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Approximate Fixed Points and Summable Almost Stability

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Abstract

In this paper, we establish some approximate fixed point theorems in the setting of *b*-metric space generalizing some of the recent results reported in the literature. Further, we study summable almost stability of iterative scheme.

Keywords: Hardy-Rogers contraction, Ciric type contraction, Approximate fixed point, b-metric space, summable almost stability.

MSC2010: 47H10, 54H25, 47H15.

1. INTRODUCTION AND PRELIMINARIES

Let $T: X \to X$ and $x^* \in X$. Then, x^* is an $\varepsilon-$fixed point or an approximate fixed point of T if

$$d(x^*, Tx^*) < \varepsilon, \quad \varepsilon>0.$$

For a given $\varepsilon,$ we denote the set of approximate fixed points of T by $F_\varepsilon(T),$ and it is given by

$$F_\varepsilon(T) = \{x^* \in X, d(x^*, Tx^*) < \varepsilon\}.$$

In many practical situations, we may need an approximate solution of the problem or it might not be possible to get an exact fixed point of a map under consideration due to certain strong restrictions on the map or on the space. An approximate solution plays an important role in such problems. Fixed point and approximate fixed points have been proved to be of prime importance in various areas such as mathematical economics, game theory, dynamic programming, nonlinear analysis and several other areas of applicable analysis. Cromme and Diener [9] generalized Brouwer's fixed point theorem to a discontinuous map and found approximate fixed points of the maps. Hou and Chen [14] extended their results to set valued maps and Espinola and Kirk [11] obtained interesting results in product spaces.Tijs et al. [30] established approximate fixed point theorems by reducing the strictness on the maps in metric space. He replaced the compactness conditions used in the Brouwer, Kakutani theorems [7-8] by boundedness conditions in finite dimensional spaces and also dropped the completeness of the metric space in Banach's contraction theorem [2]. Recently Berinde [6] obtained approximate fixed points for operators satisfying Kannan, Chatterjea and Zamfirescu type of conditions on metric spaces. In recent years, several authors studied approximate fixed point theory in different settings (see [3, 17] and the references therein).

There are plenty of iterative schemes available in literature for finding the desired solution of the problems formulated as fixed point equations. Usually, Picard iteration is used for strict contractive type operator and Krasnoselskij iteration for non-expansive or pseudo-contractive operators. Various other schemes such as Mann, Ishikawa etc are also used in the literature under different situations. It is very essential to see if these methods are numerically stable or not. Harder and Hicks [12-13] defined the stability for fixed point iterations.

Let (X,d) be a metric space, $T: X \to X$ and $x_0 \in X$. Assume that the iteration procedure

$$x_{n+1} = f(T, x_n), \quad n = 0,1,2,..., \tag{1.1}$$

where f is some function, converges to a fixed point p of T. Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X and $\{\varepsilon_n\}_{n=0}^{\infty}$ be defined by $\varepsilon_n = d(y_{n+1}, f(T, y_n))$, n=0,1,2,... The fixed point iteration is said to be T-stable or stable with respect to T if and only if

$$\lim_{n\to\infty} \varepsilon_n = 0 \Rightarrow \lim_{n\to\infty} y_n = p. \tag{1.2}$$

This concept of stability has been widely studied by various authors for different nonlinear operators for different metric spaces (see [15, 19, 24-28]). Osilike [20-22] studied the stability of Ishikawa iteration for pseudo contractive operators and introduced a weaker concept of stability. The fixed point iteration is said to be almost T-stable or almost stable with respect to T if and only if

$$\sum_{n=0}^{\infty}\varepsilon_n < \infty \Rightarrow \lim_{n\to\infty} y_n = p. \tag{1.3}$$

Berinde [4] defined almost stability in much weaker condition and established some summably almost stable fixed point procedures with respect to various contractive operator. The fixed point iteration is said to be summably almost T-stable with respect to T if and only if

$$\sum_{n=0}^{\infty}\varepsilon_n < \infty \Rightarrow \sum_{n=0}^{\infty} d(y_n, p) < \infty. \tag{1.4}$$

It is remarked that if an iteration procedure is not T-stable then it is not almost T-stable also and hence, not summably almost T-stable but converse may not be true, see Example 1.1 below (also see [Ex.1, 4]).

Example 1.1. Let $X = R.$ Define $T: X \to X$ where $Tx = \dfrac{x}{3}$ and (X, d) is a metric space with usual metric. Then to show that iteration procedure $x_{n+1} = Tx_n,\ n = 0,1,2,...$ is neither T-stable nor almost T-stable but summable almost T-stable.

Solution. Let $\{y_n\} \subseteq \{R\}$ be given by $y_n = \dfrac{n}{1+n},\ n \geq 0.$ Then,

$$\begin{aligned}
\varepsilon_n &= \left|y_{n+1} - f(T, y_n)\right| \\
&= \left|y_{n+1} - Ty_n\right| \\
&= \left|\frac{n+1}{2+n} - \frac{n}{3(1+n)}\right| \\
&\leq \frac{1}{(1+n)(2+n)}.
\end{aligned}$$

(i) $$\lim_{n\to\infty}\varepsilon_n = \lim_{n\to\infty}\frac{1}{(1+n)(2+n)} = 0.$$

But $\lim_{n\to\infty} y_n = \lim_{n\to\infty}(\dfrac{n}{1+n}) = 1 \neq 0$ (the unique fixed point of T).

Therefore, Picard iteration is not T-stable.

(ii) $$\sum_{n=0}^{\infty}\varepsilon_n = \sum_{n=0}^{\infty}\frac{1}{(1+n)(2+n)} < \infty.$$

But again, $\lim_{n\to\infty} y_n = \lim_{n\to\infty}(\dfrac{n}{1+n}) = 1 \neq 0$ (the unique fixed point of T).

Therefore, Picard iteration is not almost T-stable also.

(iii) $$\sum_{n=0}^{\infty}\varepsilon_n = \sum_{n=0}^{\infty}\frac{1}{(1+n)(2+n)} < \infty.$$

and $$\sum_{n=0}^{\infty} d(y_n, p) = \sum_{n=0}^{\infty}\left|\frac{n}{1+n} - 0\right| = \sum_{n=0}^{\infty}\frac{n}{1+n} < \infty.$$

Hence, Picard iteration is summable almost T-stable.

In this paper, we study summable almost stability of iterative schemes and establish some basic approximate fixed point results for the maps satisfying general contractive conditions in the setting of b-metric space.

The concept of b-metric space was introduced by Bakhtin [1] and popularized by Czerwik [10] and many others.

Definition 1.1 [10]. Let X be a non-empty set and $b \geq 1$ be a given real number. A function $d : X \times X \to R_+$ is said to be a b-metric if and only if for all $x, y, z \in X$, the following conditions are satisfied:

(i) $d(x, y) = 0$ iff $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, z) \leq b[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space.

Notice that when b becomes unity in condition (iii), (X, d) becomes a metric space. Therefore, the class of b-metric spaces contains that of metric spaces. This class of spaces has been wildly explored for various types of single-valued and multi-valued operators in different settings.

Now, we discuss some well known definitions of contractive conditions.

Let $T : X \to X$, if for all $x, y \in X$ there exists a constant $0 \leq a < 1$ such that

$$d(Tx, Ty) \leq a d(x, y). \tag{1.5}$$

Then T is called contraction mapping and constant 'a' is called contractivity factor for T.

If there exists a number a, $0 < a < \frac{1}{2}$ such that

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)]. \tag{1.6}$$

and

$$d(Tx,Ty) \le a[d(x,Ty)+d(y,Tx)]. \tag{1.7}$$

Then T is called Kannan and Chatterjea contractions [29] respectively.

If there exists $a,b,c \ge 0,\ a+b+c<1$ such that

$$d(Tx,Ty) \le ad(x,y)+bd(x,Tx)+cd(y,Ty) \tag{1.8}$$

then T is called a Reich contraction mapping [29].

If there exists nonnegative constant a_i satisfying $\sum_{i=1}^{5} a_i < 1$ such that

$$d(Tx,Ty) \le a_1 d(x,y)+a_2 d(x,Tx)+a_3 d(y,Ty)+a_4 d(x,Ty)+a_5 d(y,Tx) \tag{1.9}$$

then T is called a Hardy and Rogers contraction mapping [29].

If $a_4 = a_5 = 0$ in (1.9), it becomes (1.8). On putting $a_1 = a_4 = a_5 = 0$ and $a_2 = a_3 = \alpha$, we get (1.6).

If there exists nonnegative functions a_1, a_2, a_3, a_4 satisfying $\sup_{x,y\in X}\{a_1(x,y)+a_2(x,y)+a_3(x,y)+2a_4(x,y)\} \le \lambda < 1$ such that for each $x, y \in X$,

$$\begin{aligned} d(Tx,Ty) \le{}& a_1(x,y)d(x,y)+a_2(x,y)d(x,Tx)+a_3(x,y)d(y,Ty) \\ &+a_4(x,y)[d(x,Ty)+d(y,Tx)] \end{aligned} \tag{1.10}$$

then T is called a Ciric contraction mapping [29].

The following example establishes the existence of an $\varepsilon -$ fixed point.

Example 1.2. Let (X,d) be a metric space with $X = R$ and usual metric d such that

$$T_n x = \frac{x}{3} + \frac{1}{n},\ \forall x \in X,\ n \in N,\ n \ge \frac{1}{\varepsilon},\ \varepsilon > 0.$$

Then, T_n satisfy (1.9) but not (1.6), (1.7) and (1.8).

Solution. (i) Let $x = 2, y = 5,\ a_1 = 0.2,\ a_2 = 0.15,\ a_3 = 0.4,\ a_4 = 0.05$ and $a_5 = 0.1.$ So that $a_1 + a_2 + a_3 + a_4 + a_5 < 1.$ Then,

$$d(T_n x, T_n y) = |T_n x - T_n y| = \left|\frac{2}{3} + \frac{1}{n} - \frac{5}{3} - \frac{1}{n}\right| = 1.$$

Also

$$a_1 d(x,y) + a_2 d(x,T_n x) + a_3 d(y,T_n y) + a_4 d(x,T_n y) + a_5 d(y,T_n x)$$
$$= 0.2\left|2-5\right| + 0.15\left|2-\frac{2}{3}-\frac{1}{n}\right| + 0.4\left|5-\frac{5}{3}-\frac{1}{n}\right| + 0.05\left|2-\frac{5}{3}-\frac{1}{n}\right| + 0.1\left|5-\frac{2}{3}-\frac{1}{n}\right|$$
$$\geq 1 = d(T_n x, T_n y).$$

Then T_n satisfy (1.9).

Now, if we take $a = 0.15$, then from (1.6),

$$a[d(x,Tx) + d(y,Ty)] = 0.15(\left|2-\frac{2}{3}-\frac{1}{n}\right| + \left|5-\frac{5}{3}-\frac{1}{n}\right|)$$
$$\leq 1 = d(T_n x, T_n y).$$

$\therefore T_n$ does not satisfy (1.6).

Similarly, if we take $a = 0.05$, then from (1.7),

$$a[d(x,Ty) + d(y,Tx)] = 0.05(\left|2-\frac{5}{3}-\frac{1}{n}\right| + \left|5-\frac{2}{3}-\frac{1}{n}\right|)$$
$$\leq 1 = d(T_n x, T_n y).$$

So, T_n does not satisfy (1.7).

Further, if we take $a = 0.2,\ b = 0.15,\ c = 0.05$, then from (1.8),

$$ad(x,y) + bd(x,Tx) + cd(y,Ty) = 0.2\left|2-5\right| + 0.15\left|2-\frac{2}{3}-\frac{1}{n}\right| + 0.05\left|5-\frac{5}{3}-\frac{1}{n}\right|$$
$$\leq 1 = d(T_n x, T_n y).$$

Hence, T_n does not satisfy (1.8).

(ii) Now,

$$x - T_n x = x - \frac{x}{3} - \frac{1}{n} = \frac{2x}{3} - \frac{1}{n} \geq \frac{2x}{3} - \varepsilon.$$

But as

$$x \to x_\varepsilon^* = 0.$$
$$\therefore x_\varepsilon^* - T_n x_\varepsilon^* \geq -\varepsilon$$
$$\text{or } -(x_\varepsilon^* - T_n x_\varepsilon^*) \leq \varepsilon$$
$$\text{or } \left|T_n x_\varepsilon^* - x_\varepsilon^*\right| \leq \varepsilon.$$

Hence, $x_\varepsilon^* = 0$ is an $\varepsilon-$fixed point of T_n. Clearly, it is not a fixed point since $T_n x^* \neq x^* \ \forall n \in N$.

Lemma 1 [6]. Let $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$, be sequences of nonnegative numbers and $0 \le q < 1$, such that

$$a_{n+1} \le q a_n + b_n, \quad \forall n \ge 0.$$

Then, the following hold true.

(i) If $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

(ii) If $\sum_{n=0}^{\infty} b_n < \infty$, then $\sum_{n=0}^{\infty} a_n < \infty$.

2. RESULTS

Theorem 2.1. Let (X,d) be a b-metric space and $T : X \to X$ satisfies (1.9). Then for each $\varepsilon > 0$, the diameter of $\mathrm{Fix}_\varepsilon(T)$ is not longer than $\dfrac{r\varepsilon(1+r+r(a_2+a_3)+r^2(a_4+a_5))}{1-r^2a_1-r^3a_4-r^3a_5}$.

Proof. Let x and y be two $\varepsilon-$fixed points of T. Then,

$$\begin{aligned}
d(x,y) &\le r[d(x,Tx)+d(Tx,y)] \\
&\le r\varepsilon + r^2[d(Tx,Ty)+d(Ty,y)] \\
&\le r\varepsilon + r^2 d(Tx,Ty) + r^2\varepsilon \\
&\le r\varepsilon + r^2\varepsilon + r^2[a_1 d(x,y)+a_2 d(x,Tx)+a_3 d(y,Ty)+a_4 d(x,Ty)+a_5 d(y,Tx)] \\
&\le r\varepsilon + r^2\varepsilon + r^2 a_1 d(x,y) + r^2 a_2\varepsilon + r^2 a_3\varepsilon + r^2 a_4 d(x,Ty) + r^2 a_5 d(y,Tx) \\
&\le r\varepsilon + r^2\varepsilon + r^2 a_1 d(x,y) + r^2 a_2\varepsilon + r^2 a_3\varepsilon + r^3 a_4[d(x,y)+d(y,Ty)] \\
&\quad + r^3 a_5[d(y,x)+d(x,Tx)].
\end{aligned}$$

$$(1-r^2a_1-r^3a_4-r^3a_5)d(x,y) \le r\varepsilon + r^2\varepsilon + r^2a_2\varepsilon + r^2a_3\varepsilon + r^3a_4\varepsilon + r^3a_5\varepsilon.$$

or

$$d(x,y) \le \frac{r\varepsilon(1+r+ra_2+ra_3+r^2a_4+r^2a_5)}{1-r^2a_1-r^3a_4-r^3a_5}.$$

or

$$d(x,y) \le \frac{r\varepsilon(1+r+r(a_2+a_3)+r^2(a_4+a_5))}{1-r^2a_1-r^3a_4-r^3a_5}.$$

Hence proved.

Theorem 2.2. Let (X,d) be a b-metric space and $T: X \to X$ satisfies (1.10). Then for each $\varepsilon > 0$, the diameter of $\mathrm{Fix}_\varepsilon(T)$ is not longer than $\dfrac{r\varepsilon(1+r+ra_2(x,y)+ra_3(x,y)+2r^2a_4(x,y))}{1-r^2a_1(x,y)-2r^3a_4(x,y)}$.

Proof. Let x and y be two $\varepsilon-$fixed points of T. Then,

$$\begin{aligned}
d(x,y) &\le r[d(x,Tx)+d(Tx,y)] \\
&\le r\varepsilon + r^2[d(Tx,Ty)+d(Ty,y)] \\
&\le r\varepsilon + r^2 d(Tx,Ty) + r^2\varepsilon \\
&\le r\varepsilon + r^2\varepsilon + r^2[a_1(x,y)d(x,y)+a_2(x,y)d(x,Tx)+a_3(x,y)d(y,Ty) \\
&\quad + a_4(x,y)\{d(x,Ty)+d(y,Tx)\}] \\
&\le r\varepsilon + r^2\varepsilon + r^2a_1(x,y)d(x,y) + r^2a_2(x,y)\varepsilon + r^2a_3(x,y)\varepsilon \\
&\quad + r^2a_4(x,y)[d(x,Ty)+d(y,Tx)] \\
&\le r\varepsilon + r^2\varepsilon + r^2a_1(x,y)d(x,y) + r^2a_2(x,y)\varepsilon + r^2a_3(x,y)\varepsilon \\
&\quad + r^3a_4(x,y)[d(x,y)+d(y,Ty)+d(y,x)+d(x,Tx)]
\end{aligned}$$

$$(1-r^2a_1(x,y)-2r^3a_4(x,y))d(x,y) \le r\varepsilon + r^2\varepsilon + r^2a_2(x,y)\varepsilon + r^2a_3(x,y)\varepsilon + 2r^3a_4(x,y)\varepsilon.$$

or

$$d(x,y) \le \frac{r\varepsilon(1+r+ra_2(x,y)+ra_3(x,y)+2r^2a_4(x,y))}{(1-r^2a_1(x,y)-2r^3a_4(x,y))}.$$

Hence proved.

If we put $a_1 = a,\ a_2 = a_3 = a_4 = a_5 = 0$ and $r = s$ in Theorem 2.1 and $a_1(x,y) = a,\ a_2(x,y) = a_3(x,y) = a_4(x,y) = 0$ and $r = s$ in Theorem 2.2, we get Theorem 3.2 of Prasad et al. [24].

Corollary 2.1 [24]. Let (X,d) be a b-metric space and $T: X \to X$ satisfies

$$d(Tx,Ty) \le ad(x,y), \quad a \in [0,1).$$

Then for each $\varepsilon > 0$, the diameter of $\mathrm{Fix}_\varepsilon(T)$ is no longer than $\dfrac{s\varepsilon(1+s)}{(1-as^2)}$.

If we put $a_1 = 0,\ a_2 = a_3 = b, a_4 = a_5 = 0$ and $r = s$ in Theorem 2.1 and $a_1(x, y) = 0,\ a_2(x, y) = a_3(x, y) = b,\ a_4(x, y) = 0$ and $r = s$ in Theorem 2.2, we get Theorem 3.4 of Prasad et al. [24].

Corollary 2.2 [24]. Let (X, d) be a b-metric space and $T : X \to X$ satisfies

$$d(Tx, Ty) \le b[d(x, Tx) + d(y, Ty)],\quad b \in [0, \frac{1}{2}).$$

Then for each $\varepsilon > 0$, the diameter of $\mathrm{Fix}_\varepsilon(T)$ is no longer than $s\varepsilon(1 + s + 2bs)$.

If we put $a_1 = a_2 = a_3 = 0,\ a_4 = a_5 = c$ and $r = s$ in Theorem 2.1 and $a_1(x, y) = a_2(x, y) = a_3(x, y) = 0,\ a_4(x, y) = c$ and $r = s$ in Theorem 2.2, we get Theorem 3.6 of Prasad et al. [24].

Corollary 2.3 [24]. Let (X, d) be a b-metric space and $T : X \to X$ satisfies

$$d(Tx, Ty) \le c[d(x, Ty) + d(y, Tx)],\quad c \in [0, \frac{1}{2}).$$

Then for each $\varepsilon > 0$, the diameter of $\mathrm{Fix}_\varepsilon(T)$ is no longer than $\dfrac{s\varepsilon(1 + s + 2cs^2)}{1 - 2cs^3}$.

Consequently we get Theorem 3.2, 3.3 and 3.6 of Berinde [5].

Theorem 2.3. Let (X, d) be a b-metric space and $T : X \to X$ satisfies (1.9). Suppose T has a fixed point p. Let $x_0 \in X$ and $x_{n+1} = Tx_n,\ \ n \ge 0$. Then $\{x_n\}$ converges strongly to p and is summable almost stable with respect to T.

Proof. Using definition of b-metric space, we have

$$\begin{aligned} d(y_{n+1}, p) &\le r(d(y_{n+1}, Ty_n) + d(Ty_n, p)) \\ &\le r(\varepsilon_n + d(Ty_n, p)) \end{aligned}$$

using (1.9), it becomes

$$\begin{aligned} d(y_{n+1}, p) &\le r\varepsilon_n + r[a_1 d(y_n, p) + a_2 d(y_n, Ty_n) + a_3 d(p, Tp) + a_4 d(y_n, Tp) + a_5 d(p, Ty_n)] \\ &\le r\varepsilon_n + r[a_1 + a_4 + a_5] d(y_n, p). \end{aligned}$$

Hence, iteration (1.1) is summably almost stable with respect to T.

Now,

$$\begin{aligned}
\varepsilon_n &= d(y_{n+1}, Ty_n) \\
&\le r(d(y_{n+1}, p) + d(p, Ty_n)) \\
&\le r(d(y_{n+1}, p) + d(Tp, Ty_n)) \\
&\le rd(y_{n+1}, p) + r[d(Tp, Ty_n)] \\
&\le rd(y_{n+1}, p) + r[a_1 d(y_n, p) + a_2 d(y_n, Ty_n) + a_3 d(p, Tp) + a_4 d(y_n, Tp) + a_5 d(p, Ty_n)] \\
&\le rd(y_{n+1}, p) + r[a_1 + a_4 + a_5] d(y_n, p) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

That is $\{x_n\}$ converges to *p*. Hence proved.

Theorem 2.4. Let (X, d) be a *b*-metric space and $T : X \to X$ satisfy (1.10). Suppose *T* has a fixed point *p*. Let $x_0 \in X$ and $x_{n+1} = Tx_n,\ \ n \ge 0.$ Then $\{x_n\}$ converges strongly to *p* and is summable almost stable with respect to *T*.

Proof. Using definition of *b*-metric space,

$$\begin{aligned}
d(y_{n+1}, p) &\le r(d(y_{n+1}, Ty_n) + d(Ty_n, p)) \\
&\le r(\varepsilon_n + d(Ty_n, p)) \\
&\le r\varepsilon_n + r[a_1(x, y)d(y_n, p) + a_2(x, y)d(y_n, Ty_n) + a_3(x, y)d(p, Tp) + \\
&\quad a_4(x, y)(d(y_n, Tp) + d(p, Ty_n))] \\
&\le r\varepsilon_n + r[a_1(x, y) + 2a_4(x, y)]d(y_n, p).
\end{aligned}$$

Hence, iteration (1.1) is summably almost stable with respect to *T*.

Now,

$$\begin{aligned}
\varepsilon_n &= d(y_{n+1}, Ty_n) \\
&\le r(d(y_{n+1}, p) + d(p, Ty_n)) \\
&\le r(d(y_{n+1}, p) + d(Tp, Ty_n)) \\
&\le rd(y_{n+1}, p) + r[d(Tp, Ty_n)] \\
&\le rd(y_{n+1}, p) + r[a_1(x, y)d(y_n, p) + a_2(x, y)d(y_n, Ty_n) + a_3(x, y)d(p, Tp) + \\
&\qquad a_4(x, y)(d(y_n, Tp) + d(p, Ty_n))] \\
&\le rd(y_{n+1}, p) + r[a_1(x, y) + 2a_4(x, y)]d(y_n, p) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

That is, $\{x_n\}$ converges to *p*. This completes the proof.

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