Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 9 (2017), pp. 5227-5233

http://www.ripublication.com

On Invariant Best Approximation in Modular Spaces

Salwa Salman Abed

Department of Math., Coll. of Education for pure sciences,
Ibn Al-Haitham, Uinv. of Baghdad, Iraq.

Abstract

In a modular space, we introduce the concept of a best approximation and prove results about proximinal set, Chebysev set and existence invariant best approximation.

Keywords Modular spaces, best approximation, fixed points.

AMS (2010) subject classification: 46B20, 47H09.

1-INTRODUCTION

Ky Fan's best approximation theorem [1] is well known and useful in nonlinear analysis, game theory, minimax theoy, fixed point theory and variational inequalities. The importance due to its unifying nature. It is easy to derive many fixed point theorems under weaker assumptions as corollaries. Interesting extensions have been given by several researchers and variety of applications, mostly in fixed point theory and approximation theory.

In this paper, we introduce the best approximation and some other related concepts in modular spaces. Then, the conditions to existences of proximinal and Chebysev sets in finite dimension are given. Also, we prove an analogue to Ky Fan′s best approximation theorem which can be seen as an extension to Brouwer's and Schauders fixed point theorems.

2- PRELIMINARIES

Let us begin with

Definition (2.1) [2] Let $\mathcal{V}$ be a linear space over $\mathcal{K}(= \mathcal{R}\ or\ \mathcal{C})$

(1) A function $m: \mathcal{V} \to [0, \infty]$ is called modular if

(i) $(x) = 0 \Leftrightarrow x = 0$;

(ii)$m(\alpha x) = m(x)$ for $\alpha \in \mathcal{K}$ $with$ $|\alpha| = 1$, for all $x \in \mathcal{V}$;

(iii)$m(\alpha x + \beta y) \leq m(x) + m(y)$ if $\alpha, \beta \geq 0$, for all $x, y \in \mathcal{V}$.

Definition (2.2) [2] The modular m is called convex, if (iii) in definition (2.1)replaced by

(iii') $m(\alpha x + \beta y) \leq m(\alpha x) + m(\beta y)$ if $\alpha, \beta \geq 0, \alpha + \beta = 1$ for all $x, y \in \mathcal{V}$

(2) A modular m defines a corresponding modular space, i.e., the space $\mathcal{V}_m$ given by

$$\mathcal{V}_m = \{\boldsymbol{x} \in \mathcal{V} |\, m(\alpha x) \to o \ as\ \alpha \to o\}.$$

Remark (2.1) [3]

(i) Note that m is an increasing function. Suppose$0 < \alpha < \beta$. Then, condition (iii) with $y = 0$ shows that $m(\alpha x) = m(\beta y)$.

(ii) Let $(\mathcal{V}, \|\quad\|)$ be a norm space, then $\|\quad\|)$ is a convex modular on $\mathcal{V}$, but the converse is not tru*e*.

In general the modular m does not behave as a norm or a distance because it is not sub additive, but one can associate to a modular the F −norm.

Definition (2.3) [3] The m-ball, B_m centered at $x \in \mathcal{V}_m$ with radius *r* as

$$B_m = \{\boldsymbol{y} \in \mathcal{V}_m; \mathrm{m}(x - y) < r\}.$$

The class of all m-balls in a modular space$\mathcal{V}_m$ generates a topology which makes $\mathcal{V}_m$ Hausdorff topological linear space. Every m-ball is convex set, therefore every modular space locally convex Hausdorff topological vector space.

Definition (2.4) [3] Let $\mathcal{V}_m$ be a modular space.

(a) A sequence $\{x_n\} \subset \mathcal{V}_m$ is said to be m-convergent to $\in \mathcal{V}_m$, and write x_n if $m(x_n - x) \to 0$ as $n \to \infty$.

(b) A sequence $\{x_n\} \subset \mathcal{V}_m$ is said to be m-Cauch if $m(x_n - x_l) \to 0$ as $n, l \to \infty$.

(c) $\mathcal{V}_m$ is called m-complete if any m-Cauchy sequence is m-convergent.

(d) A subset $\mathcal{U} \subset \mathcal{V}_m$ is called m-closed if for any sequence$\{x_n\} \subset \mathcal{U}$ convergent to $x \in \mathcal{V}_m$, we have $x \in \mathcal{U}$.

(e) A m-closed subset $\mathcal{U} \subset \mathcal{V}_m$is called m −compact if any sequence $\{x_n\} \subset \mathcal{U}$ has a m −convergent subsequence.

(f) A subset $\mathcal{U} \subset \mathcal{V}_m$is said to be m −bounded if

$$diam\ (\mathcal{U})\ < \infty,$$

where $diam\ (\mathcal{U}) = sup_{x,y\ \in \mathcal{U}}\ m(x-y)$ is called the m −diameter of $\mathcal{U}$.

(g) The m −distance between $x\ \in\ \mathcal{V}_m$ and $\mathcal{U} \subset \mathcal{V}_m$ as

$$d_m(x,\mathcal{U}) = inf_{x\ \in \mathcal{U}}\ m(x-y).$$

3- BEST APPROXIMATION

Consider$\emptyset \neq \mathcal{U} \subset \mathcal{V}_m$, we seek a point $y \in \mathcal{U}$ which is a best approximation for a given $x\ \in\ \mathcal{V}_m$; that is $m(x-y) =\ d_m(x,\mathcal{U}\ ) = inf_{y\in\mathcal{U}}\ m(x - y)$

then y is said to be an element of best approximation of x by $\mathcal{U}$.

We shall denote by $P_{\mathcal{U}}(x)$ or $Px$the set of all elements of best approximation of x by $\mathcal{U}$. It is easy to note that

Proposition (3.1) Let $\mathcal{V}_m$be a modular space and $\emptyset \neq \mathcal{U} \subset \mathcal{V}_m$, then:

(i) $P_{\mathcal{U}}(x)$ is closed:

(ii) If $m(x) < \infty, \forall x$ then $P_{\mathcal{U}}(x)$ is bounded set:

(iii) If $\mathcal{U}$ is convex set and m is convex function then $P_{\mathcal{U}}(x)$.

Now, we define the following

Definition (3.1)

Let $\mathcal{V}_m$ be a modular space and $\mathcal{U}$ be a subset of$\mathcal{V}_m$, then:

(i) $\mathcal{U}$ is called proximinal if for all $x\ \in\ \mathcal{V}_m$, there exists a $y \in \mathcal{U}$ such that $m(x-y) =\ d_m(x,\mathcal{U}\ )$.

(ii)$\mathcal{U}$ is called Chebysev if for each $x\ \in\ \mathcal{V}_m$, there is an unique element $y \in \mathcal{U}$ such that $m(x-y) =\ d_m(x,\mathcal{U}\ )$.

Example (3.1):

Consider$\mathcal{V}_m = R^2$with convex modular function $m(x) = \sqrt{a_1^2 + a_2^2}$ where $x = (a_1, a_2)$.

Setting$x = (1{,}1) and\ y = (1{,}0)$, we have $m(x - \alpha y)\ = ((1 - \alpha, 1))\ = \sqrt{(1-\alpha)^2 + 1}$

the value, $\sqrt{(1-\alpha)^2+1}$ will be the minimum if and only if $\alpha = 1$. Thus the unique best approximation of x by $\mathcal{U}$ is the closed linear subspace spanned by y. And $\mathcal{U}$ is Chebysev set.

Example (3.2)

Let $\mathcal{V}_m, x\ and\ y$ as in example (3.2) and $m(x) = \max\{|a_1|, |a_2|\}$, then $m(x - \alpha y) = |1 - \alpha|$

this value will be the minimum if and only if $|1 - \alpha| \leq 1$ or equivalently $0 \leq \alpha \leq 2$. Thus there exists infinitely many best approximation of x by $\mathcal{U}$: The closed linear subspace spanned by y, that is $P_{\mathcal{U}}(x) = \{\alpha x: 0 \leq \alpha \leq 1\}$. And $\mathcal{U}$ is proximinal set.

Proposition (3.2)

If $\mathcal{V}_m$ is a modular space and $\mathcal{U}$ is a finite dimensional subspace of $\mathcal{V}_m$, then $\mathcal{U}$ is proximinal.

Proof:

Let $\mathcal{U}$ be a finite dimensional subspace of $\mathcal{V}_m$, and $x \notin \mathcal{U}$. As $\mathcal{U}$ is finite dimensional, $\mathcal{U}$ is locally compact space [3, PP. 92].

Clearly, $d_m(x, \mathcal{U}) \leq d_m(x, 0) = m(x)$. Hence, for any $y \in \mathcal{U}, m(x-y) \leq m(x)$ and follows from definition(2.1-iii) that $m(y) \leq 2m(x)$. Hence to find $y_o \in \mathcal{U}$ such that $m(x-y_0) = d_m(x, \mathcal{U})$, we need only consider elements from the compact set $G = \{y \in \mathcal{U}: m(y) \leq 2m(x)\}$. Since G is compact, there is $y_o \in G$ such that $m(x - y_0) = min_{y\in G}\, m(x-y) = min_{y\in \mathcal{U}}\, m(x-y) = d_m(x, \mathcal{U})$. Hence $\mathcal{U}$ is proximinal.

To present the next proposition about existence of Chebysev set, we need the following definition:

Definition (3.2) The modular m is called strictly convex, if

$$m(\alpha x + \beta y) = m(\alpha x) + m(\beta y) \text{ if } \alpha, \beta \geq 0, \alpha + \beta = 1 \text{ for all } x, y \in \mathcal{V}.$$

Lemma (3.1) For any nonzero x and y in strictly modular space, there is $\lambda > 0\ such\ that\ y = \lambda x$.

Proof:

It is clear.

Proposition (3.3) If $\mathcal{V}_m$is a strictly convex modular space and $\mathcal{U}$ is a finite dimensional subspace of$\mathcal{V}_m$, then $\mathcal{U}$ is Chebysev set.

Proof:

Suppose that y_0 and z_0 in $P_{\mathcal{U}}(x)$ for $x \notin \mathcal{U}$; $m(x-y_0) = m(x-z_0) = d_m(x,\mathcal{U})$. For a basis $\{e_1,\ldots,e_n\}$ for $\mathcal{U}$, suppose that $y_0 = \sum_1^n a_i e_i$ and $z_0 = \sum_1^n b_i e_i$, $(a_i, b_i \in \mathcal{K})$. Since

$$d_m(x,\mathcal{U}) \le m\left(x - \sum_1^n \frac{a_i + b_i}{2} e_i\right) \le \frac{1}{2} m\left(x - \sum_1^n a_i e_i\right) + \frac{1}{2} m\left(x - \sum_1^n b_i e_i\right)$$
$$= \frac{1}{2} d_m(x,\mathcal{U}) + \frac{1}{2} d_m(x,\mathcal{U}) = d_m(x,\mathcal{U}).$$

it follows that

$$m\left(x - \sum_1^n \frac{a_i+b_i}{2} e_i\right) = \frac{1}{2} m(x - \sum_1^n a_i e_i) + \frac{1}{2} m(x - \sum_1^n b_i e_i).$$

Since $\mathcal{V}_m$ is strictly convex, there exists $\lambda > 0$ such that $x - \sum_1^n a_i e_i = \lambda(x - \sum_1^n b_i e_i)$ which implies that $(1-\alpha)x \in \mathcal{V}_m$. Since $x \notin \mathcal{U}, \alpha = 1$. $\{e_1,\ldots,e_n\}$ is linearly independent, $a_i = b_i$ for $i = 1,2,\ldots,n$; therefore $y_0 = z_0$.

4- INVARIANT BEST APPROXIMATION

By the above definition of the set of best approximations $P_{\mathcal{U}}(x)$ of x by $\mathcal{U}$, the multivalued mapping $P_{\mathcal{U}}: \mathcal{V}_m \to 2^{\mathcal{V}_m}$ is said to be the metric projection on $\mathcal{V}_m$. If $\mathcal{U}$ is Chebysev set, then for each $x \in \mathcal{V}_m$, there exists a unique $y \in \mathcal{U}$ such that$P_{\mathcal{U}}(x) = y$. That is $P_{\mathcal{U}}$ is a single valued mapping. To show the relation between the best approximation and fixed point theorems, consider the function$f: \mathcal{U} \to \mathcal{V}_m$. Another type of best approximation is the existence of a point $w \in \mathcal{U}$ such that

$$m(w - f(w) = d_m(f(w),\mathcal{U}) = inf_{y \in \mathcal{U}}\, m(f(w) - y) \qquad \ldots(4.1)$$

We note that y is a solution of (4.1) if and only if y is a fixed point of $f^{\circ}P_{\mathcal{U}}$.

Theorem (4.1) Let $\emptyset \neq \mathcal{U}$ be convex subset of a modular space $\mathcal{V}_m$, $f: \mathcal{U} \to \mathcal{U}$ be the upper semi-continuous multivalued mapping such that $\emptyset \neq f(x)$ is closed and convex subset of $\mathcal{U}$ for each $x \in \mathcal{U}$ and $\overline{f(U)}$ be compact subset of $\mathcal{U}$. then f has a fixed point.

Proof:

Since any modular space is locally convex space then by theorem (2) in [5] the proof is complete.

Theorem (4.2)

Let $\emptyset \neq \mathcal{U}$ be compact convex subset of a convex modular space$\mathcal{V}_m$, $f: \mathcal{U} \to \mathcal{V}_m$ be the continuous mapping, then there exists a y_0 such that $m(y_0, f(y_0)) = d_m(f(y_0), \mathcal{U})$.

Proof**:**

Let $i: \mathcal{U} \to R^+$ be defined by $i(x) = inf\{m(y - x): y \in \mathcal{U}\}$. Since f is a continuous on $\mathcal{U}$ for each $\in \mathcal{U}$, then there exists a $y \in \mathcal{U}$ such that $i(x) = m(y - f(x))$ (because $\mathcal{U}$ is compact).

Define a multivalued mapping $T: \mathcal{U} \to 2^{\mathcal{U}}$ by:

$Tx = \left\{y \in \mathcal{U}: i(x) = m\left(y - f(x)\right)\right\} \subset \mathcal{U}. Then\ Tx \neq \emptyset$ (as above). We will prove that

(i) Tx is closed set,

(ii) Tx is convex set, and

(iii) T is upper semi − continuous mapping.

For (i)**,** suppose that z is an accumulation point of Tx, then there exists a sequence $< z_n > \subseteq Tx$ such that $z_n \to z$. We have

$$m(z - f(x)) = m\left(\lim_{n\to\infty} z_n - f(x)\right) = \lim_{n\to\infty} m(z - f(x)) = i(x)$$

thus z belongs to Tx, and then Tx is closed set.

For (ii), suppose that $0 \leq \lambda \leq 1$ and $y_1, y_2 \in Tx \subset \mathcal{U}$, since $\mathcal{U}$ is convex, then $\lambda y_1 + (1 - \lambda)\, y_2 \in \mathcal{U}$ and

$$i(x) = d_m(x, \mathcal{U}) \leq m(\lambda y_1 + (1 - \lambda) y_2 - x).$$

Now, $m(\lambda y_1 + (1 - \lambda) y_2 - x) \leq \lambda m(y_1 - x) + (1 - \lambda) m(y_2 - x) = i(x)$

so, $i(x) = m(\lambda y_1 + (1 - \lambda) y_2 - x)$ and this prove that Tx is convex set.

For (iii), let A be a closed subset of$\mathcal{U}$, we must prove that $T^{-1}(A) = \{y \in \mathcal{U} : \mathrm{T(y)} \cap \mathrm{A} \neq \emptyset\}$ is closed

subset of $\mathcal{U}$. Suppose that $x_0 \in \mathcal{U}$ be an accumulation point of $T^{-1}(A)$, then there exists a net $\langle x_\alpha \rangle \subseteq T^{-1}(A)$ and converges to x_0. This implies that there is a net $y_\alpha \in T(x_\alpha) \cap A$. That is, $y_\alpha \in A$ and$y_\alpha \in T(x_\alpha)$. So, $m(y_\alpha - f(x_\alpha)) = i(x_\alpha)$for each α. Since $\mathcal{U}$ is compact and A is closed subset of $\mathcal{U}$, then A is compact, so there is a $y_0 \in A$ and a subnet $< y_\beta >$ of $< y_\alpha >$ such that $y_\beta \to y_0$. Hence, $y_\beta \in T(x_\alpha)$, this implies that $m(y_\beta - f(x_\alpha) = i(x_\alpha)$for each β. That is $y_0 \in T(x_o) \cap A$. This implies that $y_0 \in T^{-1}(A)$. Thus, T is upper semi continuous. By theorem (4.1) we get $y_0 \in \mathcal{U}$

$\ni y_0 \in T(y_0)$, that is, $m(y_0 - f(y_0) = d_m(y_0, \mathcal{U})$.

As a consequence, we have the version of Schauder's fixed point theorem in the following

Corollary (1) Let $\emptyset \neq \mathcal{U}$ be compact convex subset of a convex modular space$\mathcal{V}_m$, $f: \mathcal{U} \to \mathcal{V}_m$ be the continuous mapping. Then f has a fixed point.

Proof:

In this case $d_m(y, \mathcal{U}) = 0$ and so $y = f(y)$.

Now, as a suggestion, one can study the random fixed points random and the invariant best approximation due in [6-7] in the setting of modular spaces. Also, the results in [8] about fixed points for condensing mapping can be studied in these spaces.

REFERENCES

[1] Singh, S.P., Watson, B. and Srivastave, P., (1997), "*Fixed Point Theory and Best Approximation*", The KKM-map Principle, Kluwer Academic Publishers

[2] Nakano, H. (1950) *Modular semi-ordered spaces*, Tokyo Mathematical Book Series, Maruzen Co. Ltd, Tokyo,Japan.

[3] Musielak J.,(1983) *Orlicz spaces and modular spaces*, Lecture Notes in Math, 1034, Springer–Verlag, Berlin, Heidelberg, New York.

[4] Narici, L. and Bekenstein E., (1985), *Topological vector space*, Marcel Dekker and INC., New York.

[5] Himmelberg C. J., (1972), *Fixed points of compact multifunction*, Journal of mathematical analysis and applications,38, 205-207.

[6] Abed, S. S. Ajeel Y. J. Alsaidy, S. K., *Results on general $\emptyset -$weakly random operators,* to appear in International Journal of Pure and Applied Mathematics IJPAM, ISSN 1311-8080.

[7] Abed, S. S., Ajeel Y. J., (2016), On invaraint random best approximations, International Journal of Recent Scientific Research Research,Vol. 7, Issue, 3, pp. 9530-9533.

[8] Abed, S.S., Luaibi, H.H., (2015)," *Fixed Point Theorems for Condensing Mappings in G-Spaces* ", Inter. J. of Math. Sci., Vol.35, Issue.1, pp.1701-1711.