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Classes of Multivalent Functions Defined by Linear Operator $L_{q,s}^{p,\alpha}(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s)$ and Sandwich Theorems

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Abstract

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n,$$

which are analytic and p-valent in the unit disk $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$.

Let

$$\begin{aligned} {}_l^p f_m(\alpha_1, \alpha_2, \ldots \alpha_l; \beta 1, \beta 2, \ldots \beta m; z) &= z^p \, {}_l F_m(\alpha_1, \alpha_2, \ldots \alpha_l; \beta 1, \beta 2, \ldots \beta m) \\ &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_l)_n}{(\beta_1)_n (\beta_2)_n \cdots (\beta_m)_n} \frac{z^{n+p}}{n!} \end{aligned}$$

$(l \le m+1; l, m \in \mathbb{N})$ We define ${}_l^{p,\alpha} f_m^{(-1)}(\alpha_1, \alpha_2, \ldots \alpha_l; \beta 1, \beta 2, \ldots \beta m; z)$ by using convolution * as

$$\begin{aligned} &{}_l^p f_m(\alpha_1, \alpha_2, \ldots \alpha_l; \beta 1, \beta 2, \ldots \beta m; z) * {}_l^{p,\alpha} f_m^{(-1)}(\alpha_1, \alpha_2, \ldots \alpha_l; \beta 1, \beta 2, \ldots \beta m; z) \\ &= z^p (1-z)^{-2(1-\alpha)} \end{aligned}$$

We use this definition to introduce an operator $L_{l,m}^{p,\alpha}: \mathcal{A}_p \to \mathcal{A}_p$ as

$$L_{l,m}^{p,\alpha}(\alpha_1, \alpha_2, \ldots \alpha_l; \beta 1, \beta 2, \ldots \beta m) f(z)$$
$$= {}_l^{p,\alpha} f_m^{(-1)}(\alpha_1, \alpha_2, \ldots \alpha_l; \beta 1, \beta 2, \ldots, \beta m; z) * f(z)$$

Using this operator, we define some new classes of p-valent functions and study their properties by using cetain first order differential subordination and superordination. Also certain inclusion relations are established and an integral transform is discussed.

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1. Introduction

Let $\mathcal{H}$ be the class of functions analytic in $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$ and $\mathcal{H}(a, n)$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$ Let A_p denote the class of normalized analytic p-valent functions in the unit disk $\mathbb{U}$ that have the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n. \tag{1.1}$$

For functions $f(z) = z + \sum_{n=p+1}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=p+1}^{\infty} b_n z^n$ in the class A_p, the Hardamard Product is defined as follows.

$$(f * g)(z) = z + \sum_{n=p+1}^{\infty} a_n b_n z^n, \quad z \in \mathbb{U}. \tag{1.2}$$

For $\alpha_j \in \mathbb{C}(j = 1, 2, \ldots, l)$ and $\beta_j \in \mathbb{C} - 0, -1, -2, \ldots (j = 1, 2, \ldots, m)$, the generalized hypergeometric function ${}_lF_m(\alpha_1, \alpha_2, \ldots \alpha_l; \beta_1, \beta_2, \ldots \beta_m : z)$ is defined by the series

$${}_lF_m(\alpha_1, \alpha_2, \ldots \alpha_l; \beta_1, \beta_2, \ldots \beta_m : z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_l)_n}{(\beta_1)_n (\beta_2)_n \cdots (\beta_m)_n} \frac{z^{n+p}}{n!}$$

$(l \leq m + 1; l, m \in \mathbb{N}_0)$
where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)(a+2)\cdots(a+n-1) \quad for n \in \mathbb{N}$$

and it is 1 for n=0. Corresponding to the function

$$ {}_l^p f_m(\alpha_1, \alpha_2, \ldots \alpha_l; \beta_1, \beta_2, \ldots \beta_m; z) = z^p \, {}_lF_m(\alpha_1, \alpha_2, \ldots \alpha_l; \beta_1, \beta_2, \ldots \beta_m) $$

$$ = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n \cdots (\alpha_l)_n}{(\beta_1)_n(\beta_2)_n \cdots (\beta_m)_n} \frac{z^{n+p}}{n!} $$

We define ${}_l^{p,\alpha} f_m^{(-1)}(\alpha_1, \alpha_2, \ldots \alpha_l; \beta_1, \beta_2, \ldots \beta_m; z)$ by using convolution * as

$$ {}_l^p f_m(\alpha_1, \alpha_2, \ldots \alpha_l; \beta_1, \beta_2, \ldots \beta_m; z) * {}_l^{p,\alpha} f_m^{(-1)}(\alpha_1, \alpha_2, \ldots \alpha_l; \beta_1, \beta_2, \ldots \beta_m; z) $$
$$ = z^p(1-z)^{-2(1-\alpha)}, \, for \alpha < 1 $$

We use this definition to introduce an operator $L_{l,m}^{p,\alpha} : \mathcal{A}_p \to \mathcal{A}_p$ as

$$ \begin{aligned} & L_{l,m}^{p,\alpha}(\alpha_1, \alpha_2, \ldots \alpha_l; \beta_1, \beta_2, \ldots \beta_m) f(z) \\ &= {}_l^{p,\alpha} f_m^{(-1)}(\alpha_1, \alpha_2, \ldots \alpha_l; \beta_1, \beta_2, \ldots \beta_m; z) * f(z) \\ &= z^p + \sum_{n=p+1}^{\infty} \frac{(\beta_1)_{n-p}(\beta_2)_{n-p} \cdots (\beta_m)_{n-p}}{(\alpha_1)_{n-p}(\alpha_2)_{n-p} \cdots (\alpha_l)_{n-p}} (2-2\alpha)_{n-p} a_n z^n \end{aligned} \tag{1.3} $$

For $l = m + 1$ and $\alpha_2 = \beta_1, \alpha_3 = \beta_2, \ldots, \alpha_l = \beta_m$, we note that, $L_{l,m}^{1,0}(1, \alpha_2, \ldots \alpha_l; \beta_1, \ldots \beta_m) f(z) = z f'(z)$ and $L_{l,m}^{1,0}(2, \alpha_2, \ldots \alpha_l; \beta_1, \ldots \beta_m) f(z) = f(z)$ To make the notation simple we write,

$$ L_{l,m}^{p,\alpha}[\alpha_1] f(z) = L_{l,m}^{p,\alpha}(\alpha_1, \alpha_2, \ldots \alpha_l; \beta_1, \beta_2, \ldots \beta_m) f(z) $$

From (1.3), it is obtained that

$$ \alpha_1 L_{l,m}^{p,\alpha}[\alpha_1] f(z) = z(L_{l,m}^{p,\alpha}[\alpha_1 + 1] f(z))' + (\alpha_1 - p)(L_{l,m}^{p,\alpha}[\alpha_1 + 1] f(z)) \tag{1.4} $$

Relation (1.4) plays an important role in obtaining our results.
An analytic function f is said to be subordinate to another analytic function g, written as

$$ f(z) \prec g(z), \quad (z \in \mathbb{U}) $$

if there exists a Schwarz function w, which is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathbb{U}$ such that $f(z) = g(w(z))$.

A function $f \in \mathcal{A}_p$ is said to be in the class $H_{a,c,p}(A, B)$ if it satisfies the following subordination.

$$ \frac{L_p(a+1, c) f(z)}{L_p(a, c) f(z)} \prec 1 + \frac{A - B}{a} \frac{z}{1 + Bz}, (z \in \mathbb{U} : a \neq 0 : -1 \leq B < A \leq 1) $$

where $L_p(a, c)$ is the familiar Carlson and Shaffer operator [3] given by $L_p(a, c) := L_{2,1}^{p,\frac{1}{2}}(c, 1; a)$. This class $H_{a,c,p}(A, B)$ was introduced by Liu and Owa [5] and they have proved the following.

Theorem 1.1. (Liu and Owa [[5],Theorem 1,p.1715]) Let $a \geq \dfrac{A-B}{1-B}$, then

$$H_{a+1,c,p}(A,B) \subset H_{a,c,p}(A,B).$$

Theorem 1.2. (Liu and Owa[[5],Theorem 3,p.1717]) Let $f(z) \in \mathcal{A}_p$. Then $f \in H_{a,c,p}(A,B)$ if and only if

$$F(z) = \frac{a}{z^{a-p}} \int_0^z t^{a-p-1} f(t)dt \in H_{a+1,c,p}(A,B)$$

For two analytic functions f and F, we say that F is superordiante to f if f is subordiante to F. Recently Miller and Mocanu [7] considered certain second order differential superordinations. Using the results of Miller and Mocanu [7], Bulboaca have considered certain classes of first order differentail superordinations [2] and superordination preserving integral operator [1]. In present investigation, on the lines of Kumar and Taneja [4], we define more general classes of p-valent functions which we define below using subordination and superordination.

Definition 1.3. A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{A}^\alpha(p,n,\alpha_1,\alpha_2,\ldots,\alpha_l;\beta_1,\beta_2,\ldots,\beta_m : \varphi)$ if it satisfies the following subordination:

$$\frac{L_{l,m}^{p,\alpha}[\alpha_1]f(z)}{L_{l,m}^{p,\alpha}[\alpha_1+1]f(z)} \prec \varphi(z). \tag{1.5}$$

and it is said to be in $\overline{\mathcal{A}^\alpha}(p,n,\alpha_1,\alpha_2,\ldots,\alpha_l;\beta_1,\beta_2,\ldots,\beta_m : \varphi)$ if f satisfies the following superordination:

$$\varphi(z) \prec \frac{L_{l,m}^{p,\alpha}[\alpha_1]f(z)}{L_{l,m}^{p,\alpha}[\alpha_1+1]f(z)} \tag{1.6}$$

where $\varphi(z)$ is analytic in $\mathbb{U}$ and $\varphi(0)=1$.

To make the notation simple, we also write

$$\mathcal{A}^\alpha(p,n,\alpha_1 : \varphi) := \mathcal{A}^\alpha(p,n,\alpha_1,\alpha_2,\ldots,\alpha_l;\beta_1,\beta_2,\ldots,\beta_m : \varphi)$$

and

$$\overline{\mathcal{A}^\alpha}(p,n,\alpha_1 : \varphi) := \overline{\mathcal{A}^\alpha}(p,n,\alpha_1,\alpha_2,\ldots,\alpha_l;\beta_1,\beta_2,\ldots,\beta_m : \varphi)$$

Also we define the class $\mathcal{A}^\alpha(p,n,\alpha_1 : \varphi_1,\varphi_2)$ by the following:

$$\mathcal{A}^\alpha(p,n,\alpha_1 : \varphi_1,\varphi_2) = \overline{\mathcal{A}^\alpha}(p,n,\alpha_1 : \varphi_1) \cap \mathcal{A}^\alpha(p,n,\alpha_1 : \varphi_2)$$

Definition 1.4. [[7], Definition 2.p.817] Denote by $\mathcal{Q}$, the set of all functions $f(z)$ that are analytic and injective on $\mathbb{U} - E(f)$, where

$$E(f) = \left\{\zeta \in \partial\mathbb{U} : z \overset{\lim}{\to} \zeta \ f(z) = \infty\right\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U} - E(f)$.

Lemma 1.5. [6][12. Theorem 3.4h.p.132]. Let $\psi(z)$ be univalent in the unit disk $\mathbb{U}$ amd let ϑ and φ be analytic in a domain $D \supset \psi(\mathbb{U})$ with $\varphi(w) \neq 0$ when $w \in \psi(\mathbb{U})$.

$$Q(z) := z\psi'(z)\varphi(\psi(z))$$

$$h(z) := \vartheta(\psi(z)) + Q(z)$$

Suppose that

(1) $Q(z)$ is starlike in $\mathbb{U}$ and

(2) $\mathcal{R}\left(\dfrac{zh'(z)}{Q(z)}\right) > 0$ for $z \in \mathbb{U}$

If $q(z)$ is analytic in $\mathbb{U}$, with $q(0) = \psi(0)$, $q(\mathbb{U}) \subset D$ and

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(\psi(z)) + z\psi'(z)\varphi(\psi(z)) \tag{1.7}$$

then $q(z) \prec \psi(z)$ and $\psi(z)$ is the best dominant.

Lemma 1.6. [2] Let $\psi(z)$ be univalent in the unit disk $\mathbb{U}$ amd let ϑ and φ be analytic in a domain $D \supset \psi(\mathbb{U})$. Suppose that

(1) $\mathcal{R}\left(\dfrac{\vartheta'(\psi(z))}{\varphi(\psi(z))}\right) > 0$ for $z \in \mathbb{U}$

(2) $z\psi'(z)\varphi(\psi(z))$ is starlike in $\mathbb{U}$.

If $q(z) \in \mathcal{H}(\psi(0), 1) \cap \mathcal{Q}$, with $q(\mathbb{U}) \subseteq D$ and $\vartheta(q(z)) + zq'(z)\vartheta(q(z))$ is univalent in $\mathbb{U}$, then

$$\vartheta(\psi(z)) + z\psi'(z)\varphi(\psi(z)) \prec \vartheta(q(z)) + zq'(z)\varphi(q(z)) \tag{1.8}$$

implies $\psi(z) \prec q(z)$ and $\psi(z)$ is the best subordinant.

2. Results involving Linear Operator

By using Lemma 1.5, we prove the result of subordination.

Theorem 2.1. Let $\psi(z)$ be univalent in $\mathbb{U}$, $\psi(0) = 1$. Assume that $z\psi'(z)/\psi$ is starlike in $\mathbb{U}$ and $\mathcal{R}(\alpha_1\psi(z)) > 0$. Let $\chi(z)$ be defined by

$$\chi(z) := \frac{1}{\alpha_1 - 1}\left[\alpha_1\psi(z) - 1 + \frac{z\psi'(z)}{\psi(z)}\right] \text{ where } \alpha_1 \neq 1 \tag{2.1}$$

If $f \in \mathcal{A}^\alpha(p, n, \alpha_1 - 1 : \chi)$ then $f \in \mathcal{A}^\alpha(p, n, \alpha_1 : \psi)$.
If $f \in \overline{\mathcal{A}^\alpha}(p, n, \alpha_1 - 1 : \chi)$ and

$$0 \neq \frac{L_{l,m}^{p,\alpha}[\alpha_1]f(z)}{L_{l,m}^{p,\alpha}[\alpha_1 + 1]f(z)} \in \mathcal{H}(1, 1) \cap \mathcal{Q}, \tag{2.2}$$

$\dfrac{L_{l,m}^{p,\alpha}[\alpha_1 - 1]f(z)}{L_{l,m}^{p,\alpha}[\alpha_1]f(z)}$ is univalent in $\mathbb{U}$ then $f \in \overline{\mathcal{A}^\alpha}(p, n, \alpha_1 : \psi)$.

Proof. We define a function

$$q(z) = \frac{L_{l,m}^{p,\alpha}[\alpha_1]f(z)}{L_{l,m}^{p,\alpha}[\alpha_1 + 1]f(z)} \tag{2.3}$$

Clearly $q(z)$ is analytic in $\mathbb{U}$.
Taking logarithmic derivative of both sides, we get,

$$\frac{zq'(z)}{q(z)} = \frac{z\left(L_{l,m}^{p,\alpha}[\alpha_1]f(z)\right)'}{L_{l,m}^{p,\alpha}[\alpha_1]f(z)} - \frac{z\left(L_{l,m}^{p,\alpha}[\alpha_1 + 1]f(z)\right)'}{L_{l,m}^{p,\alpha}[\alpha_1 + 1]f(z)} \tag{2.4}$$

Now, by using the identity (1.4), we get,

$$\frac{1}{\alpha - 1}\left[\alpha_1 q(z) - 1 + \frac{zq'(z)}{q(z)}\right] = \frac{L_{l,m}^{p,\alpha}[\alpha_1 - 1]f(z)}{L_{l,m}^{p,\alpha}[\alpha_1]f(z)} \tag{2.5}$$

Since $f \in \mathcal{A}^\alpha(p, n, \alpha_1 - 1 : \chi)$, we get,

$$\frac{L_{l,m}^{p,\alpha}[\alpha_1 - 1]f(z)}{L_{l,m}^{p,\alpha}[\alpha_1]f(z)} \prec \chi(z)$$

then, from (2.5) and definition of χ,

$$\alpha_1 q(z) + \frac{zq'(z)}{q(z)} \prec \alpha_1\psi(z) + \frac{z\psi'(z)}{\psi(z)} \tag{2.6}$$

Now, we define

$$\vartheta(w) := \alpha_1 w \text{ and } \varphi(w) := \frac{1}{w}$$

Note that $\varphi(w) \neq 0$ and $\vartheta(w)$, $\varphi(w)$ are analytic in $C - \{0\}$. Set

$$Q(z) := \frac{z\psi'(z)}{\psi(z)} \tag{2.7}$$

$$h(z) := \vartheta(\psi(z)) + Q(z) = \alpha_1\psi(z) + \frac{z\psi'(z)}{\psi(z)} \tag{2.8}$$

By hypothesis of Theorem 2.1, $Q(z)$ is starlike and

$$\mathcal{R}\left\{\frac{zh'(z)}{Q(z)}\right\} = \mathcal{R}\left\{\alpha_1\psi(z) + 1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{z\psi'(z)}{\psi(z)}\right\} > 0$$

Clearly $q(0) = \psi(0)$
By applying Lemma 1.5, we obtain that $q(z) \prec \psi(z)$ or

$$\frac{L_{l,m}^{p,\alpha}[\alpha_1]f(z)}{L_{l,m}^{p,\alpha}[\alpha_1 + 1]f(z)} \prec \psi(z)$$

which shows that $f \in \mathcal{A}^{\alpha}(p, n, \alpha_1 : \psi)$. The other half of theorem (2.1) follows by applying lemma (1.4). ■

Using Theorem (2.1), we get the following "Sandwich Theorem".

Corollary 2.2. Let $\psi_i(z)$ be univalent in $\mathbb{U}$, $\psi_i(0) = 1$, $(i = 0, 1)$. Further assume that $\dfrac{z\psi_i'(z)}{\psi_i(z)}$ is starlike univalent in $\mathbb{U}$ and $\mathcal{R}\{\alpha_1\psi_i(z)\} > 0$, $(i = 0, 1)$. If $f \in \mathcal{A}^{\alpha}(p, n, \alpha_1 - 1 : \psi_1, \psi_2)$ satisfies (2.2) then $f \in \mathcal{A}^{\alpha}(p, n, \alpha_1 : \psi_1, \psi_2)$. where

$$\chi_i(z) := \frac{1}{\alpha_1 - 1}\left[\alpha_1\psi_i(z) - 1 + \frac{z\psi_i'(z)}{\psi_i(z)}\right] \text{ where } \alpha_1 \neq 1 \text{ and } i = 1, 2$$

Theorem 2.3. Let $\psi(z)$ be univalent in $\mathbb{U}$, $\psi(0) = 1$, and λ be a complex number. Assume that $\dfrac{z\psi'}{(\lambda + p - \alpha_1 + \alpha_1\psi)}$ is starlike in $\mathbb{U}$ and $\mathcal{R}\{\lambda + p - \alpha_1 + \alpha_1\psi\} > 0$. Define the functions F and h by

$$F(z) := \frac{\lambda + p}{z^{\lambda}} \int_0^z t^{\lambda - 1} f(t)dt \tag{2.9}$$

$$h(z) := \psi(z) + \frac{z\psi'(z)}{\lambda + p - \alpha_1 + \alpha_1\psi(z)}$$

If $f \in \mathcal{A}^{\alpha}(p, n, \alpha_1 : h)$, then $F \in \mathcal{A}^{\alpha}(p, n, \alpha_1 : \psi)$. If $f \in \overline{\mathcal{A}^{\alpha}}(p, n, \alpha_1 : h)$,

$$0 \neq \frac{L_{l,m}^{p,\alpha}[\alpha_1]F(z)}{L_{l,m}^{p,\alpha}[\alpha_1 + 1]F(z)} \in \mathcal{H}(1, 1) \cap \mathcal{Q} \tag{2.10}$$

and $\dfrac{L_{l,m}^{p,\alpha}[\alpha_1]f(z)}{L_{l,m}^{p,\alpha}[\alpha_1 + 1]f(z)}$ is univalent in $\mathbb{U}$ then $F \in \overline{\mathcal{A}^{\alpha}}(p, n, \alpha_1 : \psi)$.

Proof. Form the definition of $F(z)$ and equation (1.4), we get the following equations

$$(\lambda + p)L_{l,m}^{p,\alpha}[\alpha_1]f(z) = \lambda L_{l,m}^{p,\alpha}[\alpha_1]F(z) + z\left(L_{l,m}^{p,\alpha}[\alpha_1]F(z)\right)' \tag{2.11}$$

and

$$(\lambda + p)L_{l,m}^{p,\alpha}[\alpha_1 + 1]f(z) = \alpha_1 L_{l,m}^{p,\alpha}[\alpha_1]F(z) + (\lambda + p - \alpha_1)L_{l,m}^{p,\alpha}[\alpha_1 + 1]F(z) \tag{2.12}$$

We define a function $q(z)$ by

$$q(z) = \frac{L_{l,m}^{p,\alpha}[\alpha_1]F(z)}{L_{l,m}^{p,\alpha}[\alpha_1 + 1]F(z)} \tag{2.13}$$

Clearly $q(z)$ is not analytic in $\mathbb{U}$. Using equation (2.12), we can write,

$$\frac{(\lambda + p)L_{l,m}^{p,\alpha}[\alpha_1 + 1]f(z)}{L_{l,m}^{p,\alpha}[\alpha_1 + 1]F(z)} = \lambda + p - \alpha_1 + \alpha_1 q(z) \tag{2.14}$$

Taking logarithmic differentiation of both sides and using equation (2.11), we have

$$\frac{L_{l,m}^{p,\alpha}[\alpha_1]f(z)}{L_{l,m}^{p,\alpha}[\alpha_1 + 1]f(z)} = q(z) + \frac{zq'(z)}{\lambda + p - \alpha_1 + \alpha_1 q(z)} \tag{2.15}$$

Since $f \in \mathcal{A}^{\alpha}(p, n, \alpha_1 : h)$, we have form (2.15),

$$q(z) + \frac{zq'(z)}{\lambda + p - \alpha_1 + \alpha_1 q(z)} \prec \psi(z) + \frac{z\psi'(z)}{\lambda + p - \alpha_1 + \alpha_1 \psi(z)} \tag{2.16}$$

Now, we define

$$\vartheta(w) := w \text{ and } \varphi(w) := \frac{1}{\lambda + p - \alpha_1 + \alpha_1 w}$$

Clearly $\varphi(w) \neq 0$ and $\vartheta(w), \varphi(w)$ are analytic in $\mathbb{C} - \left\{\frac{\alpha_1 - \lambda - p}{\alpha_1}\right\}$. Set

$$Q(z) := \frac{z\psi'(z)}{\lambda + p - \alpha_1 + \alpha_1 \psi(z)}$$

$$h(z) := \vartheta(\psi(z)) + Q(z) = \psi(z) + \frac{z\psi'(z)}{\lambda + p - \alpha_1 + \alpha_1 \psi(z)}$$

By hypothesis of theorem (2.3), $Q(z)$ is starlike and

$$\mathcal{R}\left\{\frac{zh'(z)}{Q(z)}\right\} = \mathcal{R}\left\{\lambda + p - \alpha_1 + \alpha_1\psi(z) + 1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{\alpha_1 z\psi'(z)}{\lambda + p - \alpha_1 + \alpha_1\psi(z)}\right\} > 0$$

By applying lemma (1.3), we obtain that

$$q(z) \prec \psi(z)$$

or

$$\frac{L_{l,m}^{p,\alpha}[\alpha_1]F(z)}{L_{l,m}^{p,\alpha}[\alpha_1 + 1]F(z)} \prec \psi(z)$$

which implies $F \in \mathcal{A}^{\alpha}(p, n, \alpha_1 : \psi)$. The second half of the Theorem (2.3) follows by a similar application of Lemma (1.4). ■

Using theorem (2.3), we have the following result:

Corollary 2.4. Let ψ_i be univalent in $\mathbb{U}$, $\psi_i(0) = 1 (i = 1, 2)$ and λ be a complex number. Assume that $\frac{z\psi_i'}{(\lambda + p - \alpha_1 + \alpha_1\psi_i)}$ is starlike in $\mathbb{U}$ and $\mathcal{R}\{\lambda + p - \alpha_1 + \alpha_1\psi_i(z)\} > 0 (i = 1, 2)$. If $f \in \mathcal{A}^{\alpha}(p, n, \alpha_1 : h_1, h_2)$ satisfies (2.10) then the function F defined by (2.9) belong to $\mathcal{A}^{\alpha}(p, n, \alpha_1 : \psi_1, \psi_2)$ where

$$h(z) := \psi_i(z) + \frac{z\psi_i'(z)}{\lambda + p - \alpha_1 + \alpha_1\psi_i(z)} (i = 1, 2)$$

Theorem 2.5. Let $f(z) \in \mathcal{A}_p$ and $\alpha_1 \neq 1$. Define F by

$$F(z) := \frac{\alpha_1}{z^{\alpha_1 - p}} \int_0^z t^{\alpha_1 - p - 1} f(t) dt \tag{2.17}$$

Then $f \in \mathcal{A}^{\alpha}(p, n, \alpha_1 : \varphi)$ if and only if $F \in \mathcal{A}^{\alpha}(p, n, \alpha_1 - 1 : \frac{1 - \alpha_1\varphi}{1 - \alpha_1})$. Also $f \in \overline{\mathcal{A}^{\alpha}}(p, n, \alpha_1 : \varphi)$ if and only if $F \in \overline{\mathcal{A}^{\alpha}}(p, n, \alpha_1 - 1 : \frac{1 - \alpha_1\varphi}{1 - \alpha_1})$.

Proof. From (2.17), we have

$$\alpha_1 f(z) = (\alpha_1 - p)F(z) + zF'(z) \tag{2.18}$$

In the same way, for operator (1.3) we can write,

$$\alpha_1 L_{l,m}^{p,\alpha}[\alpha_1]f(z) = (\alpha_1 - p)L_{l,m}^{p,\alpha}[\alpha_1]F(z) + z\left(L_{l,m}^{p,\alpha}[\alpha_1]F(z)\right)' \tag{2.19}$$

From (2.17), we get,

$$L_{l,m}^{p,\alpha}[\alpha_1 + 1]f(z) = L_{l,m}^{p,\alpha}[\alpha_1]F(z) \tag{2.20}$$

and

$$\alpha_1 L_{l,m}^{p,\alpha}[\alpha_1]f(z) = z\left(L_{l,m}^{p,\alpha}[\alpha_1 + 1]f(z)\right)' + (\alpha_1 - p)L_{l,m}^{p,\alpha}[\alpha_1 + 1]f(z)$$

$$= z\left(L_{l,m}^{p,\alpha}[\alpha_1]F(z)\right)' + (\alpha_1 - p)L_{l,m}^{p,\alpha}[\alpha_1]F(z)$$

$$= (\alpha_1 - 1)L_{l,m}^{p,\alpha}[\alpha_1 - 1]F(z) - (\alpha_1 - 1 - p)L_{l,m}^{p,\alpha}[\alpha_1]F(z) + (\alpha_1 - p)L_{l,m}^{p,\alpha}[\alpha_1]F(z)$$

$$= (\alpha_1 - 1)L_{l,m}^{p,\alpha}[\alpha_1 - 1]F(z) + L_{l,m}^{p,\alpha}[\alpha_1]F(z) \tag{2.21}$$

Therefore, form (2.20) and (2.21), we have,

$$\frac{L_{l,m}^{p,\alpha}[\alpha_1 - 1]F(z)}{L_{l,m}^{p,\alpha}[\alpha_1]F(z)} = \frac{1}{(\alpha_1 - 1)}\left\{\frac{\alpha_1 L_{l,m}^{p,\alpha}[\alpha_1]f(z)}{L_{l,m}^{p,\alpha}[\alpha_1 + 1]f(z)} - 1\right\}$$

and the desired result follows from it. ■

Corollary 2.6. Let $f(z) \in \mathcal{A}_p$ and $\alpha_1 \neq 1$.Then $f \in \mathcal{A}^\alpha(p, n, \alpha_1 : \varphi_1, \varphi_2)$ if and only if F given by (2.17) is in $\mathcal{A}^\alpha(p, n, \alpha_1 - 1 : \frac{1 - \alpha_1\varphi_1}{1 - \alpha_1}, \frac{1 - \alpha_1\varphi_2}{1 - \alpha_1})$.

References

[1] T. Bulboaca, *A class of superordination-preserving integral operators*, Indag. Math. (N.S.). **13(3)**(2002), 301–311.
[2] T. Bulboaca, *Classes of first-order differential superordinations*, Demonstr. Math., **35** (2)(2002), 287–292.
[3] B.C. Carlson and D.B. Shaffer, *Starlike and prestarlike hypergeometric functions*, SIAM J. Math. Anal., **15(4)**(1984), 737–745.
[4] S.S. Kumar and H.C. Taneja, *Classes of Multivalent Funcitons Defined by Dziok-Srivastava Linear Operator and Multivalent Transformation*, KYUNGPOOK, Math., **46(4)**(2006), 97–109.
[5] J.L. Liu and S. Owa, *On the class of multivalent functions involving certain linear operator*, Indian J. Pure Appl. Math., **33(11)**(2002), 1713–1722.
[6] S.S. Miller and P.T. Mocanu, *Differential Subordinations: Theory and Applications*, Pure and Applied Mathematics No.225 Dekker, New York, (2000).
[7] S.S. Miller and P.T. Mocanu, *Subordinants of differential superordinatins*, Complex Variables, **48** (10)(2003), 815–826.