Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 9 (2017), pp. 5961-5964

http://www.ripublication.com

Semi-ergodic Transformations

Nanang Susyanto

Department of Mathematics, Universitas Gadjah Mada
Sekip Utara, Yogyakarta, Indonesia

Abstract

We propose a new concept of generalizing ergodic property, which we call the semi-ergodic transformation. This generalization is motivated by the ergodicity in infinite measure spaces. The relation between ergodic and semi-ergodic is studied that is completed by some simple examples. Finally, the main theorem that guarantees any transformation to be semi-ergodic is provided.

MSC 2010 subject classifications: Primary 28D05; Secondary 28D15

Keywords and phrases: ergodic, semi-ergodic, measure theory, preserved transformation, infinite measure spaces.

1 INTRODUCTION

Let (X, Ω, μ) be a measure space and the transformation $T: X \mapsto X$ is nonsingular. If there is a set A with $\mu(A) > 0$, $A \neq X$, and $T^{-1}(A) = A$ then the transformation T can be studied on the set A and $X \backslash A$ separately. Otherwise, if there is no A with that properties then the transformation T is called **ergodic**; see e.g. Walter (1981) and Lasota and Mackey (1994).

Definition 1.1. *(A. Lasota and M.C. Mackey, 1994) Let (X, Ω, μ) be a measure space and the nonsingular transformation $T: X \mapsto X$ is given. The transformation T is called ergodic if the property*

$$A \in \Omega \Rightarrow \mu(A) > 0 \text{ or } \mu(X \backslash A) = 0 \quad (1)$$

holds. In other words, transformation T is ergodic if all invariant sets are trivial subset of X.

A simple example of ergodic transformation is $T: \mathbb{Z} \mapsto \mathbb{Z}$ where $\mathbb{Z}$ is the set of all integers defined by

$$T\,(x) = x + 1$$

for every $x \in \mathbb{Z}$ with the counting measure μ. Put an abritrary invariant set A with $\mu(\text{A}) > 0$. Then $A \neq \emptyset$, so there exists $a \in A$. Since $a \in A$ and $T^{-1}(A) = A$ then $\{a - 1\} = T^{-1}(\{\text{a}\}) \subset A$, so $a - 1 \in A$. On the other hand, we have $T\,(A) = A$, so $a + 1 = T(a) \in A$. Inductively, we have $\ldots, a - 2, a - 1, a, a + 1, a + 2, \ldots \in A$. It follows that $A = \mathbb{Z}$.

Definition (1.1) is extended for infinite measure spaces by Arzumanian, Eigen and Hajian (2015). Their definition needs a condition called σ −finite measure, i.e., there exist $\{A_n\}_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} A_n = X$ and $\mu(A_n) < \infty$ for every n; see e.g. Wheeden (1977). In the present paper, we will not restrict the (infinite) measure to be σ −finite. It means that we will be working in any measure spaces.

The rest of this paper is organized as follows. Section 2 gives the definition of semi-ergodic transformation completed by some simple examples. After exploring the relation between ergodic and semi-ergodic transformations, the main theorem that guarantees semi-ergodic property is provided at the end of Section 2. Finally, some conclusions are drawn in Section 3.

2 MAIN RESULTS

Generalizing the definition of ergodic transformation, we will introduce the definition of semi-ergodic transformation as follows.

Definition 2.1. *Let (X, Ω, μ) be a measure space and the nonsingular transformation $T: X \mapsto X$ is given. Transformation T is called semi-ergodic*

$$A \in \Omega \Rightarrow \mu(\text{A}) = 0 \text{ or } \mu(\text{A}) = \text{X}. \tag{2}$$

If $\mu(X) < \infty$ then T is called finite semi-ergodic and if $\mu(X) = \infty$ then T is called infinite semi-ergodic.

Of course every ergodic transformation is also semi-ergodic since the definition of semi-ergodic is a generalization from the definition of ergodic. Precisely, it is stated in the following theorem.

Theorem 2.1. *Every ergodic transformation $T: X \mapsto X$ is semi-ergodic.*

Proof. *Put an arbitrary invariant set A with $\mu(X) > 0$. Since T is ergodic then*

$\mu(X\backslash A) = 0$. *We have* $X = A \cup (X\backslash A)$ *and* $A \cap (X\backslash A) = \emptyset$. *This implies*

$$\mu(X) = \mu\big(A \cup (X\backslash A)\big) = \mu(A) + \mu(X\backslash A) = \mu(A).$$

This leads to the conclusion that T is semi-ergodic. □

It would not be interesting if every semi-ergodic transformation was also ergodic. An example of semi-ergodic transformation that is not ergodic is provided in the following example.

Example 2.1. Consider the Borel Space $(\mathbb{R}, \Omega, \mu)$ where μ is the Lebesgue Measure and the transformation $T: \mathbb{R} \mapsto \mathbb{R}$ defined by

$$T(x) = \begin{cases} 2x \text{ if } x \geq 0 \\ \dfrac{x}{2} \text{ if } x < 0. \end{cases}$$

Transformation T is not ergodic because there are 3 invariant sets with positive measure, i.e., $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}\backslash\mathbb{R}^+$ where $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$. However, it is easy to see that T is semi-ergodic because

$$\mu(\mathbb{R}) = \mu(\mathbb{R}^+) = \mu(\mathbb{R}\backslash\mathbb{R}^+) = \infty.$$

The following theorem proves the equivalence between ergodic and semi- ergodic in finite measure spaces.

Theorem 2.2. *Every finite semi-ergodic transformation* $T: X \mapsto X$ *is ergodic.*

Proof. Put an arbitrary invariant set A such that $\mu(A) > 0$. Since T is semi- ergodic then $\mu(A) = \mu(X)$. We have $X = A \cup (X\backslash A)$ and $A \cap (X\backslash A) = \emptyset$, so $\mu(X) = \mu\big(A \cup (X\backslash A)\big) = \mu(A) + \mu(X\backslash A)$. Since $\mu(A) = \mu(X)$ then $\mu(X\backslash A) = 0$. □

Theorem 2.2 suggests us to only consider infinite semi-ergodic transformations. The following theorem can be used to characterize the infinite semi- ergodic property of transformation.

Theorem 2.3. *Let* (X, Ω, μ) *be an infinite measure space and the nonsingular transformation* $T: X \mapsto X$ *is given. If for every measurable function* $f: X \mapsto \mathbb{R}$ *the relation*

$$f\big(T(x)\big) = f(x) \textit{ for almost all } x \in X \tag{3}$$

implies f *to be constant almost everywhere, then T is semi-ergodic.*

Proof. Assume T is not semi-ergodic, so T is not ergodic. Then there is a nontrivial invariant set A. Put $f = 1_A$. Since A is nontrivial, f is not constant. Thus, $T^{-1}(A) = A$ implies

$$f\big(T(x)\big) = 1_A(x) = 1_{T^{-1}(A)}(x) = 1_A(x) = f(x) \text{ a.e.,}$$

that contradicts with (3). □

Unfortunately, the converse of Theorem 2.3 is not always true that can be seen from the following example that will close this section.

Example 2.2. Consider the Borel Space $(\mathbb{R}, \Omega, \mu)$ where μ is the Lebesgue Measure and transformation $T: \mathbb{R} \mapsto \mathbb{R}$ is defined by

$$T(x) = \begin{cases} x + 2\pi \text{ if } x \geq 0 \\ -x - 2\pi \text{ if } x < 0. \end{cases}$$

It is easy to see that *T* is semi-ergodic. Now, put $f(x) = \cos x$. Condition in Theorem 2.3 is not satisfied since

$$f(T(x)) = \begin{cases} f(x + 2\pi) = cos\,(x + 2\pi) = \cos x = f(x) \text{ if } x \geq 0 \\ f(-x - 2\pi) = cos\,(-x - 2\pi) = \cos(-x) = \cos x = f(x) \text{ if } x < 0 \end{cases}$$

holds but f is obviously not constant.

3 CONCLUSION

We have defined a new concept called *semi-ergodic transformations.* It has been shown that ergodic and semi-ergodic transformation are equivalent in the finite measure spaces but it is not necessarily to be true in infinite measure spaces. Finally, we have provided a characterization when a transformation is semi-ergodic.

REFERENCES

[1] Arzumanian, V., Eigen, S., and Hajian, A. (2015) Notes on Ergodic Theory in Infinite Measure Spaces. *Armen. J. Math.,* 2, 97-120.

[2] Lasota, Andrzej and Mackey, M.C. (1994) *Chaos, Fractals, and Noise.* Springer, New York.

[3] Walter, Peter. (1981) *An Introduction to Ergodic Theory.* Springer, New York.

[4] Wheeden, Ricard L. (1977) *Measure and Integral.* Marcel Dekker, New York