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Deterministic and Stochastic Optimal Control Analysis of an SIR Epidemic model

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Abstract

This paper considers a optimal control analysis of a non -linear dynamical system of linear quadratic control. It is shown the difference between the stochastic and deterministic control system and for the occurrence of symmetry breaking as a function of the noise is included to formulate the stochastic model. For the Deterministic optimal control problem existence of optimal control is proved and it is solved by using Pontryagins Maximum Principle. The stochastic optimal control problem is discussed by using Stochastic Maximum Principle and the results are obtained numerically through simulation. In order to solve the stochastic optimal control problem numerically, we use an approximation based on the solution of the deterministic model.

Keywords: Epidemics, Optimal control theory, Pontryagins Maximum Principle, Stochastic Maximum Principle.

1. INTRODUCTION

The progress of an communicable disease throughout the particular population is highly amenable to mathematical modeling. Many pioneers are give basic foundation to mathematical modeling of epidemics like Kermack and MeKendrick [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. These initial models, and many subsequent revisions and improvements are operated on the principle that individuals in the given population can be divided by there epidemiological status as susceptible to infection, infectious and recovered. The SIR model is the one of the fundamental and simplest of all epidemiological models. The problem of optimal control of non-linear system in the presence of noise occurs in many area of science and engineering. In the absence of noise, the optimal control problem can be solved by two methods: using Pontriyagins Maximum Principle (PMP) [13], which is similar to the Hamiltonian equation of motion and it is ordinary differential equations and another is the Hamiltonian Jacobi Bellman (HJB) equation, which is partial differential equation [2].

In the presence of Standard Weiner noise, the PMP formulation can be generalized and yields the set of stochastic differential equations, due to boundary conditions it is difficult to solve at initial and final time [3]. In this situation, it is quite direct to solve mathematically by using HJB equation when noise is included in the model. However in general it is difficult to get results for deterministic or stochastic HJB equation numerically through simulation due to the curse of dimensionality. We are designed the stochastic model by perturbation of parameter and performed optimal control analysis using stochastic maximum principle to study stochastic optimal control problem of SIR type epidemic model with media awareness programs and to obtain solution of stochastic optimal control problem numerically it is used the approximation of the solution of deterministic model and this approximation includes the complete nonlinear stochasticity. The efficiency of the optimal controls are observed by numerical simulation. The results are presented and compared through numerical simulations. The rest of the paper continues with six sections; Section 2 deals with formulation of mathematical model, in Section 3 and 4 the model with optimal control is analyzed for both deterministic and stochastic model respectively, in Section 5 has the numerical results discussion and concluding remarks are presented in Section 6.

2. DETERMINISTIC MODEL

A non-linear dynamical system of linear quadratic control of the SIR epidemic with media awareness program is used for the deterministic and stochastic optimal control analysis. It is assumed that once the infected individuals recovered from the disease they will get permanent immunity against disease. In the absence of media coverage we assume a classical bilinear incidence with the rate at which new infected arise given by $\beta x_1 x_2$, where β being the infection coefficient. In the presence of media coverage, social distancing mechanism comes into effect. The reporting by media is assumed to be an increasing function of the number of infective cases present, as a consequence at the contact rate between susceptible and infective individuals there is a decreasing function of the number of infectives cases present. It is considered similar non-linear function as in J.Cui and Wenbin Liu [8] and denote the effective contact rate as $P(I) = \beta - \pi f(I)$, where π is maximum reduced effective contact rate due to mass media alert in the presence of infective. It is also assumed that $\beta \geq \pi$ and it is chosen the media coverage function as, $f(I) = I/1+I$.

In view of the above facts, the dynamics of model is governed by the following system of nonlinear ordinary differential equations: The model sub-divides the total population in a region at time t, denoted by $N(t)$, into the following sub populations of unaware susceptible $x_1(t)$, and the infectives $x_2(t)$, and that of recovered individuals $x_3(t)$ from infective class and vaccinated individuals from susceptible class. The total variable population at time t is given by, $N(t) = x_1(t) + x_2(t) + x_3(t)$.

Table 1: Parameters

Parameters	Explanation
Λ	Constant rate at which number of susceptible increasing continuously.
β	Contact rate of susceptible with infectives.
π	Rate of awareness.
ϕ	Rate of vaccination of susceptibles.
γ	Recovery rate of infectives.
δ	Disease induced death rate.
μ	Natural death rate from each class.

$$\frac{dx_1}{dt} = \Lambda - \left(\beta - \frac{\pi x_2}{1+x_2}\right) x_1(t)x_2(t) - (\phi+\mu)x_1(t)$$
$$\frac{dx_2}{dt} = \left(\beta - \frac{\pi x_2}{1+x_2}\right) x_1(t)x_2(t) - (\gamma+\delta+\mu)x_2(t) \quad (1)$$
$$\frac{dx_2}{dt} = \phi x_1(t) + \gamma x_2(t) - \mu x_3(t)$$

where $x_1(0) > 0$, $x_2(0) > 0$, $x_3(0) \geq 0$. For the construction of optimal control problem, the following controls have been used.

1. Let $u_1(t) \in [0, u_{1max}]$ is the time dependent control refers to strengthening effort made on vaccination program.

2. Let $u_2(t) \in [0, u_{2max}]$ is the time dependent control refers to controlling effort that alters infection cases receiving treatment per unit time.

Then the model system (1) converts to,

$$\frac{dx_1}{dt} = \Lambda - \left(\beta - \frac{\pi x_2}{1+x_2}\right)x_1(t)x_2(t) - ((1+u_1(t))\phi + \mu)x_1(t)$$
$$\frac{dx_2}{dt} = \left(\beta - \frac{\pi x_2}{1+x_2}\right)x_1(t)x_2(t) - ((1+u_2(t))\gamma + \delta + \mu)x_2(t) \quad (2)$$
$$\frac{dx_2}{dt} = (1+u_1(t))\phi x_1(t) + (1+u_2(t))\gamma x_2(t) - \mu x_3(t)$$

where $x_1(0) > 0$, $x_2(0) > 0$, $x_3(0) \geq 0$.

To show the existence of the feasible set of system (2) which attracts all solutions initiation in the interior of positive orthant, we have to prove that the system (2) is dissipative, that is, all solutions are uniformly bounded in a proper subset $\Omega \in R_3^+$.

Using the fact that $N = x_1(t) + x_2(t) + x_3(t)$, the system (1) reduce to the following system:

$$\begin{aligned}\frac{dN}{dt} &= \Lambda - \mu N(t) - \delta x_2(t) \\ &\leq \Lambda - \mu N(t)\end{aligned} \quad (3)$$

After solving equation (3), we have

$$N(t) \leq N(0)e^{-\mu t} + \frac{\Lambda}{\mu}(1 - e^{-\mu t}) \quad (4)$$

where $N(0)$ is the sum of initial values $x_1(0)$, $x_2(0)$, $x_3(0)$. Now from equation (3) $\lim t \to \infty, as\ N \to \frac{\Lambda}{\mu}$ then $\frac{\Lambda}{\mu}$ is the upper bound of N. Therefore the region of attraction is given by the set:

$$\Omega = \left\{(x_1, x_2, x_3) \in R_+^3 : 0 \leq x_1 + x_2 + x_3 \leq N \leq \frac{\Lambda}{\mu}\right\} \quad (5)$$

and attracts all solutions initiation in the interior of positive orthant. The equilibria for the system (2) is obtained by equating each of the derivative to zero. By doing so it is found that model system (2) has two non-negative equilibria i.e. disease free equilibrium (DEF) $E_0\left(\frac{\Lambda}{\mu}, 0, 0\right)$ and endemic equilibrium (EE) $E_1\left(x_1^*, x_2^*, x_3^*\right)$. To define local stability of E_0, the Jacobian matrix of model system (2) is evaluated at DFE isgiven by

$$J(E_0)=\begin{pmatrix} -\mu & 0 & 0 \\ 0 & \dfrac{\beta\Lambda}{\mu}-((1+u_2)\gamma+\delta+\mu) & 0 \\ \phi(1+u_1) & \dfrac{-\phi(1+u_1)(1+u_2)\gamma\beta\Lambda}{\mu^2} & -\mu \end{pmatrix}$$

The eigen values of the Jacobian matrix are $-\mu, \dfrac{\beta\Lambda}{\mu}-((1+u_2)\gamma+\delta+\mu), -\mu$. Except second eigen value other eigen values of the Jacobian matrix at E_0 are negative, thus if

$$\frac{\beta\Lambda}{\mu((1+u_2)\gamma+\delta+\mu)}<0 \tag{6}$$

E_0 is asymptotically stable. Hence if $R_0<1$ then the disease cannot invade the population and if $R_0>1$ then invasion of the disease is always possible, where R_0 is defined by

$$R_0=\frac{\beta\Lambda}{\mu((1+u_2)\gamma+\delta+\mu)} \tag{7}$$

The endemic equilibrium $E_1\left(x_1^*, x_2^*, x_3^*\right)$ exist with positive component provided, $R_0>1$. A stability analysis of SIR model of same kind had been discussed by many researchers (See, Driessche [11] and Cui J. [12]), hence in the following sections optimal control analysis is discussed.

3. OPTIMAL CONTROL ANALYSIS

The successful intervention strategy is one which decrease the number of infections in a minimum cost (Kar and Jana 2013 [14]). An effective way of drawing a best strategy for information circulation is the optimal control theory [4, 15]. Therefore public health policy maker sought to minimize the infection with a minimum cost of controlling the disease. Then the objective functional is

$$J(u_1,u_2)=\int_0^{t_f}\left(A_1x_2(t)+A_2\frac{u_1^2(t)}{2}+A_3\frac{u_2^2(t)}{2}\right)dt \tag{8}$$

subject to system (2). We consider the quadratic cost functional such that where A_1, A_2, A_3 are positive constants. This performance involves minimizing the number of infections as well as the cost of controls u_1 and u_2 , the cost includes the expenses made for implementation of controls. The optimal control $u_1^*(t)$ and $u_2^*(t)$ are such that

$$J(u_1^*(t)u_2^*(t)) = \min\{J(u_1,u_2)\,|\,u_1,u_2 \in U\} \tag{9}$$

where the set of controls are defined as

$$U = \{u_i(t) : 0 \le u_i(t) \le 1,\ i = 1,2; 0 \le t \le t_f, u_i(t)\, is\ Lebesque\ measurable\} \tag{10}$$

3.1. Existence of control problem

In this section, we consider control system (2) with initial conditions at $t = 0$ to show the existence of control problem. Note that for bounded Lebesque measurable controls and non-negative initial conditions, non-negative bounded solutions to the state system exists. Let us go back to the optimal control problem (2) and (8). In order to find optimal solution first we should find the Lagrangian and Hamiltonian for for the optimal problem is given by

$$L = A_1 x_2(t) + A_2 \frac{u_1^2(t)}{2} + A_3 \frac{u_2^2(t)}{2}$$

We seek for the minimal value of the Lagrangian. To do this we define the Hamiltonian H for the control problem as follows:

$$H = L(x_2,u_1,u_2) + \lambda_1 \frac{dx_1}{dt} + \lambda_2 \frac{dx_2}{dt} + \lambda_3 \frac{dx_3}{dt} \tag{11}$$

For existence of given control problem we state and prove the following theorem.

Theorem 3.2: There exist an optimal control $u^* = (u_1^*(t), u_2^*(t))$ such that $J(u_1^*(t)u_2^*(t)) = \min\{J(u_1,u_2)\,|\,u_1,u_2 \in U\}$

Subject to the control system (2) with initial condition at t=0.

Proof: To prove the existence of an optimal control problem, used the result in (D.L.Lukes [16]). Note that the control and state variables are non-negative values. In the minimization of optimal control problem, the objective functional in u_1 and u_2 has to be satisfied by necessary convexity. By definition the set of control variables $u_1, u_2 \in U$ is also convex and closed. The optimal system is bounded which determines the compactness needed for the existence of the optimal control. In addition the integrand in the objective functional (8), $A_1 x_2(t) + A_2 \frac{u_1^2(t)}{2} + A_3 \frac{u_2^2(t)}{2}$ is convex on the control set U. Also we can easily see that, there exist a constant $\eta > 1$ and positive numbers θ_1, θ_2 such that

$$J(u_1,u_2) \geq \theta_1\left(|u_1|^2+|u_2|^2\right)^{\eta/2} - \theta_2$$

because the state variable are bounded, which completes the existence of an optimal control. After establishing existence of an optimal control, to obtain the optimal solution of ordinary differential equations for the adjoint variables, under boundary conditions and characterization of an optimal control, we used the Pontryagin's Maximum Principle [15] is follow.

If (x,u) is an optimal solution of an optimal control problem, then there exist a non-trivial vector function $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ satisfies the following inequalities,

$$\frac{dx}{dt} = \frac{\partial H(t,u,\lambda)}{\partial \lambda}$$

$$0 = \frac{\partial H(t,u,\lambda)}{\partial \lambda}$$

$$\frac{d\lambda}{dt} = \lambda' = -\frac{\partial H(t,u,\lambda)}{\partial x}$$

Now, it is applied to the necessary conditions to the Hamiltonian H in (11).

Theorem 3.2: Let x_1^*, x_2^* and x_3^* be optimal solutions of states with associated optimal variables $(u_1^*(t),\ u_2^*(t))$ for the optimal control problem (2) and (8). Then there exist adjoint variables $\lambda_i, i=1,2,3.$ Satisfying

$$\begin{aligned}
\lambda_1' &= (\lambda_1-\lambda_2)\left(\beta - \frac{\pi \mathrm{x}_2}{1+\mathrm{x}_2}\right)\mathrm{x}_2 + (1+\mathrm{u}_1(\mathrm{t}))\phi(\lambda_1-\lambda_3) - \mu\lambda_1 \\
\lambda_2' &= -A_1 + \left(\beta x_1 - \frac{\pi x_1 \mathrm{x}_2(1+2x_2)}{(1+\mathrm{x}_2)^2}\right)(\lambda_1-\lambda_2) + (1+\mathrm{u}_2(\mathrm{t}))\phi(\lambda_2-\lambda_3) - (\phi+\mu)\lambda_2 \qquad (12) \\
\lambda_3' &= \lambda_3\mu
\end{aligned}$$

with transversality conditions $\lambda_i(t_f),\ i=1,2,3.$ Further

$$u_1^* = \max\left\{\min\left[\frac{x_1(t)\phi(\lambda_1-\lambda_3)}{A_2}, 1\right], 0\right\} \qquad (13)$$

And

$$u_2^* = \max\left\{\min\left[\frac{x_2(t)\gamma(\lambda_2-\lambda_3)}{A_3},1\right],0\right\} \tag{14}$$

Proof: To determine the adjoint equations and boundary conditions, we use the Hamiltonian (11), By differentiating the Hamiltonian equation with respect to x_1, x_2 and x_3 we obtain system (12). By solving the equations $\frac{\partial H}{\partial u_1}$ and $\frac{\partial H}{\partial u_2}$ it is obtained the solution of u_1 and u_2 respectively, on the interior of control set and using the optimality conditions and the property of the control space U, we can derive (13) and (14).

For the characterization of optimal control, it is considered the formulas (13) and (14). To solve the optimality system, we use initial and transversality conditions together with the characterization of the optimal control (u_1^*, u_2^*) given by (13) and (14). In addition to the second derivative of the Lagrangian with respect to u_1 and u_2 respectively, are positive, which shows that the optimal problem is minimum at controls u_1 and u_2. By substituting the values of u_1^* and u_2^* in the control system (2), we get the following system.

$$\frac{dx_1}{dt} = \Lambda - \left(\beta - \frac{\pi x_2}{1+x_2}\right)x_1(t)x_2(t) - \left(\left(1+\max\left\{\min\left[\frac{x_1(t)\phi(\lambda_1-\lambda_3)}{A_2},1\right],0\right\}\right)\phi + \mu\right)x_1(t)$$

$$\frac{dx_2}{dt} = \left(\beta - \frac{\pi x_2}{1+x_2}\right)x_1(t)x_2(t) - \left(\left(1+\max\left\{\min\left[\frac{x_2(t)\gamma(\lambda_2-\lambda_3)}{A_3},1\right],0\right\}\right)\gamma + \delta + \mu\right)x_2(t)$$

$$\frac{dx_2}{dt} = \left(1+\max\left\{\min\left[\frac{x_1(t)\phi(\lambda_1-\lambda_3)}{A_2},1\right],0\right\}\right)\phi x_1(t) + \left(1+\max\left\{\min\left[\frac{x_2(t)\gamma(\lambda_2-\lambda_3)}{A_3},1\right],0\right\}\right)\gamma x_2(t) - \mu x_3(t)$$

(15)

With H^* at $(t, x_1, x_2, x_3, u_1, u_2, \lambda_1, \lambda_2, \lambda_3)$

$$L = A_1 x_2(t) + A_2\frac{u_1^2(t)}{2} + A_3\frac{u_2^2(t)}{2}$$

We seek for the minimal value of the Lagrangian. To do this we define the Hamiltonian H for the control problem as follows:

$$H = A_1x_2(t) + A_2\frac{\left(\max\left\{\min\left[\frac{x_1(t)\phi(\lambda_1-\lambda_3)}{A_2},1\right],0\right\}\right)^2}{2} + A_3\frac{\left(\max\left\{\min\left[\frac{x_2(t)\gamma(\lambda_2-\lambda_3)}{A_3},1\right],0\right\}\right)^2}{2} + \lambda_1\frac{dx_1}{dt} + \lambda_2\frac{dx_2}{dt} + \lambda_3\frac{dx_3}{dt}$$

(16)

To find out the optimal controls and states, it is solved numerically, the above system (15) and equation (16).

4. STOCHASTIC MODEL

An important component in an ecosystem is that system is inevitably affected by environmental noise taking into account the effect of randomly fluctuating environment, it is assumed that fluctuating environment with manifest themselves as fluctuation in the parameters β,

$$\beta \to \beta + \varepsilon\eta(t)$$

where, $\eta(t) \sim N(0,1)$ and ε is the intensity of white noise and $dW(t) = \eta(t)dt$, where W(t) is standard Weiner process with W(0)=0, and with intensity of white noise $\varepsilon^2 > 0$. The stochastic version corresponding to the deterministic model (2) is described by the following set of stochastic differential equations:

$$\begin{aligned}
\frac{dx_1}{dt} &= \Lambda - \left(\beta - \frac{\pi x_2}{1+x_2}\right)x_1(t)x_2(t) - ((1+u_1(t))\phi + \mu)x_1(t) - \varepsilon x_1(t)x_2(t)dW(t) \\
\frac{dx_2}{dt} &= \left(\beta - \frac{\pi x_2}{1+x_2}\right)x_1(t)x_2(t) - ((1+u_2(t))\gamma + \delta + \mu)x_2(t) + \varepsilon x_1(t)x_2(t)dW(t) \qquad (17) \\
\frac{dx_2}{dt} &= (1+u_1(t))\phi x_1(t) + (1+u_2(t))\gamma x_2(t) - \mu x_3(t)
\end{aligned}$$

where $x_1(0) > 0$, $x_2(0) > 0$, $x_3(0) \geq 0$.

Let (Ω, F, P) be a complete probability space with a filtration $\{F_t\}_{t\geq 0}$ satisfying the conditions (i.e. it is increasing and right continuous while F_0 contains all P-null sets). Let W(t) be a one-dimensional Wiener process defined on filtered complete probability space $(\Omega, F, \{F_t\}_{t\geq 0}, P)$. For some $n \in N$, some $x_0 \in R^n$, and an n-dimensional Wiener process W(t), consider the general n dimensional stochastic differential equation. For simplicity of description, we define the vector.

$x(t) = [x_1(t), x_2(t), x_3(t)]'$ and $u(t) = [u_1(t), u_2(t)]'$

$$dx(t) = f(x(t), u(t))dt + g(x(t))dw(t), \qquad (18)$$

with the initial conditions

$$x(0) = [x_1(0), x_2(0), x_3(0)]' = x_0$$

where f and g are vectors with components such that

$$f_1(x(t),u(t)) = \Lambda - \left(\beta - \frac{\pi x_2}{1+x_2}\right)x_1(t)x_2(t) - ((1+u_1(t))\phi + \mu)x_1(t)$$

$$f_2(x(t),u(t)) = \left(\beta - \frac{\pi x_2}{1+x_2}\right)x_1(t)x_2(t) - ((1+u_2(t))\gamma + \delta + \mu)x_2(t)$$

$$f_3(x(t),u(t)) = (1+u_1(t))\phi x_1(t) + (1+u_2(t))\gamma x_2(t) - \mu x_3(t)$$

$$g_1 = -\varepsilon x_1 x_2, \quad g_2 = \varepsilon x_1 x_2, g_3 = 0.$$

We consider the quadratic cost functional such that

$$J(u) = \frac{1}{2}E\left\{\int_0^{t_f}\left(A_1 x_2(t) + A_2\frac{u_1^2(t)}{2} + A_3\frac{u_2^2(t)}{2}\right)dt + k_1x_1^2 + k_2x_2^2 + k_3x_3^2\right\}$$

where $A_1, A_2, A_3, k_1, k_2, k_3$ are positive constants. Our goal is to find an optimal control $u^*(t) = (u_1^*(t), u_2^*(t))'$ such that

$J(u^*) \leq J(u), \forall\ u_1(t), u_2(t) \in U$

where U is an admissible control set defined by

$U = \{u_1(t), u_2(t) | 0 \leq u_1(t) \leq u_{1max}, 0 \leq u_2(t) \leq u_{2max} \forall\, t \in (0, t_f]\}$, (22)

where u_{1max}, $u_{2max} \in \Re^+$ are constants. In order to use the stochastic maximum principle [17], first we define the Hamiltonian $Hm(x, u, p, q)$ in such a way that

$$H(x,u,p,q) = \langle f(x,u), p\rangle + l(x,u) + \langle g(x), q\rangle \qquad (23)$$

where $\langle .,. \rangle$ denotes a Euclidian inner product. $p = [p_1, p_2, p_3]'$ and $q = [q_1, q_2, q_3]'$ are adjoint vectors. It follows from the maximum principle that

$$dx^*(t) = \frac{\partial H(x^*,u^*,p,q)}{\partial p}dt + g(x^*(t))dW(t) \qquad (24)$$

$$dp(t) = -\frac{\partial H(x^*,u^*,p,q)}{\partial x}dt + q(t)dW(t) \qquad (25)$$

$$H_m(x^*,u^*,p,q) = \max_{u\in U} H_m(x^*,u^*,p,q) \qquad (26)$$

where $x^*(t)$ is an optimal trajectory of $x(t)$. The initial and terminal conditions of (24) and (25) are

$$x^*(0) = x_0 \qquad (27)$$

$$p(t_f) = -\frac{\partial h(x^*(t_f))}{\partial x} \quad (28)$$

Since equation (26) implies that the optimal control $u^*(t)$ is a function of $p(t)$, $q(t)$ and $x^*(t)$, we have

$$u^*(t) = \phi(x^*, p, q) \quad (29)$$

where φ is determined by (26), hence equation (24) and (25) can be rewritten as

$$dx^*(t) = \frac{\partial H(x^*, u^*, p, q)}{\partial p} dt + g(x^*(t))dW(t) \quad (30)$$

$$dp(t) = -\frac{\partial H(x^*, u^*, p, q)}{\partial x} dt + q(t)dW(t) \quad (31)$$

Thus the Hamiltonian is defined by

$$\begin{aligned} H &= A_1 x_2(t) + A_2 \frac{u_1^2(t)}{2} + A_3 \frac{u_2^2(t)}{2} \\ &+ p_1\left(\Lambda - \left(\beta - \frac{\pi x_2}{1+x_2}\right) x_1(t) x_2(t) - ((1+u_1(t))\phi + \mu) x_1(t) \right) \\ &+ p_2\left(\beta - \left(\frac{\pi x_2}{1+x_2}\right) x_1(t) x_2(t) - ((1+u_2(t))\gamma + \delta + \mu) x_2(t) \right) \\ &+ p_3\big((1+u_1(t))\phi x_1(t) + (1+u_2(t))\gamma x_2(t) - \mu x_3(t)\big) \\ &- q_1 \varepsilon x_1 x_2 + q_2 \varepsilon x_1 x_2 \end{aligned} \quad (32)$$

It follows from the stochastic maximum principle that

$$dp(t) = -\frac{\partial H(x^*, u^*, p, q)}{\partial x} dt + q(t)dW(t) \quad (33)$$

That is

$$\begin{aligned} dp_1(t) &= \left\{ p_1\left(\left(\beta - \frac{\pi x_2}{1+x_2}\right) x_2(t) - ((1+u_1(t))\phi + \mu)\right) - p_2\left(\left(\beta - \frac{\pi x_2}{1+x_2}\right) x_2(t) + q_1 \varepsilon x_2 - q_2 \varepsilon x_2\right)\right\} dt + q_1 dW(t) \\ dp_2(t) &= \left\{ p_1\left(\left(\beta x_1 - \frac{\pi x_2(2x_2 + x_2^2)}{1+x_2}\right) - p_2\left(\beta x_1 - \frac{\pi x_2(2x_2 + x_2^2)}{1+x_2}\right) - ((1+u_2)\gamma + \delta + \mu)\right) + A_1 + q_1 \varepsilon x_1 - q_2 \varepsilon x_1 \right\} dt + q_2 dW(t) \\ dp_3(t) &= p_3 \mu \end{aligned} \quad (34)$$

With the initial and terminal conditions

$$x_1^*(0)=x_{10}, x_2^*(0)=x_{20}, x_3^*(0)=x_{30},$$
$$dp(t_f)=-\frac{\partial h(x^*(t_f))}{\partial x} \qquad (35)$$

And

$$h(x_1(t),x_2(t))=\frac{k_1}{2}x_1^2(t)+\frac{k_2}{2}x_2^2(t)+\frac{k_3}{2}x_3^2(t)$$
$$\therefore\ p_1(t_f)=-k_1x_1, p_2(t_f)=-k_2x_2,\ p_3(t_f)=-k_3x_3,$$
$$p_1(t_f)=-k_1, p_2(t_f)=-k_2,\ p_3(t_f)=-k_3. \qquad (36)$$

Then by differentiating Hamiltonian equation with respect to u_1 and u_2 we get the optimal controls u_1^* and u_2^*.

$$u_1^*=\max\left\{\min\left[\frac{x_1(t)\phi(p_1-p_3)}{A_2},1\right],0\right\} \qquad (37)$$

And

$$u_2^*=\max\left\{\min\left[\frac{x_2(t)\gamma(p_2-p_3)}{A_3},1\right],0\right\} \qquad (38)$$

In the next section it is discussed the numerical simulation of results of deterministic and stochastic optimal control problem.

5. NUMERICAL ANALYSIS

In this section, the numerical simulations of the deterministic optimal control problem (2) and corresponding stochastic optimal control problem (17) is discussed. Parameter values used in the simulation are Λ=40, β=0.0003, π=0.0002, γ=0.045, ϕ=0.65, δ=0.02, μ=0.05, A_1=100, A_2=150, A_3=100 with the initial conditions $x_1(0)$=1,00,000, $x_2(0)$=100 and $x_3(0)$=0 and an intensity of noise for stochastic model is assumed as ε=0.005. Numerical solutions to the optimal control problem comprising of the state equation (2) and adjoint system (12) are carried out using parameters considered above and in the discussion that follows for the stochastic optimality control problem, it is used the results of deterministic control problem to find an approximate numerical solution for the stochastic control problem (See, Witbooi et al. [18]). In particular, it is used a proxy for $\lambda_i, i=1,2,3$ in the calculation of $u_1(t)$ and $u_2(t)$. It is noted that the presence of $x_1(t)$ and $x_2(t)$ in $u_1(t)$ and $u_2(t)$, includes the stochasticity in the adjoint system (in the stochastic case). Using the method mentioned in [19], from this results are obtained by simulations, which is shown in Figure 2.

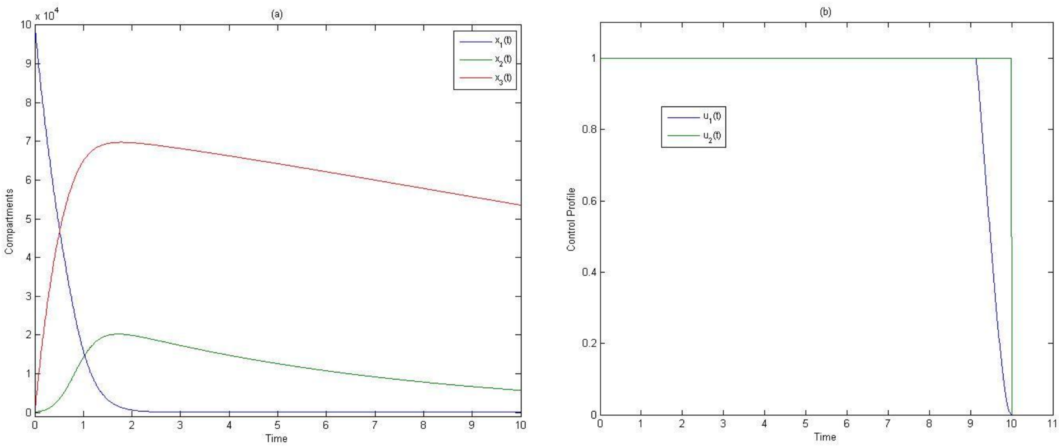

Fig. 1: Simulation of deterministic model solution (a) and control profile u(t) (b).

The time series plot illustrates the dynamics of the population in each compartment with respect to time (in days), Figure 1(a) shows the time series plot for the deterministic epidemic model under the time dependent controls $u_1(t)$ and $u_2(t)$ and control profile of the same model in Figure1(b). Further it is evident from the Figure 1(b) that it is optimal to use control u_1 up to 9 units of time at maximum rate and then it gradually decreases to lower bound up to 10 units of time and the time dependent control u_2 refers to controlling effort that alters infection cases receiving treatment per unit time is results optimally if it is used from initial to final time at maximum rate.

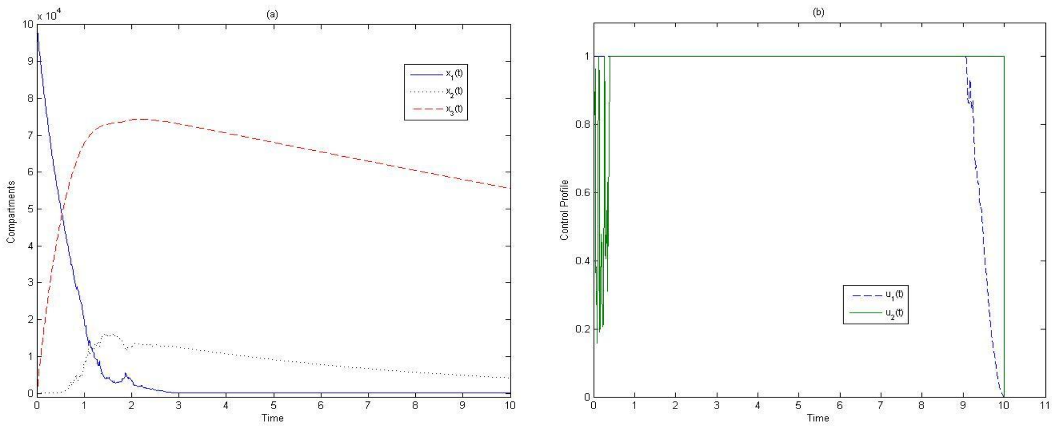

Fig. 2: Simulation of Stochastic model solution (a) and control profile u(t)(b).

Figure 2 illustrates the stochastic model solutions(a) and control profile of the same model using the parameter values and initial conditions same as deterministic model. It is observed from Figure 2(a) that the stochastic model solution also depicts same scenario as that of deterministic model solution under the time dependent control u_1 and u_2 and also control profile u_1 and u_2 exhibits same state of affairs as deterministic model in Figure 2(b). An important point to note about our approximation is that it fully accommodates the stochasticity. Further it is evident from the Figure 2(b) that it is optimal to use strengthening effort made on vaccination program up to 9 units of time at maximum rate and gradually decreases to lower bound up to 10 units of time. The control u_2 that is controlling effort that alters infection cases receiving treatment per unit time in Figure 2(b), it is varying from initial point of time and later it is at upper bound till final time.

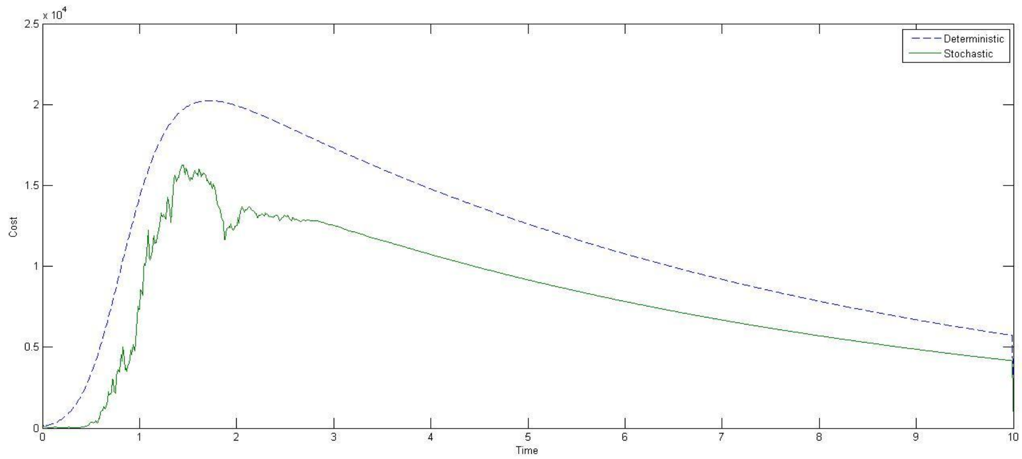

Fig. 3: Simulation of Deterministic and Stochastic cost functional.

Figure 3 shows the simulation for cost functional of deterministic and stochastic model. From the figure it is clear that the similar kind of flow is found in both cost curves, which agrees that the infections directly influencing on cost functional but we can observe the stochastic model cost it is observed that inclusion of noise give exact results than deterministic model also it is observed that the cost for stochastic problem is less than the deterministic problem.

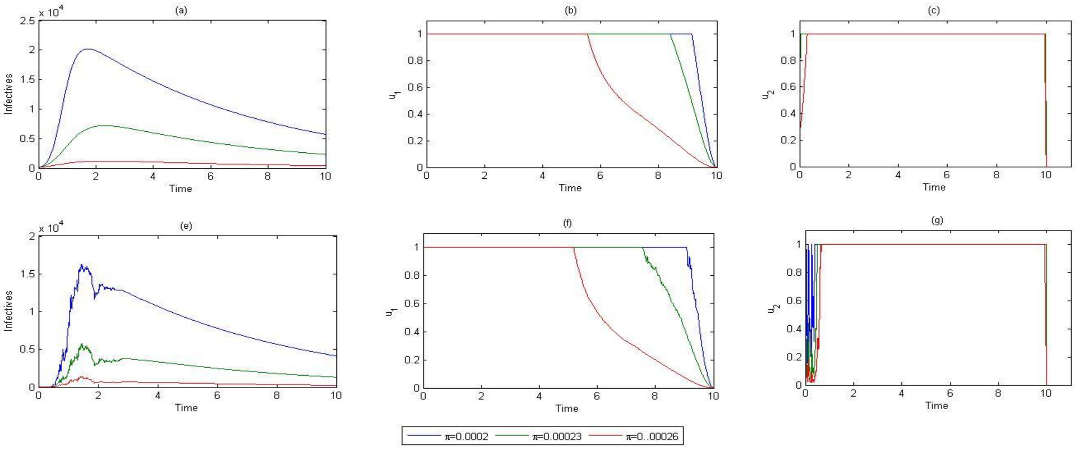

Fig.4: Simulation of infective trajectory, control profile u_1 and u_2 for varying rate of parameter π

Figure 4 shows the simulation of deterministic and stochastic model for infectives and controls u_1 and u_2, when rate of awareness varies. From the Figure 4 (a),(b),(c) it is very clear that as the rate of awareness increases the number of infections are decreasing and usage of controls u_1 and u_2 is also decreasing. Similar kind of variation is found in the stochastic curves of infectives and controls u_1 and u_2 see Figure 4 (d),(e),(f).

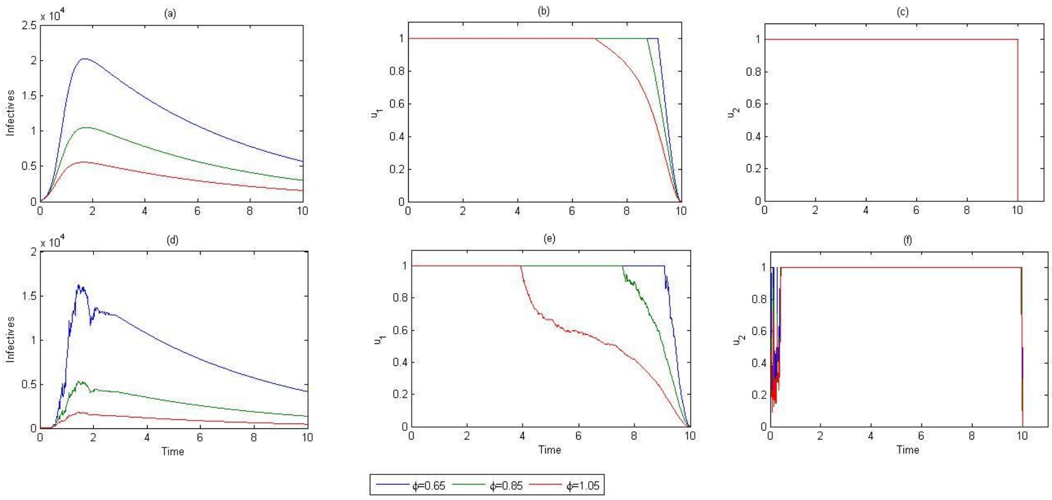

Fig. 5: Simulation of infective trajectory, control profile u_1 and u_2 for varying rate of parameter ϕ.

From the Figure 5 it seen that as the vaccination rate ϕ increases the number of infections are decreasing and control u_1 usage is decreasing for higher vaccination rate where as control u_2 is at maximum rate till final time.

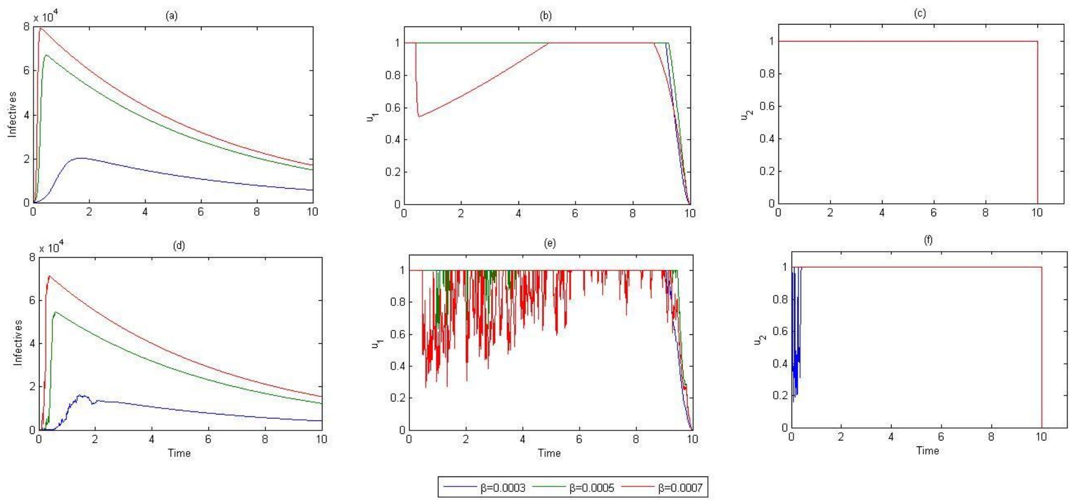

Fig.6: Simulation of infective trajectory, control profile u_1 and u_2 for varying rate of parameter β .

Figure 6 shows the variation in the number of infections and controls for varying infection rate. It is quite straightforward that as infection rate β increases, number of infections are increases, but the control u_1 is decreasing for higher infection rate and control u_2 is at maximum rate from initial to final time in both deterministic and stochastic case.

6. CONCLUSION

The present study considered the optimal control analysis of both deterministic and stochastic modeling of infectious disease by taking effects of strengthening effort made on vaccination program and controlling effort that alters infection cases receiving treatment per unit time strategies on the epidemic into account. Optimal control strategy under the quadratic cost functional using Pontrygin's Maximum Principle and Solutions are derived for both deterministic and stochastic optimal control problem respectively. To solve stochastic optimal problem, it is used the Stochastic Maximum Principle, however it ought to be pointed out that for stochastic optimality system, it may be difficult to obtain the numerical results because of dimensionality. For the analysis of the stochastic optimal control problem, the results of deterministic control problem are used to find an approximate numerical solution for the stochastic control problem. Outputs of the simulations shows that non-pharmaceutical interventions and vaccination place important role in the minimization of infectious population with minimum cost. Numerical simulation of stochastic system enables to measure the feasibility of option followed. A formal approach to the numerical simulation of the stochastic optimal control problem is far more complex and labor intensive and our method is a workable approximate alternative.

REFERENCES

[1] Kermack WO, McKendrick AG, Contribution to the mathematical theory of epidemics. Proc R Soc Lond A, 1927, 115:700-721.

[2] Bootsma MCJ, Ferguson NM. The effect of public health measures on the 1918 influenza pandemic in U.S. cities. Proceedings of the National Academy of Sciences of the United States of America, 2007;104(18):7588-7593.

[3] Tchuenche JM, Khamis SA, Agusto FB. Optimal Control and Sensitivity Analysis of an Influenza Model with Treatment and Vaccination. Acta Biotheor, 2011; 59:1-28.

[4] Okosun KO, Makinde OD, Takaidza I. Impact of Optimal Control on the Treatment of HIV/AIDS and Screening of Unaware Infectives. Applied Mathematical Modelling, 2013; 37: 3802-3820.

[5] Joshi HR. Optimal Control of an HIV Immunology Model. Optimal Control Applications and Methods, 2002; 23:199-213.

[6] Blayneh K, Cao Y, Kwon HD. Optimal control of vector-borne disease: Treatment and prevention. Discrete and continuous Dynamical systems series B, 2009 ;11: 587-611.

[7] Gani SR., Taslim AS., and Kadi AS. The transmission dynamics of pandemic influenza A/H1N1 2009–2010 in India,Current Science 2011; 101(08):1065-1072.

[8] Yiping L., Jing-an C. The impact of media convergence on the dynamics of infectious diseases, International Journal of Biomathematics, 2008;1:65–74.

[9] Jing-an C., Tao X., Huaiping Zhu. An SIS infection model incorporating media coverage. The Rocky Mountain Journal of Mathematics, 2008;38(5):1323–1334.

[10] Tornatore E., Buccellato SM., Vetro P., On a stochastic disease model with vaccination, Rend. Circ. Mat. Palermo 55(2)(2006) 223-240.

[11] Driessche VD., Watsmough J., Reproduction number and sub-threshold endemic equilibria for compartmental models of disease transmission, Math Biosci. 2002;180:29–48.

[12] Jing-an C., Yonghong S., Huaiping Z., The impact of media on the control of infectious disease. Journal of Dynamics and Differential Equations, 2008;20(1):31–53.

[13] Pontryagin LS., Boltyanskii VG., Gamkrelidze RV. and Mishchenko EF.,

1962, "The Mathematical Theory of Optimal Processes," Wiley, New York.

[14] Kar TK, Jana SA., Theoretical study on mathematical modelling of an infectious disease with application of optimal control, 2013; 111, 37–50.

[15] Fleming WH., Rishel RW., 1975, "Deterministic and Stochastic Optimal Control," Springer Verlag, New York.

[16] Lukes DL: Differential Equations: Classical to Controlled. New York:Academic Press; 1982.

[17] Kunita H.,: Lecture on stochastic flows and Applications, Springer-Verlag (1986).

[18] Witbooi PJ., Grant E., Muller, and Grth J.Van Schallkwyk, 2015, "Vaccination Control in a Stochastic SVIR Epidemic Model," Hindawi.

[19] Higham, D., An algorithmic introduction to numerical simulation of stochastic differential equations, SIAM Rev. 43 (2001) 525–546.