(s)ds + \frac{at + b}{a + b}\int_0^1 k(s)z(s)ds + \frac{at + b}{a + b}\mu$$

The proof is complete. ■

Lemma 3.3. The function $G(t, s)$ defined by (3.3) satisfies

$$0 \leq G(t, s) \leq \frac{(at + b)(1 - s)^{\alpha-1}}{(\alpha + b)\Gamma(\alpha)}, \ t, s \in [0, 1],$$

and $\max\limits_{t\in[0,1]} G(t, s) = G(s, s), \ s \in [0, 1]$.

Proof. From (3.3) we easily see that

$$G(t, s) \leq \frac{(at + b)(1 - s)^{\alpha-1}}{(\alpha + b)\Gamma(\alpha)}, \ t, s \in [0, 1].$$

For $0 \leq s \leq t \leq 1$, by (3.3) we have

$$\begin{aligned} \frac{\partial G(t, s)}{\partial t} &= \frac{1}{\Gamma(\alpha)}\left(\frac{a(1 - s)^{\alpha-1}}{a + b} - \frac{\alpha - 1}{(t - s)^{2-\alpha}}\right) \\ &\leq \frac{(1 - s)^{\alpha-1}}{\Gamma(\alpha)}\left(\frac{a}{a + b} - (\alpha - 1)\right) \\ &< 0 \end{aligned} \tag{3.4}$$

which implies that

$$G(s, s) > G(t, s) > G(1, s) = 0, \quad 0 \leq s < t < 1. \tag{3.5}$$

So, the continuity of $G(t, s)$ leads to $0 \leq G(t, s) \leq G(s, s), \quad 0 \leq s \leq t \leq 1$. It is clear by (3.3) that also $0 \leq G(t, s) \leq G(s, s)$ for $0 \leq t \leq s \leq 1$. This end the proof. ■

Lemma 3.4. A function $y(t)$ is a solution of fractional differential equation 1.1 and 1.2 if and only if $y(t)$ is a solution of the integral equation

$$y(t) = \int_0^1 G(t, s) f(s, y(s), {}^C D^{\alpha}_{0+} y(s))ds + \frac{at + b}{a + b}\int_0^1 k(s)z(s)ds + \frac{at + b}{a + b}\mu. \tag{3.6}$$

Proof. According to Lemma 2.8, it is evident that the solution of fractional differential equation 1.1 and 1.2 is the solution of the integral equation 3.6. On the other hand, if $y \in C[0, 1]$ is the solution 3.6, then

$$\begin{aligned} y'(t) &= -\frac{1}{\Gamma(\alpha - 1)}\int_0^t (t - s)^{\alpha-2} f(s, y(s), {}^C D^{\alpha}_{0+} y(s))ds \\ &\quad + \frac{a}{(a + b)\Gamma(\alpha)}\int_0^1 (1 - s)^{\alpha-1} f(s, y(s), {}^C D^{\alpha}_{0+} y(s))ds \\ &\quad + \frac{a}{(a + b)}(\int_0^1 k(s)z(s)ds + \mu) \end{aligned}$$

it is easy to see that $y' \in AC[0, 1]$ and $y \in AC^2[0, 1]$. From Lemma2.2 we obtain that ${}^C D^{\alpha}_{0+} y$ exists almost everywhere on [0, 1]. Noting that

$$\begin{aligned} y''(t) &= -\frac{d}{dt}\Big(\frac{1}{\Gamma(\alpha-1)}\int_0^1 (t-s)^{\alpha-2} f(s, y(s), {}^C D^{\alpha}_{0+} y(s)\Big) \\ &= -\frac{d}{dt} I^{\alpha-1} f(t, y(t), {}^C D^{\alpha}_{0+} y(t)) \end{aligned}$$

we can conclude that ${}^C D^{\alpha}_{0+} y(t) = -f(t, y(t), {}^C D^{\alpha}_{0+} y(t))$ and y is the solution of fractional differential equation 1.1 and 1.2. The proof is complete. ■

First we list the following hypotheses:

(H1) The function $f : J \times E \times E \to E$ and $g : J \times E \to E$ are continuous.

(H2) There exist constants $K > 0$ and $0 < L < 1$ such that

$$\|f(t, u, v) - f(t, \bar{u}, \bar{v})\| \leq K\|u - \bar{u}\| + L\|v - \bar{v}\|$$

for any $u, \bar{u}, v, \bar{v} \in E$ and $t \in J$.

(H3) There exist a constant $N > 0$ such that

$$\|g(t, u) - g(t, \bar{u})\| \leq N\|u - \bar{u}\|.$$

We are now in a position to state and prove our existence result for the problem (1.1)-(1.2) based on concept of measures of noncompactness and Darbo's fixed point theorem.

Theorem 3.5. Assume (H1)-(H3) hold. If

$$\frac{Kg^*}{(1-L)(\Gamma(\alpha+1))} + g^* N N_0 < 1 \tag{3.7}$$

then the BVB (1.1)-(1.2) has at least one solution on J.

Proof. Transform the problem (1.1)-(1.2) into a fixed point problem. Consider the operator $T : C(J, E) \to C(J, E)$ defined by

$$Ty(t) = \int_0^1 G(t, s)h(s)ds + \frac{at+b}{a+b}\int_0^1 k(s)z(s)ds + \frac{at+b}{a+b}\mu, \ t \in J \tag{3.8}$$

where $h \in C(J, E)$ be such that $h(t) = f(t, y(t), h(t))$. Clearly, the fixed point of operator T are solution of problem (1.1)-(1.2). We shall show that T satisfies the assumption of Darbo's fixed point theorem. The proof will be given in several claims.

Claim 1: T is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C(J, E)$. If $t \in J$ we have

$$\|T(y_n)(t) - T(y)(t)\| \leq \int_0^1 |G(t,s)|\|h_n(s) - h(s)\|ds \quad (3.9)$$
$$+ \int_0^1 |k(s)|\|z_n(s) - z(s)\|ds \quad (3.10)$$

where $h_n, h, z_n, z \in C(J, E)$ such that $h_n(t) = f(t, y(t), h_n(t)), h(t) = f(t, y(t), h(t))$, $z_n(t) = g_n(t, y_n(t))$ *and* $z(t) = g(t, y(t))$.

By (H2) and (H3), we have

$$\begin{aligned} \|h_n(t) - h(t)\| &= \|f(t, y(t), h_n(t)) - f(t, y(t), h(t))\| \\ &\leq K\|y_n - y\| + L\|h_n(t) - h(t)\| \end{aligned}$$

Then

$$\|h_n(t) - h(t)\| \leq \frac{K}{1-L}\|y_n - y\|.$$

and

$$\|z_n(t) - z(t)\| \leq N\|y_n - y\|.$$

Since $y_n \to y$, then we get $h_n(t) \to h(t)$ anbd $z_n(t) \to z(t)$ as $n \to \infty$ for each $t \in J$. And let $\eta > 0$ ans $\nu > 0$ such that for each $t \in J$, we have $\|h_n(t)\| \leq \eta$, $\|h(t)\| \leq \eta$,$\|z_n(t)\| \leq \nu$ and $\|z(t)\| \leq \nu$. Then we have

$$\begin{aligned} G(t,s)\|h_n(s) - h(s)\| &\leq \frac{(at+b)(1-s)^{\alpha-1}}{(a+b)\Gamma(\alpha)}\|h_n(s) + \|h(s)\| \\ &\leq 2\eta\frac{(at+b)(1-s)^{\alpha-1}}{(a+b)\Gamma(\alpha)}. \end{aligned}$$

also

$$\|z_n(t) - z(t)\| \leq 2\nu k^*\frac{(at+b)}{(a+b)}$$

here $k^* = \|k(s)\|$.

For each $t \in J$, the function $s \to \frac{(at+b)}{(a+b)}\left[\frac{2\eta(1-s)^{\alpha-1}}{\Gamma(\alpha)} + 2\nu k^*\right]$ is integrable on $[0, t]$, then the Lebesgue Dominaled convergence theorem and (3.9) imply that

$$\|T(y_n)(t) - T(y(t))\| \to 0, \quad \text{as } n \to \infty$$

and hence

$$\|T(y_n)(t) - T(y(t))\|_{[0,1]} \to 0, \quad \text{as } n \to \infty.$$

Consequently, N is continuous. Let the constant R such that

$$R \geq \frac{f^*g^* + g^*\mu}{(1-L)\Gamma(\alpha+1)[1-g^*NN_0] - Kg^*} \tag{3.11}$$

where $f^* = \sup_{t\in J}\|f(t,0,0)\|$, $g^* = \max_{t\in J}\frac{at+b}{a+b}$, $\|z(s)\| = \|g(t,y(s))\| \leq N\|y(s)\|$ and $N_0 = \int_0^1 k(s)ds$.

Define $D_R = \{y \in C(J,E) : \|y\| \leq R\}$. It is clear that D_R is a bounded, closed and convex subset of $C(J,E)$.

Claim 2: $T(D_R) \subset D_R$

Let $y \in D_R$ we show that $T_y \in D_R$. If $t \in [0,1]$ then $\|T(y)(t)\|$ i.e. we have

$$\begin{aligned}
\|T(y)(t)\| &= \max_{t\in[0,1]}\left[\int_0^1 G(t,s)h(s)ds + \frac{at+b}{a+b}\int_0^1 k(s)z(s)ds + \frac{at+b}{a+b}\mu\right] \\
&\leq \int_0^1 \frac{(at+b)(1-s)^{\alpha-1}}{(\alpha+b)}\|h(s)\|ds + \frac{at+b}{a+b}\int_0^1 k(s)\|z(s)\|ds \\
&\quad + \frac{at+b}{a+b}\mu,
\end{aligned} \tag{3.12}$$

By (H2) and (H3), we have for each $t \in J$

$$\begin{aligned}
\|h(t)\| &\leq \|f(t,y,h(t)) - f(t,0,0)\| + \|f(t,0,0)\| \\
&\leq K\|y\| + L\|h(t)\| + f^* \\
&\leq KR + L\|h(t)\| + f^*
\end{aligned}$$

Then $\|h(t)\| \leq \dfrac{f^*KR}{1-L} := M$. Thus (3.11) and (3.12) imply that

$$\|Ty(t)\| \leq \frac{M(at+b)}{(a+b)\Gamma(\alpha+1)} + \frac{(at+b)NRN_0}{a+b} + \frac{(at+b)\mu}{a+b} \leq R$$

from which it follows that for each $t \in [0,1]$ we have $\|Ty(t)\| \leq R$ which implies that $T(D_R) \subset D_R$.

Claim 3: $T(D_R)$ is bounded and equicontinuous.

By Claim 2 we have $T(D_R) = \{T(y) : y \in D_R\} \subset D_R$. Thus for each $u \in D_R$ we have $\|T(y)\|_{[0,1]} \leq R$ which means that $T(D_R)$ is bounded. Let $t_1, t_2 \in [0,1]$, $t_1 < t_2$ and let $y \in D_R$. Then

$$\begin{aligned}\|T(y)(t_2)-T(y)(t_1)\| &\leq \left\| \int_0^1 [G(t_2,s)-G(t_1,s)]h(s)ds \right.\\ &\quad \left. +\frac{a}{a+b}\int_0^1 k(s)z(s)(t_2-t_1)ds+\frac{a}{a+b}(t_2-t_1)\right\| \\ &\leq M\int_0^1 [G(t_2,s)-G(t_1,s)]ds \\ &\quad +\frac{aNR}{a+b}\int_0^1 k(s)(t_2-t_1)ds+\frac{a}{a+b}(t_2-t_1)\end{aligned}$$

As $t_2 \to t_1$, the right hand side of the above inequality tends to zero.

Claim 4: The operator $T : D_R \to D_R$ is a strict set contraction. Let $V \subset D_R$, if $t \in J$, we have

$$\begin{aligned}\alpha(T(y)(t)) &= \alpha\left((Ty)(t),\, y\in V\right)\\ &\leq \alpha\left(\int_0^1 G(t,s)h(s)ds+\frac{at+b}{a+b}\int_0^1 k(s)z(s)ds+\frac{at+b}{a+b}\mu,\;\; y\in V\right)\end{aligned}$$

Then by Lemma (2.5) imply that, for each $s \in J$,

$$\begin{aligned}\alpha\left(\{h(s),\, y\in V\}\right) &= \alpha\left(\{f(s,y(s),h(s)),\, y\in V\}\right)\\ &\leq K\alpha\left(\{y(s),\, y\in V\}\right)+L\alpha\left(\{h(s),\, y\in V\}\right)\end{aligned}$$

Thus,

$$\begin{aligned}\alpha\left(\{y(s),\, y\in V\}\right) &\leq \frac{K}{1-L}\alpha\left(y(s),\, y\in V\right)\\ \alpha\left(g(y(s)),\, y\in V\right) &= \alpha\left(z(s),\, z\in V\right)\\ &= N\alpha\left(y(s),\, y\in V\right)\end{aligned}$$

Then

$$\begin{aligned}\alpha(T(V)(t)) &\leq \frac{K}{1-L}\int_0^1 G(t,s)\{\alpha(y(s))ds,\, y\in V\}\\ &\quad +\frac{at+b}{a+b}N\int_0^1 G(t,s)\{\alpha(y(s))ds,\, y\in V\}\\ &\leq \left(\frac{K}{1-L}\int_0^1 G(t,s)ds+N\frac{at+b}{a+b}\int_0^1 k(s)ds\right)\{\alpha(y(s))ds,\, y\in V\}\\ &\leq \frac{kg^*+g^*NN_0(1-L)(\Gamma(\alpha+1))}{\Gamma(\alpha+1)(1-L)}\alpha(V)\end{aligned}$$

Therefore

$$\alpha_c(TV) \leq \frac{kg^* + g^* N N_0(1-L)(\Gamma(\alpha+1))}{\Gamma(\alpha+1)(1-L)} \alpha_C(V)$$

So, by (3.7), the operator T is set contraction. As a consequence of Theorem Darbo's, we deduce that T has s fixed point which is solution to the problem (1.1)-(1.2). This completes the proof. ■

4. Example

Consider the following fractional boundary value problem

$$\begin{aligned} {}^C D_{0^+}^{3/2} y(t) &= \frac{1}{200}(t\sin(y) - y\cos(t)) \\ &\quad + \frac{1}{2}\sin({}^C D_{0^+}^{3/2} y(t)) + \frac{1}{2}, \quad \text{for each } t \in [0,1], \end{aligned} \tag{4.1}$$

$$ay(0) - 4y^{'}(0) = 0; \quad y(1) = \int_0^1 \frac{1}{100} s y(s) ds + \mu \tag{4.2}$$

where

$$\begin{aligned} f(t,u,v) &= \frac{1}{200}(t\sin(y) - y\cos(t)) + \frac{1}{2}\sin({}^C D_{0^+}^{3/2} y(t)) \\ &\quad + \frac{1}{2}, \ t \in [0,1], \ u, v \in C(J,E) \end{aligned}$$

$k(s) = s$, $g(s,u) = \frac{y(s)}{100}$, $g \in C(E,E)$, $\alpha = 1.5$, $a = 1$, $b = 4$. Evidently, $\frac{a}{a+b} < \alpha - 1$.

For any $u, \bar{u}, v, \bar{v} \in E$ and $t \in [0,1]$

$$\begin{aligned} \|f(t,u,v) - f(t,\bar{u},\bar{v})\| &\leq \frac{1}{200}|t|\|\sin u - \sin\bar{u}\| + \frac{1}{200}|\cos t|\|u - \bar{u}\| \\ &\quad + \frac{1}{2}\|\sin v - \sin\bar{v}\| \\ &\leq \frac{1}{100}\|u - \bar{u}\| + \frac{1}{2}\|v - \bar{v}\| \end{aligned}$$

Hence condition (H1) and (H3) is satisfied with $K = \frac{1}{100}$, $L = \frac{1}{2}$ and $N = \frac{1}{100}$, and the condition

$$\frac{kg^* + g^* N N_0(1-L)(\Gamma(\alpha+1))}{\Gamma(\alpha+1)(1-L)} < 1$$

are satisfied with $\alpha = 1.5$, $N_0 = \frac{1}{2}$ and $g^* \leq 1$. It follows from Theorem (3.5) that the (4.1)-(4.2) has at least one solution on J.

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