Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 9 (2017), pp. 4425–4436

http://www.ripublication.com/gjpam.htm

Location Domination Number of Graph obtained by the Fusion of Single Vertex

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Abstract

A locating-dominating set is a subset of the vertex set which detect and identify the position of any vertex uniquely. In this paper, the precise method for finding location domination number of a graph obtained by vertex fusion is determined.

AMS subject classification: 05C69.
Keywords: Dominating set, Locating-Dominating set, Vertex fusion.

1. Introduction

Locating-domination sets have application in finding fault diagnosis in multi processor systems and it is done by modelling as a graph where vertices are processors, edges are links between processors.

In 1962, Oystein Ore [7] defined that a subset S of the vertex set $V(G)$ of a graph G is a dominating set if each vertex $v \in V(G) - S$ is adjacent to atleast one vertex in S. Then domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G.

For any vertex v in $V(G) - S$, $S(v)$ is the set of vertices in S which are adjacent to v. Locating-dominating set was introduced by Slater [10, 11] which is defined as follows. A dominating set S is defined to be a locating-dominating set if $S(v) \neq S(w)$, for any $v, w \in V(G) - S$. A locating dominating set is denoted by LD-set. The minimum cardinality of LD-set in G is called the location-domination number of G and it is denoted by $RD(G)$. An LD-set with $RD(G)$ elements is called as a referencing-dominating set or an RD-set.

For simple graph $G = \big(V(G), E(G)\big)$, open neighbourhood $N_G(v)$ of a vertex $v \in V(G)$ is $N_G(v) = \big\{u \in V(G) \mid uv \in E(G)\big\}$.
And $N_G[v] = \big\{u \in V(G) \mid uv \in E(G)\big\} \cup \big\{v\big\}$ is the closed neighbourhood of $v \in V(G)$.

Colboum et al. [4] prove that the Locating-dominating set is NP-complete. Slater formulated linear-time algorithm for solving Locating-Dominating-Set in tree [11] and found locating-dominating set for path, cycle [10]. A LD-set will be both locating and dominating set but the converse need not be true. Slater [8, 11] determined that if RD-set has n vertices then the given can have atmost $n + 2^n - 1$ vertices and also shown that for any trees T_n with n vertices, $RD(T_n) > \frac{n}{3}$.

A pair of vertices u and v in a graph are said to be fused (merged or identified) if u and v are replaced by a single new vertex such that every edge incident on u or v is incident on this new vertex.

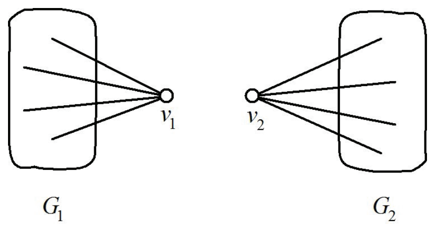

Figure 1: Graph G_1 and G_2 with vertex v_1 and v_2

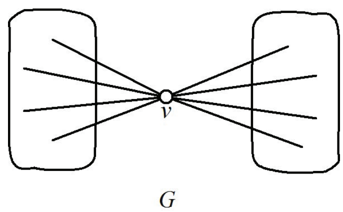

Figure 2: Graph G obtained by fusion of the vertices v_1 and v_2

2. Graph obtained by Vertex Fusion

Let G is the graph obtained from G_1 and G_2 by vertex fusion of vertices $v_1 \in G_1$ and $v_2 \in G_2$ to form a single new vertex v and it is shown in Fig. 2. Let us denote this graph G by $G = G_1\{v_1\} \bowtie_f G_2\{v_2\}$. Let S be any RD-set of G, then

$$S = S \cap V(G) = S \cap \big(V(G_1) \cup V(G_2)\big) = \big(S \cap V(G_1)\big) \cup \big(S \cap V(G_2)\big) \tag{2.1}$$

and

$$RD(G) = |S| = \big|S \cap V(G_1)\big| + \big|S \cap V(G_2)\big| - \big|\big(S \cap V(G_1)\big) \cap \big(S \cap V(G_2)\big)\big| \tag{2.2}$$

As $v = v_1 = v_2$ is the only vertex which is in $V(G_1)$ as well as $V(G_2)$ and for any vertex $u \in V(G) - \{v\}$, if $u \in V(G_1)$ implies that $u \notin V(G_2)$, we have that

$$\big|\big(S \cap V(G_1)\big) \cap \big(S \cap V(G_2)\big)\big| \leq 1 \tag{2.3}$$

Theorem 2.1. Let G_1, G_2 be any two graphs and G be a graph obtained by fusion of vertices $v_1 \in G_1$ and $v_2 \in G_2$ to form a single vertex v. Then

$$RD(G_1) + RD(G_2) - 2 \leq RD(G) \leq RD(G_1) + RD(G_2)$$

.

Proof. Let us first prove that $RD(G) \leq RD(G_1) + RD(G_2)$.
To prove $RD(G) \leq RD(G_1) + RD(G_2)$ it is enough to show that G has an LD-set with cardinality $RD(G_1) + RD(G_2)$. Let $S_1,\ S_2$ be the RD-set of $G_1,\ G_2$ respectively. Consider $S = S_1 \cup S_2$ then

$$S(u) = \begin{cases} S_1(u), & u \in V(G_1) - S_1 \\ S_2(u), & u \in V(G_2) - S_2 \end{cases}$$

If S is an LD-set then $|S| \leq |S_1| + |S_2| = RD(G_1) + RD(G_2)$. Suppose S is not the LD-set of G, there exist $u, w \in V(G) - S$ such that $S(u) = S(w)$. Clearly if $u \in V(G_1) - S_1$ then $w \in V(G_2) - S_2$, otherwise $S(u) = S(w)$ leads to $S_1(u) = S_1(w)$ or $S_2(u) = S_2(w)$ that is S_1, S_2 cannot be the LD-set of G_1, G_2 respectively. Therefore $S_1(u) = S_2(w)$ and combining with the fact that $v = v_1 = v_2$ is the only possible vertex in S_1 and S_2 leads to $S_1(u) = S_2(w) = \{v\}$. Hence $v \in S$ and $S \cup \{u\}$, $S \cup \{w\}$ are the LD-set of G. But as $|S| = |S_1| + |S_2| - |S_1 \cap S_2| = RD(G_1) + RD(G_2) - 1$, we have $RD(G) \leq |S \cup \{u\}| = RD(G_1) + RD(G_2)$.

Now we will prove that $RD(G_1)+RD(G_2)-2 \leq RD(G)$. Suppose G has an RD-set S such that $RD(G) = RD(G_1) + RD(G_2) - 3$. As $\big|\big(S \cap V(G_1)\big) \cap \big(S \cap V(G_2)\big)\big| \leq 1$ by equation (2.3) we have two cases.

Case 1: $v \in S$, i.e. $|(S \cap V(G_1)) \cap (S \cap V(G_2))| = 1$.
Therefore (2.1) becomes

$$RD(G_1) + RD(G_2) - 3 = RD(G) = |S \cap V(G_1)| + |S \cap V(G_2)| - 1$$

This implies that $|S \cap V(G_1)| + |S \cap V(G_2)| = RD(G_1) + RD(G_2) - 2$. Thus there are two chances, either $|S \cap V(G_1)| \leq RD(G_1) - 1$ or $|S \cap V(G_2)| \leq RD(G_2) - 1$.

Without loss of generality let us take $|S \cap V(G_1)| \leq RD(G_1) - 1$. But as S is the RD-set of G and $v \in S$, we must have $S \cap V(G_1)$, $S \cap V(G_2)$ is the LD-set of G_1, G_2 respectively. Thus G_1 has an LD-set $S \cap V(G_1)$ with $|S \cap V(G_1)| < RD(G_1)$, this contradict the definition of RD-set. Hence $RD(G) \geq RD(G_1) + RD(G_2) - 2$.

Case 2: $v \notin S$, i.e. $\big|\big(S \cap V(G_1)\big) \cap \big(S \cap V(G_2)\big)\big| = 0$.
Equation (2.1) becomes,

$$RD(G_1) + RD(G_2) - 3 = RD(G) = |S \cap V(G_1)| + |S \cap V(G_2)|$$

Thus either $|S \cap V(G_1)| \leq RD(G_1) - 2$ or $|S \cap V(G_2)| \leq RD(G_2) - 2$ and let us take $|S \cap V(G_1)| \leq RD(G_1) - 2$. Therefore $S \cap V(G_1)$ is not LD-set of G_1.

As S is the RD-set of G, $S \cap V(G_1)$ must be the LD-set of $G_1 - \{v_1\}$. Hence $\big(S \cap V(G_1)\big) \cup \{v_1\}$ is the LD-set of G_1 with cardinality less than or equal to $RD(G_1) - 1$, this violate the RD-set definition. So $RD(G) \geq RD(G_1) + RD(G_2) - 2$. Therefore from both Case 1 and 2, we attain that $RD(G_1) + RD(G_2) - 2 \leq RD(G)$. ■

Theorem 2.2. Let G_1, G_2 be any two graphs and $G = G_1\{v_1\} \bowtie_f G_2\{v_2\}$. For $i = 1, 2$, let G_i have an RD-set S_i such that $v_i \in S_i$ and $S_i(u_i) \neq \{v_i\}$ for all $u_i \in V(G_i) - S_i$. Then $RD(G) = RD(G_1) + RD(G_2) - 2$ if and only if for $i = 1, 2$, $S_i - \{v_i]$ is the RD-set of $G_i - \{v_i\}$ and $N_{G_i}(v_i) \cap S_i \neq \Phi$.

Proof. Let S be the RD-set with $RD(G) = RD(G_1) + RD(G_2) - 2$.
Hence (2.2) becomes

$$\big|S \cap V(G_1)\big| + \big|S \cap V(G_2)\big| - \big|\big(S \cap V(G_1)\big) \cap \big(S \cap V(G_2)\big)\big| \\ = RD(G_1) + RD(G_2) - 2 \quad (2.4)$$

with $\big|\big(S \cap V(G_1)\big) \cap \big(S \cap V(G_2)\big)\big| \leq 1$ by (2.3).

We have to prove that $S_i - \{v_i\}$ is the RD-set of $G_i - \{v_i\}$ and $N_{G_i}(v_i) \cap S_i \neq \Phi$ for $i = 1, 2$.

Case 1: $\big|\big(S \cap V(G_1)\big) \cap \big(S \cap V(G_2)\big)\big| = 1$.

Then $v \in S \cap V(G_1)$ and $v \in S \cap V(G_2)$. i.e., $v \in S$. Hence $S \cap V(G_1)$, $S \cap V(G_2)$ must be the LD-set of G_1, G_2 respectively. Otherwise S cannot be the RD-set of G. Therefore $RD(G_1) \leq \big|S \cap V(G_1)\big|$ and $RD(G_2) \leq \big|S \cap V(G_2)\big|$.

By substituting above inequality, (2.4) transforms to

$$RD(G_1) + RD(G_2) - 2 \geq RD(G_1) + RD(G_2) - 1$$

This is contradiction implies that $\big|\big(S \cap V(G_1)\big) \cap \big(S \cap V(G_2)\big)\big| \neq 1$.
Therefore $\big|\big(S \cap V(G_1)\big) \cap \big(S \cap V(G_2)\big)\big| = 0$.

Case 2: $\big|\big(S \cap V(G_1)\big) \cap \big(S \cap V(G_2)\big)\big| = 0$, i.e., $v \notin S$.
Equation (2.4) becomes,

$$\big|S \cap V(G_1)\big| + \big|S \cap V(G_2)\big| = RD(G_1) + RD(G_2) - 2 \quad (2.5)$$

Which implies $|S \cap V(G_1)| \leq RD(G_1)$, otherwise S cannot be the RD-set of G.

If $|S \cap V(G_1)| = RD(G_1)$, by (2.5) $|S \cap V(G_2)| = RD(G_2) - 2$. As S is the RD-set which implies $S \cap V(G_2)$ must be atleast a LD-set of $G_2 - \{v_2\}$. Hence $\big(S \cap V(G_2)\big) \cup \big\{v_2\big\}$ will be the LD-set of G_2. Thus $\big|\big(S \cap V(G_2)\big) \cup \big\{v_2\big\}\big| \geq RD(G_2)$, i.e. $RD(G_2) - 2 + 1 \geq RD(G_2)$ this is impossible. Therefore $|S \cap V(G_1)| \neq RD(G_1)$.

If $|S \cap V(G_1)| = RD(G_1) - 1$, then by (2.5) $|S \cap V(G_2)| = RD(G_2) - 1$. Therefore $S \cap V(G_1)$ is not RD-set of G_1. But as S is the RD-set of G, $S \cap V(G_1)$ must be the LD-set of $G_1 - \{v_1\}$. Hence $S_1 = \big(S \cap V(G_1)\big) \cup \big\{v_1\big\}$ must be the LD-set of G_1. And as $|S_1| = RD(G_1)$, S_1 is the RD-set of G_1. Hence we have an RD-set S_1 of G_1 such that $S_1 - \{v_1\}$ is the RD-set of $G_1 - \{v_1\}$.

Similarly we can show that $S_2 = \big(S \cap V(G_2)\big) \cup \big\{v_2\big\}$ is the RD-set of G_2 such that $S_2 - \{v_2\}$ is the RD-set of $G_2 - \{v_2\}$.

As S is the RD-set, there exist some vertex in S which is adjacent to v. i.e., $S \cap N_G(v) \neq \Phi$. Therefore atleast $S \cap N_{G_1}(v_1) \neq \Phi$ or $S \cap N_{G_2}(v_2) \neq \Phi$.

Claim: $S \cap N_{G_1}(v_1) \neq \Phi$ and $S \cap N_{G_2}(v_2) \neq \Phi$.

Suppose $S \cap N_{G_1}(v_1) \neq \Phi$ and $S \cap N_{G_2}(v_2) = \Phi$. Then

$$S(v) = S(v_1) = \big(S \cap N_{G_1}(v_1)\big) \cup \big(S \cap N_{G_2}(v_2)\big) = S \cap N_{G_1}(v_1)$$

As S is the RD-set, $S(u) \neq S(w)$ for all $u, w \in V(G_1) - \big(S \cap V(G_1)\big)$ and $S(v_1) = S \cap N_{G_1}(v_1) \neq S(u)$ for all $u \in V(G_1) - \big(S \cap V(G_1)\big)$. Therefore $S \cap V(G_1)$ is the LD-set of G_1 with $|S \cap V(G_1)| \geq RD(G_1)$. i.e. $RD(G_1) - 1 \geq RD(G_1)$ and it cannot be. Therefore $S \cap N_{G_1}(v_1) \neq \Phi$ and $S \cap N_{G_2}(v_2) \neq \Phi$.

Thus for $i = 1, 2$, G_i has an RD-set S_i with $v_i \in S_i$, such that $S_i - \{v_i\}$ is the RD-set of $G_i - \{v_i\}$ and $N_{G_i}(v_i) \cap S_i \neq \Phi$.

Conversely, let us assume that for $i = 1, 2$, G_i has an RD-set S_i with $v_i \in S_i$, such that $S_i - \{v_i\}$ is the RD-set of $G_i - \{v_i\}$ and $N_{G_i}(v_i) \cap S_i \neq \Phi$. And we will prove that $RD(G) = RD(G_1) + RD(G_2) - 2$.

Let $S = (S_1 \cup S_2) - \{v\}$. Then

$$S(u) = \begin{cases} S_1(u) - \{v\}, & u \in V(G_1) - S_1 \\ S_2(u) - \{v\}, & u \in V(G_2) - S_2 \end{cases} \text{ and}$$

$$S(v) = \big(N_{G_1}(v_1) \cap S_1\big) \cup \big(N_{G_2}(v_2) \cap S_2\big) \neq \Phi$$

as $N_{G_i}(v_i) \cap S_i \neq \Phi$ for $i = 1, 2$. Also

$$|S| = \big|S_1 - \{v\}\big| + \big|S_2 - \{v\}\big| = RD(G_1) + RD(G_2) - 2.$$

If $S(u) = S(w)$ implies that

$$\big(S_1(u) - \{v\}\big) \cup \big(S_2(u) - \{v\}\big) = \big(S_1(w) - \{v\}\big) \cup \big(S_2(w) - \{v\}\big)$$

As $u \in V(G_1) - S_1$ or $u \in V(G_2) - S_2$, without loss of generality let us take $u \in V(G_1) - S_1$ which implies

$$S_1(u) - \{v\} = \big(S_1(w) - \{v\}\big) \cup \big(S_2(w) - \{v\}\big).$$

As $\big(S_1 - \{v\}\big) \cap \big(S_2 - \{v\}\big) = \Phi$, $S_2(w) - \{v\} = \Phi$ which implies $w \notin V(G_2) - S_2$. Therefore $S_1(u) - \{v\} = S_1(w) - \{v\}$, i.e. $S_1 - \{v_1\}$ is not the RD-set of $G_1 - \{v_1\}$. This contradicts our assumption. Therefore

$$S(u) \neq S(w) \text{ for all } u, w \in V(G) - \big(S \cup \{v\}\big) \tag{2.6}$$

Now, we have to prove that $S(v) \neq S(u)$ for all $u \in V(G) - \big(S \cup \{v\}\big)$.
As $S(v) = \big(N_{G_1}(v_1) \cap S_1\big) \cup \big(N_{G_2}(v_2) \cap S_2\big)$, implies $S(v) \subseteq S_1$ and $S(v) \subseteq S_2$.

If $S(u) = S(v)$ then $S(u) \subseteq S_1$ and $S(u) \subseteq S_2$ that is u is adjacent to vertices of S_1 and S_2. But v is only vertex which is adjacent to $V(G_1)$ and $V(G_2)$. Therefore

$$S(u) \neq S(v) \text{ for all } u \in V(G) - \big(S \cup \{v\}\big) \tag{2.7}$$

Combining (2.6) and (2.7) we have, $S(u) \neq S(w)$ for all $u, w \in V(G) - S$. Hence S is the LD-set of G with cardinality $RD(G_1) + RD(G_2) - 2$. Thus by Theorem 2.1, S must be the RD-set of G.

Therefore $RD(G) = RD(G_1) + RD(G_2) - 2$. ■

Theorem 2.3. Let G_1, G_2 be any two graphs and $G = G_1\{v_1\} \bowtie_f G_2\{v_2\}$. If for both $i = 1$ and 2, v_i does not belongs to anyone of the RD-set of G_i then
$RD(G) = RD(G_1) + RD(G_2)$.

Proof. Let S be the RD-set of G. If $v \in S$, then $S \cap V(G_1)$ and $S \cap V(G_2)$ are not RD-set of G_1 and G_2 respectively. But as S is the RD-set of G and $v \in S$, $S \cap V(G_1)$ and $S \cap V(G_2)$ are LD-set of G_1 and G_2. Hence $RD(G_1) + 1 \leq |S \cap V(G_1)|$ and $RD(G_2) + 1 \leq |S \cap V(G_2)|$.
Therefore (2.2) becomes, $RD(G) \geq RD(G_1) + RD(G_2) + 1$. This contradicts the Theorem 2.1. Hence $v \notin S$ and (2.2) transforms to

$$RD(G) = |S \cap V(G_1)| + |S \cap V(G_2)| \tag{2.8}$$

We have to prove that $|S| = RD(G_1)+RD(G_2)$ and we prove this result by contradiction method by assuming that G has an RD-set S with
$RD(G) = |S| \leq RD(G_1)+ RD(G_2) - 1$.

Hence (2.8) implies $|S \cap V(G_1)| + |S \cap V(G_2)| \leq RD(G_1) + RD(G_2) - 1$. i.e. either $|S \cap V(G_1)| \leq RD(G_1) - 1$ or $|S \cap V(G_2)| \leq RD(G_2) - 1$. Without loss of generality let us take that $|S \cap V(G_1)| \leq RD(G_1) - 1$.

Now let us denote $S \cap V(G_1)$ by S_1. Clearly, S_1 is not the RD-set of G_1. Therefore there exist $u, w \in V(G_1) - S_1$ such that $S_1(u) = S_1(w)$.

Claim: Either $u = v_1$ or $w = v_1$.

If $u, w \neq v_1$, then

$$S(u) = \big(S(u) \cap V(G_1)\big) \cup \big(S(u) \cap V(G_2)\big).$$

And as v is the only vertex common to $V(G_1)$ and $V(G_2)$ and $v \notin S$,
$S(u) = S_1(u) \cup \Phi = S_1(u)$. Similarly we can show that $S(w) = S_1(w)$.

Therefore $S(u) = S_1(u) = S_1(w) = S(w)$, that is S is not an LD-set. This contradiction implies that either $u = v_1$ or $w = v_1$.

$\Longrightarrow G_1 - \{v_1\}$ has an LD-set S_1 such that $|S_1| \leq RD(G_1) - 1$

$\Longrightarrow G_1$ has an LD-set $S_1 \cup \{v_1\}$ such that

$$|S_1 \cup \{v_1\}| \leq RD(G_1) \tag{2.9}$$

As $RD(G_1)$ is the minimum cardinality of any LD-set of G_1,

$$RD(G_1) \leq |S_1 \cup \{v_1\}| \tag{2.10}$$

From (2.9) and (2.10), $RD(G_1) = |S_1 \cup \{v_1\}|$. Hence G_1 has an RD-set which contains the vertex v_1. This is a contradiction to our hypothesis that G_1 has no RD-set which contains v_1. Hence $|S \cap V(G_1)| \geq RD(G_1)$. Similarly we can prove that $|S \cap V(G_2)| \geq RD(G_2)$. Hence (2.8) becomes,

$$RD(G) = |S \cap V(G_1)| + |S \cap V(G_2)| \geq RD(G_1) + RD(G_2) \tag{2.11}$$

Therefore by Theorem 2.1 and (2.11) we get that $RD(G) = RD(G_1) + RD(G_2)$. ■

Theorem 2.4. Let G_1, G_2 be any two graphs and $G = G_1\{v_1\} \bowtie_f G_2\{v_2\}$. Let either G_1 or G_2 has atleast one RD-set S_1 or S_2 respectively such that $v_1 \in S_1$ or $v_2 \in S_2$ (say $v_1 \in S_1$ and v_2 does not belong to any RD-set of G_2). If $S_1 - \{v_1\}$ is the RD-set of $G_1 - \{v_1\}$ for some RD-set S_1 then $RD(G) = RD(G_1) + RD(G_2) - 1$ otherwise $RD(G) = RD(G_1) + RD(G_2)$.

Proof. We prove this theorem in two cases, by taking that G_1 has an RD-set S_1 such that $S_1 - \{v_1\}$ is the RD-set of $G_1 - \{v_1\}$ and G_1 has no RD-set S_1 such that $S_1 - \{v_1\}$ is the RD-set of $G_1 - \{v_1\}$.

Case 1: G_1 has an RD-set S_1 such that $S_1 - \{v_1\}$ is the RD-set of $G_1 - \{v_1\}$.
Let $S = (S_1 \cup S_2) - \{v\}$ then

$$S(u) = \big(S_1(u) \cup S_2(u)\big) - \{v\} = \big(S_1(u) - \{v\}\big) \cup S_2(u)$$

Therefore

$$S(u) = \begin{cases} S_1(u) - \{v_1\}, & u \in V(G_1) - S_1 \\ S_2(u), & u \in V(G_2) - S_2 \end{cases} \text{ and } S(v) \supseteq S_2(v_2) = S_2(v)$$

We will show that S is an RD-set of G. $S(u) = S_2(u) \neq S_2(v) \subseteq S(v)$ for all $u \in V(G_2) - S_2$ and $S(v) \neq S(u)$ for all $u \in V(G_1) - (S_1 - \{v\})$ as $S_2(v) \subseteq S(v)$.

If $S(u) = S(w)$ where $u, w \neq v$ then,

$$\big(S_1(u) - \{v\}\big) \cup S_2(u) = \big(S_2(w) - \{v\}\big) \cup S_2(w)$$

As $\big(S_1 - \{v\}\big) \cap S_2 = \Phi$, $S_1(u) - \{v\} = S_1(w) - \{v\}$ and $S_2(u) = S_2(w)$.

Without loss of generality let assume that $u \in V(G_1) - S_1$, then u is not adjacent to any vertices in S_2. Hence $S_2(u) = \Phi$ and as $S_2(u) = S_2(w)$, $S_2(w) = \Phi$ which implies w is not adjacent to any vertices in S_2. Therefore $w \notin V(G_2) - S_2$, i.e. $w \in V(G_1)$. Hence if $u \in V(G_1) - S_1$ then $w \in V(G_1) - S_1$.

$\Longrightarrow S_1(u) - \{v\} = S_1(w) - \{v\}$
$\Longrightarrow S_1 - \{v\}$ is not an RD-set of $G_1 - \{v\}$.

This is a contradiction, therefore $S(u) \neq S(w)$ for all $u, w \in V(G) - S$. Hence $S = (S_1 \cup S_2) - \{v\}$ is an LD-set with $|S| = |S_1| + |S_2| - 1$.

Therefore $RD(G) \leq RD(G_1) + RD(G_2) - 1$. By Theorem 2.1 and Theorem 2.2 we have $RD(G) = RD(G_1) + RD(G_2) - 1$.

Case 2: G_1 has no RD-set S_1 such that $S_1 - \{v_1\}$ is the RD-set of $G_1 - \{v_1\}$.

By Theorem 2.2, $RD(G_1) + RD(G_2) - 1 \leq RD(G) \leq RD(G_1) + RD(G_2)$ and we have to prove $RD(G) = RD(G_1) + RD(G_2)$. We prove this by the method of contradiction by assume that G has an RD-set S such that $RD(G) = RD(G_1) + RD(G_2) - 1$. By (2.2),

$$RD(G_1) + RD(G_2) - 1 = |S \cap V(G_1)| + |S \cap V(G_2)| - \left|\big(S \cap V(G_1)\big) \cap \big(S \cap V(G_2)\big)\right| \quad (2.12)$$

Since v is only vertex which is in $V(G_1)$ and $V(G_2)$,

$$\left|\big(S \cap V(G_1)\big) \cap \big(S \cap V(G_2)\big)\right| \leq 1$$

Case 2.1: $\left|\big(S \cap V(G_1)\big) \cap \big(S \cap V(G_2)\big)\right| = 1$
i.e. $v \in S \cap V(G_2)$. Hence $S \cap V(G_2)$ is not the minimal LD-set of G_2 as RD-set G_2 does not contains $v_2 = v$. Hence $|S \cap V(G_1)| \geq RD(G_2) + 1$ and $S \cap V(G_1)$ must be the LD-set of G_1, otherwise S cannot be the RD-set of G, so $|S \cap V(G_1)| \geq RD(G_1)$.

Thus (2.12) becomes $RD(G_1) + RD(G_2) - 1 \geq RD(G_1) + RD(G_2)$. This contradiction implies that $RD(G) \neq RD(G_1) + RD(G_2) - 1$.

Case 2.2: $\left|\big(S \cap V(G_1)\big) \cap \big(S \cap V(G_2)\big)\right| = 0$
i.e. $v \notin S$. As S is the RD-set of G, $S \cap V(G_2)$ must atleast be the LD-set of $G_2 - \{v_2\}$. And $|S \cap V(G_2)| \geq RD(G_2)$, if not i.e., $|S \cap V(G_2)| < RD(G_2)$. Let us take the least possibility that $|S \cap V(G_2)| = RD(G_2) - 1$, then $\big(S \cap V(G_2)\big) \cup \{v_2\}$ must be the LD-set of G_2 and $\left|\big(S \cap V(G_2)\big) \cup \{v_2\}\right| = RD(G_2)$. Hence $\big(S \cap V(G_2)\big) \cup \{v_2\}$ is the RD-set of G_2, which contradict the hypothesis that none of the RD-set of G_2 contain v_2. Therefore $|S \cap V(G_2)| \geq RD(G_2)$.

Now (2.12) becomes, $RD(G_1) + RD(G_2) - 1 = |S| \geq |S \cap V(G_1)| + RD(G_2)$. This implies that

$$|S \cap V(G_1)| \leq RD(G_1) - 1 \quad (2.13)$$

Hence $S \cap V(G_1)$ is not an LD-set of G_1. But as S is the RD-set of G, $S \cap V(G_1)$ must be LD-set of $G_1 - \{v_1\}$. Hence $\big(S \cap V(G_1)\big) \cup \{v_1\}$ is the LD-set of G_1. Therefore

$$RD(G_1) \leq \left|\big(S \cap V(G_1)\big) \cup \{v_1\}\right| \quad (2.14)$$

But by (2.13),

$$\left|\big(S \cap V(G_1)\big) \cup \{v_1\}\right| = |S \cap V(G_1)| + |\{v_1\}| \leq RD(G_1) \quad (2.15)$$

From (2.14) and (2.15),

$$\left|\big(S \cap V(G_1)\big) \cup \{v_1\}\right| = RD(G_1).$$

i.e., G_1 has an RD-set $\big(S \cap V(G_1)\big) \cup \{v_1\}$ such that $S \cap V(G_1)$ is the LD-set of $G_1 - \{v_1\}$. But $S \cap V(G_1)$ is the minimal LD-set of $G_1 - \{v_1\}$, if not G_1 would have an RD-set with cardinality less than $RD(G_1)$. Hence $S \cap V(G_1)$ is the RD-set of $G_1 - \{v_1\}$ with the property that $\big(S \cap V(G_1)\big) \cup \{v_1\}$ is the RD-set of G_1, this is a contradiction to our assumption. Therefore $RD(G) \neq RD(G_1) + RD(G_2) - 1$.

Hence from both Case 2.1 and Case 2.2, $RD(G)$ cannot be equal to
$RD\,(G_1) + RD\,(G_2) - 1$. Thus $RD\,(G) = RD\,(G_1) + RD\,(G_2)$. ■

Theorem 2.5. Let G_1, G_2 be any two graphs and $G = G_1\{v_1\} \bowtie_f G_2\{v_2\}$. For $i = 1, 2$, let G_i have an RD-set S_i such that $v_i \in S_i$. If for every RD-set S_i, there exist a vertex $u_i \in V(G_i) - S_i$ such that $S_i(u_i) = \{v_i\}$ for both $i = 1$ and 2 then
$RD(G) = RD(G_1) + RD(G_2)$.

Proof. Let S be the RD-set of G and we have to prove that $|S| = RD(G_1) + RD(G_2)$.

By Theorem 2.2, $RD(G_1) + RD(G_2) - 1 \leq RD(G) \leq RD(G_1) + RD(G_2)$. Let us assume that $|S| = RD(G_1) + RD(G_2) - 1$ and we will show that it contradict our hypothesis that for every RD-set S_i, $S_i(u_i) = \{v_i\}$ for a unique vertex $u_i \in V(G_i) - S_i$ for $i = 1, 2$. By (2.2),

$$RD(G_1) + RD(G_2) - 1 = |S \cap V(G_1)| + |S \cap V(G_2)| - \Big|\big(S \cap V(G_1)\big) \cup \big(S \cap V(G_2)\big)\Big| \quad (2.16)$$

By (2.3), $\Big|\big(S \cap V(G_1)\big) \cup \big(S \cap V(G_2)\big)\Big| \leq 1$. So we prove the theorem by considering two cases, with $\Big|\big(S \cap V(G_1)\big) \cup \big(S \cap V(G_2)\big)\Big| = 0$ and $\Big|\big(S \cap V(G_1)\big) \cup \big(S \cap V(G_2)\big)\Big| = 1$.

Case 1: $\Big|\big(S \cap V(G_1)\big) \cup \big(S \cap V(G_2)\big)\Big| = 0$, i.e. $v \notin S$.

Then by (2.16), $RD(G_1) + RD(G_2) - 1 = |S| = |S \cap V(G_1)| + |S \cap V(G_2)|$. Thus either $|S \cap V(G_1)| \leq RD(G_1) - 1$ or $|S \cap V(G_2)| \leq RD(G_2) - 1$.

Without loss of generality let us take $|S \cap V(G_1)| \leq RD(G_1) - 1$, hence $S \cap V(G_1)$ is not the LD-set of G_1. But $S \cap V(G_1)$ must be the RD-set of $G_1 - \{v_1\}$, as S is the RD-set of G. Thus $S_3 = \big(S \cap V(G_1)\big) \cup \{v_1\}$ is the RD-set of G_1 with
$S_3(u) \subseteq \big(S(u) \cap V(G_1)\big) \cup \{v_1\}$ and $S(u) \neq \Phi$ for all $u \in V(G_1) - S_3$.

Hence $S_3(u) \neq (v_1)$ for all $u \in V(G_1) - S_3$, this contradict the hypothesis. Thus $|S| \neq RD(G_1) + RD(G_2) - 1$.

Case 2: $\Big|\big(S \cap V(G_1)\big) \cup \big(S \cap V(G_2)\big)\Big| = 1$, i.e. $v \in S$.
Therefore (2.16) becomes

$$RD(G_1) + RD(G_2) - 1 = |S| = |S \cap V(G_1)| + |S \cap V(G_2)| - 1$$

If $|S \cap V(G_1)| < RD(G_1)$ then $S \cap V(G_1)$ is not the LD-set of G_1 and $G_1 - \{v_1\}$ as $v_1 \in S \cap V(G_1)$. Otherwise $S \cap V(G_1)$ will be the LD-set of G_1 with cardinality less than $RD(G_1)$. So S cannot be the LD-set of G. Therefore $|S \cap V(G_1)| = RD(G_1)$ and similarly $|S \cap V(G_2)| = RD(G_2)$.

As S is the RD-set of G and $v \in S$, for $i = 1, 2$, $S \cap V(G_i)$ must be the LD-set of G_i, but as $|S \cap V(G_i)| = RD(G_i)$, $S \cap V(G_i)$ must be the RD-set of G_i. So, for

$i = 1$ and 2, $S_i = S \cap V(G_i)$ is the RD-set G_i with $v_i \in S_i$. So by the hypothesis of the theorem, there exist $u_i \in V(G_i) - S_i$ such that $S_i(u_i) = \{v_i\}$. Therefore

$$S(u_1) = \big(S(u_1) \cap V(G_1)\big) \cup \big(S(u_2) \cap V(G_2)\big) = S_1(u_1) = \{v_1\} = \{v\}$$

Similarly $S(u_2) = S_2(u_2) = \{v_2\} = \{v\}$. As S is the RD-set of G, $S(u_1) \neq S(u_2)$. This contradict the hypothesis that $S_1(u_1) = \{v_1\}$ and $S_2(u_2) = \{v_2\}$. Therefore $|S| \neq RD(G_1) + RD(G_2) - 1$.

Thus from both cases we infer that $RD(G)$ must be equal to $RD(G_1) + RD(G_2)$. ■

Theorem 2.6. Let G_1, G_2 be any two graphs and $G = G_1\{v_1\} \bowtie_f G_2\{v_2\}$. For $i = 1, 2$, let G_i have an RD-set S_i such that $v_i \in S_i$. If G_2 has an RD-set S_2 such that $S_2(u_2) \neq \{v_2\}$ for all $u_2 \in V(G_2) - S_2$ and G_1 have no RD-set S_1 such that $S_1(u_1) \neq \{v_1\}$ for all $u_1 \in V(G_1) - S_1$ then $RD(G) = RD(G_1) + RD(G_2) - 1$.

Proof. By Theorem 2.2, $RD(G_1) + RD(G_2) - 1 \leq RD(G)$. Now let us show that G has an RD-set with cardinality $RD(G_1) + RD(G_2) - 1$.

Let $S = S_1 \cup S_2$, where S_1 and S_2 are the RD-set of G_1 and G_2 respectively with $v_1 \in S_1$, $v_2 \in S_2$ and satisfying the condition that G_2 has an RD-set S_2 such that $S_2(u_2) \neq \{v_2\}$ for all $u_2 \in V(G_2) - S_2$ and G_1 have no RD-set S_1 such that $S_1(u_1) \neq \{v_1\}$ for all $u_1 \in V(G_1) - S_1$. Then

$$S(u) = \begin{cases} S_1(u), & u \in V(G_1) - S_1 \\ S_2(u), & u \in V(G_2) - S_2. \end{cases}$$

We have to prove that S is an LD-set of G. Suppose not, i.e. S is not the LD-set of G, then there exist some vertices $u, w \in V(G) - S$ such that $S(u) = S(w)$.

Case 1: $u, w \in V(G_1) - S_1$

Then $S_1(u) = S(u) = S(w) = S_1(w)$. Thus S_1 cannot be the LD-set of G_1. This contradiction implies that $S(u) \neq S(w)$.

Case 2: $u,\ w \in V(G_2) - S_3$. Similarly to Case 1 we can show that $S(u) \neq S(w)$.

Case 3: $u \in V(G_1) - S_1$ and $w \in V(G_2) - S_2$ then,

$$S_1(u) = S(u) = S(w) = S_2(w)$$

As $S_1 \cap S_2 = \{v\}$, we have $S_1(u) = S_2(w) = \{v\}$. Hence $S_2(w) = \{v\} = \{v_2\}$, i.e. there exist an vertex $w \in V(G_2) - S_2$ such that $S_2(w) = \{v_2\}$. This contradict the hypothesis implies that $S(u) \neq S(w)$.

Hence $S = S_1 \cup S_2$ is the LD-set of G, as for all $u, w \in V(G) - S$, $S(u) \neq S(w)$. But as $|S| = |S_1| + |S_2| - |S_1 \cap S_2| = RD(G_1) + RD(G_2) - 1$ and S is the LD-set, S must be the RD-set of G. ■

Remark 2.7. Let G_1, G_2 be any two graphs and $G = G_1\{v_1\}\bowtie_f G_2\{v_2\}$. For $i = 1, 2$, let G_i have an RD-set S_i such that $v_i \in S_i$ and $S_i(u_i) \neq \{v_i\}$ for all $u_i \in V(G_i) - S_i$. Suppose for either $i = 1$ or $i = 2$, if G_i does not have any RD-set S_i satisfying the both conditions that

(i) $S_i - \{v_i\}$ is the RD-set of $G_i - \{v_i\}$

(ii) $N_{G_i}(v_i) \cap S_i \neq \Phi$

then by Theorem 2.2, $RD(G_1) + RD(G_2) - 1 \leq RD(G)$. And likewise to Theorem 2.6, we can prove that $S = S_1 \cup S_2$ is the RD-set of G with $|S| = RD(G) = RD(G_1) + RD(G_2) - 1$.

Acknowledgment

This research is supported by UGC scheme RGNF. Award letter F1-17.1/2014-15/RGNF-2014-15-SC-TAM-80373/(SAIII/Website).

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