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The Growth of Iterated Entire Functions in Terms of Order

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Abstract

In this paper we consider $f(z)$ and $g(z)$ are two transcendental entire functions defined in the open complex plane C and taking iteration of these functions we study the growth of iterated entire functions in terms of order which improve some earlier results. We also give example supporting theorem.

AMS subject classification: 30D35.
Keywords: Entire function, meromorphic function, growth, iteration, order, lower order.

1. Introduction, Definitions and Notation

Let $f(z)$ and $g(z)$ be two transcendental entire functions defined in the open complex plane C. It is well known [2], {[12], p-67, Th-1.46} that

$$\lim_{r\to\infty} \frac{T(r, f_o g)}{T(r, f)} = \infty \quad \text{and} \quad \lim_{r\to\infty} \frac{T(r, f_o g)}{T(r, g)} = \infty.$$

In 1985 Singh [8] proved some comparative growth properties of $\log T(r, f_o g)$ and $T(r, f)$ also he raised the problem of investigating the comparative growth of $\log T(r, f_o g)$ and $T(r, g)$.

Song and Yang [10] established that f and g are any two transcendental entire functions of positive lower order and finite order then

$$\lim_{r\to\infty} \frac{\log\log M(r, f_o g)}{\log\log M(r, f)} = \infty = \lim_{r\to\infty} \frac{\log\log M(r, f_o g)}{\log\log M(r, g)}. \tag{1.1}$$

In 1991 Singh and Baloria [9] asked whether for sufficiently large $R = R(r)$

$$\limsup_{r\to\infty} \frac{\log\log M(r, f_o g)}{\log\log M(R, f)} < \infty \text{ and } \limsup_{r\to\infty} \frac{\log\log M(r, f_o g)}{\log\log M(R, g)} < \infty.$$

Singh and Baloria [9] proved under the assumptions of (1.1) that for each positive constant A

$$\lim_{r\to\infty} \frac{\log\log M(r, f_o g)}{\log\log M(r^A, f)} = \infty = \lim_{r\to\infty} \frac{\log\log M(r, f_o g)}{\log\log M(r^A, g)}.$$

They also proved that if $R = \exp(r^{\rho_f})$, where $\rho_f (< \infty)$ is the order of f,

$$\liminf_{r\to\infty} \frac{\log\log M(r, f_o g)}{\log\log M(R, f)} = 0 \text{ for } \rho_f > \rho_g.$$

Lahiri and Sharma [5] worked on the question of Singh and Baloria, and proved the following theorems.

Theorem A. Let f, g be two entire functions such that $0 < \lambda_f \le \rho_f < \infty$ and $0 < \lambda_g \le \rho_g < \infty$. Then for any positive number A and every real number α

$$\text{(i)} \quad \lim_{r\to\infty} \frac{\log\log M(r, f_o g)}{\{\log\log M(r^A, f)\}^{1+\alpha}} = \infty,$$

$$(ii) \quad \lim_{r\to\infty} \frac{\log\log M(r, f_o g)}{\{\log\log M(r^A, g)\}^{1+\alpha}} = \infty.$$

Theorem B. Let f, g be two entire functions of finite orders and $\lambda_f > 0$. Then for $p > 0$ and each $\alpha \in (-\infty, \infty)$

$$\lim_{r\to\infty} \frac{\{\log\log M(r, f_o g)\}^{1+\alpha}}{\log\log M(\exp(r^p), f)} = 0 \text{ if } p > (1+\alpha)\rho_g.$$

Theorem C. Let f be meromorphic and g be entire such that $0 < \lambda_f \le \rho_f < \infty$ and $\rho_g < \infty$. Then for $p > 0$ and each $\alpha \in (-\infty, \infty)$

$$\lim_{r\to\infty} \frac{\{\log T(r, f_o g) + \log\log M(r, g)\}^{1+\alpha}}{\log T(\exp(r^p), f)} = 0 \text{ if } p > (1+\alpha)\rho_g.$$

Theorem D. Let f be meromorphic and g be entire such that $0 < \rho_g < \infty$ and $0 < \lambda_f$. Then

$$\limsup_{r\to\infty} \frac{\log T(r, f_o g)}{\log T(\exp(r^{\mu}), g)} = \infty \text{ where } 0 < \mu < \rho_g.$$

In 1997 Lahiri and Banerjee [4] form the iterations of $f(z)$ with respect to $g(z)$ as follows:

$$\begin{aligned}
f_1(z) &= f(z) \\
f_2(z) &= f(g(z)) = f(g_1(z)) \\
f_3(z) &= f(g(f(z))) = f(g_2(z)) = f(g(f_1(z))) \\
\ldots & \quad \ldots \quad \ldots \\
\ldots & \quad \ldots \quad \ldots \\
f_n(z) &= f(g(f........(f(z) \text{ or } g(z))........)), \\
&\quad \text{according as } n \text{ is odd or even,} \\
&= f(g_{n-1}(z)) = f(g(f_{n-2}(z)))
\end{aligned}$$

and so

$$\begin{aligned}
g_1(z) &= g(z) \\
g_2(z) &= g(f(z)) = g(f_1(z)) \\
\ldots & \quad \ldots \\
\ldots & \quad \ldots \\
g_n(z) &= g(f_{n-1}(z)) = g(f(g_{n-2}(z))).
\end{aligned}$$

Clearly all $f_n(z)$ and $g_n(z)$ are entire functions.

The following definitions are well known.

Definition 1.1. The order ρ_f and lower order λ_f of a meromorphic function f is defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}$$

and

$$\lambda_f = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

If f is an entire function then

$$\rho_f = \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r}$$

and

$$\lambda_f = \liminf_{r\to\infty} \frac{\log\log M(r, f)}{\log r}.$$

Notation 1.2. **[7]** $log^{[0]}x = x$, $exp^{[0]}x = x$ and for positive integer m, $log^{[m]}x = log(log^{[m-1]}x)$, $exp^{[m]}x = exp(exp^{[m-1]}x)$.

In this paper, we investigate the comparative growth of iterated entire functions which generalize some earlier results. we do not explain the standard notations and definitions of the theory of entire and meromorphic functions as those are available in [3], [11] and [12].

2. Lemmas

The following lemmas will be needed in the sequel.

Lemma 2.1. **[3]** Let $f(z)$ be an entire function. For $0 \leq r < R < \infty$, we have

$$T(r, f) \leq \log^{+} M(r, f) \leq \frac{R+r}{R-r} T(R, f).$$

Lemma 2.2. **[2]** If f and g are any two entire functions, for all sufficiently large values of r,

$$M\left(\frac{1}{8} M\left(\frac{r}{2}, g\right) - |g(0)|, f\right) \leq M(r, f_o g) \leq M(M(r, g), f)$$

Lemma 2.3. **[1]** If f is meromorphic and g is entire, for all sufficiently large values of r

$$T(r, f_o g) \leq \{1 + o(1)\} \frac{T(r, g)}{\log M(r, g)} T(M(r, g), f)$$

Lemma 2.4. **[6]** Let $f(z)$ and $g(z)$ be two entire functions. Then we have

$$T(r, f(g)) \geq \frac{1}{3} \log M\left(\frac{1}{8} M\left(\frac{r}{4}, g\right) + O(1), f\right)$$

Lemma 2.5. Let $f(z)$ and $g(z)$ be two non constant entire functions such that $0 < \lambda_f \leq \rho_f < \infty$ and $0 < \lambda_g \leq \rho_g < \infty$. Then for any ε $(0 < \varepsilon < \min\{\lambda_f, \lambda_g\})$

$$\log^{[n]} M(r, f_n) \leq \begin{cases} (\rho_f + \varepsilon) \log M(r, g) + O(1) & \text{when } n \text{ is even} \\ (\rho_g + \varepsilon) \log M(r, f) + O(1) & \text{when } n \text{ is odd} \end{cases}$$

and

$$\log^{[n]} M(r, f_n) \geq \begin{cases} (\lambda_f - \varepsilon) \log M\left(\frac{r}{2^{n-1}}, g\right) + O(1) & \text{when } n \text{ is even} \\ (\lambda_g - \varepsilon) \log M\left(\frac{r}{2^{n-1}}, f\right) + O(1) & \text{when } n \text{ is odd,} \end{cases}$$

for all large values of r.

Proof. For $\varepsilon(>0)$ we get from Lemma 2.2 for all large values of r,

$$\begin{array}{rrcl} & M(r, f_n) & = & M(r, f(g_{n-1})) \\ & & \leq & M(M(r, g_{n-1}), f). \\ \therefore & \log\ M(r, f_n) & \leq & \log M(M(r, g_{n-1}), f) \\ & & \leq & [M(r, g_{n-1})]^{\rho_f+\varepsilon}. \\ \therefore & \log^{[2]}\ M(r, f_n) & \leq & (\rho_f+\varepsilon)\log M(r, g_{n-1}) \\ & & \leq & (\rho_f+\varepsilon)\log M(M(r, f_{n-2}), g) \\ & & \leq & (\rho_f+\varepsilon)[M(r, f_{n-2})]^{\rho_g+\varepsilon}. \\ \therefore & \log^{[3]}\ M(r, f_n) & \leq & \log(\rho_f+\varepsilon)+(\rho_g+\varepsilon)\log M(r, f(g_{n-3})) \\ & & \leq & O(1)+(\rho_g+\varepsilon)\log M(M(r, g_{n-3}), f) \\ & & \leq & (\rho_g+\varepsilon)[M(r, g_{n-3})]^{\rho_f+\varepsilon}+O(1). \\ \therefore & \log^{[4]}\ M(r, f_n) & \leq & (\rho_f+\varepsilon)\log[M(r, g_{n-3}))]+O(1) \\ & \ldots. & & \ldots. \qquad \ldots. \\ & \ldots. & & \ldots. \qquad \ldots. \\ \therefore & \log^{[n]}\ M(r, f_n) & \leq & (\rho_f+\varepsilon)\log M(r, g)+O(1) \quad \text{when } n \text{ is even.} \end{array}$$

Similarly

$$\log^{[n]}\ M(r, f_n) \leq (\rho_g+\varepsilon)\log M(r, f)+O(1) \quad \text{when } n \text{ is odd.}$$

Again, let λ_f and λ_g be the lower order of f and g respectively, then from Lemma 2.2 we have for all large values of r and any ε $(0<\varepsilon<\min\{\lambda_f, \lambda_g\})$,

$$\begin{array}{rrcl} & M(r, f_n) & = & M(r, f(g_{n-1})) \\ & & \geq & M\left(\frac{1}{8}M\left(\frac{r}{2}, g_{n-1}\right)-|g_{n-1}(0)|, f\right) \\ & & \geq & M\left(\frac{1}{16}M\left(\frac{r}{2}, g_{n-1}\right), f\right). \\ \therefore & \log M(r, f_n) & \geq & \log M\left(\frac{1}{16}M\left(\frac{r}{2}, g_{n-1}\right), f\right) \\ & & \geq & \left[\frac{1}{16}M\left(\frac{r}{2}, g_{n-1}\right)\right]^{\lambda_f-\varepsilon}. \\ \therefore & \log^{[2]} M(r, f_n) & \geq & (\lambda_f-\varepsilon)\log M\left(\frac{r}{2}, g_{n-1}\right)+(\lambda_f-\varepsilon)\log\frac{1}{16} \end{array}$$

$$
\begin{aligned}
&= (\lambda_f - \varepsilon)\log M\left(\frac{r}{2}, g(f_{n-2})\right) + O(1)\\
&\geq (\lambda_f - \varepsilon)\log M\left(\frac{1}{8}M\left(\frac{r}{2^2}, f_{n-2}\right) - |f_{n-2}(0)|, g\right) + O(1)\\
&\geq (\lambda_f - \varepsilon)\left[\frac{1}{16}M\left(\frac{r}{2^2}, f_{n-2}\right)\right]^{\lambda_g - \varepsilon} + O(1).\\
\therefore\ \log^{[3]} M(r, f_n) &\geq \log(\lambda_f - \varepsilon) + (\lambda_g - \varepsilon)\log\frac{1}{16}M\left(\frac{r}{2^2}, f_{n-2}\right) + O(1)\\
&\geq (\lambda_g - \varepsilon)\log M\left(\frac{r}{2^2}, f_{n-2}\right) + O(1)\\
\cdots\ &\quad \cdots \quad \cdots \quad \cdots\\
\cdots\ &\quad \cdots \quad \cdots \quad \cdots\\
\therefore\ \log^{[n]} M(r, f_n) &\geq (\lambda_f - \varepsilon)\log M\left(\frac{r}{2^{n-1}}, g\right) + O(1) \quad \text{when } n \text{ is even.}
\end{aligned}
$$

Similarly

$$\log^{[n]} M(r, f_n) \geq (\lambda_g - \varepsilon)\log M\left(\frac{r}{2^{n-1}}, f\right) + O(1) \quad \text{when } n \text{ is odd.}$$

This proves the lemma. ■

Lemma 2.6. Let $f(z)$ and $g(z)$ be two non constant entire functions such that $0 < \rho_f < \infty$ and $0 < \rho_g < \infty$. Then for all sufficiently large r and $\varepsilon > 0$,

$$\log^{[n-1]} T(r, f_n) \leq \begin{cases} (\rho_f + \varepsilon)\log M(r, g) + O(1) & \text{when n is even}\\ (\rho_g + \varepsilon)\log M(r, f) + O(1) & \text{when n is odd.}\end{cases}$$

The lemma follows from Lemma 2.1 and Lemma 2.5.

Lemma 2.7. Let $f(z)$ and $g(z)$ be two non constant entire functions such that $0 < \lambda_f < \infty$ and $0 < \lambda_g < \infty$. Then for any ε $(0 < \varepsilon < \min\{\lambda_f, \lambda_g\})$,

$$\log^{[n-1]} T(r, f_n) \geq \begin{cases} (\lambda_f - \varepsilon)\log M\left(\frac{r}{4^{n-1}}, g\right) + O(1) & \text{when n is even}\\ (\lambda_g - \varepsilon)\log M\left(\frac{r}{4^{n-1}}, f\right) + O(1) & \text{when n is odd}\end{cases}$$

for all sufficiently large values of r.

Proof. Since for any ε $(0 < \varepsilon < \min\{\lambda_f, \lambda_g\})$ we get from Lemma 2.1 and Lemma 2.4

for all large values of r

$$
\begin{aligned}
T(r, f_n) &= T(r, f(g_{n-1})) \\
&\geq \frac{1}{3}\log M\left(\frac{1}{8}M\left(\frac{r}{4}, g_{n-1}\right) + O(1), f\right) \\
&\geq \frac{1}{3}\left[\frac{1}{8}M\left(\frac{r}{4}, g_{n-1}\right) + O(1)\right]^{\lambda_f - \varepsilon} \\
&\geq \frac{1}{3}\left[\frac{1}{9}M\left(\frac{r}{4}, g_{n-1}\right)\right]^{\lambda_f - \varepsilon},
\end{aligned}
$$

$$
\begin{aligned}
\text{that is, } \log T(r, f_n) &\geq (\lambda_f - \varepsilon)\log M\left(\frac{r}{4}, g_{n-1}\right) + O(1) \\
&\geq (\lambda_f - \varepsilon) T\left(\frac{r}{4}, g_{n-1}\right) + O(1) \\
&\geq (\lambda_f - \varepsilon)\frac{1}{3}\log M\left(\frac{1}{8}M\left(\frac{r}{4^2}, f_{n-2}\right) + O(1), g\right) + O(1) \\
&\geq (\lambda_f - \varepsilon)\frac{1}{3}\left[\frac{1}{8}M\left(\frac{r}{4^2}, f_{n-2}\right) + O(1)\right]^{\lambda_g - \varepsilon} + O(1) \\
&\geq (\lambda_f - \varepsilon)\frac{1}{3}\left[\frac{1}{9}M\left(\frac{r}{4^2}, f_{n-2}\right)\right]^{\lambda_g - \varepsilon} + O(1).
\end{aligned}
$$

$$
\begin{aligned}
\text{Therefore, } \log^{[2]} T(r, f_n) &\geq (\lambda_g - \varepsilon)\log M\left(\frac{r}{4^2}, f_{n-2}\right) + O(1) \\
\cdots\cdots &\quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \\
\cdots\cdots &\quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \\
\text{So, } \log^{[n-1]} T(r, f_n) &\geq (\lambda_f - \varepsilon)\log M\left(\frac{r}{4^{n-1}}, g\right) + O(1) \quad \text{when } n \text{ is even.}
\end{aligned}
$$

Similarly

$$\log^{[n-1]} T(r, f_n) \geq (\lambda_g - \varepsilon)\log M\left(\frac{r}{4^{n-1}}, f\right) + O(1) \quad \text{when } n \text{ is odd.}$$

This proves the lemma. ■

3. Theorems

Theorem 3.1. Let f and g be two non constant entire functions such that $0 < \lambda_f \leq \rho_f < \infty$ and $0 < \lambda_g \leq \rho_g < \infty$. Then for any positive number A and every real number α

$$\text{(i)} \lim_{r\to\infty} \frac{\log^{[n]} M(r, f_n)}{\{\log\log M(r^A, f)\}^{1+\alpha}} = \infty,$$

and

$$\text{(ii)} \lim_{r\to\infty} \frac{\log^{[n]} M(r, f_n)}{\{\log\log M(r^A, g)\}^{1+\alpha}} = \infty.$$

Proof. If $\alpha \leq -1$ then the theorem is trivial. So we suppose that $\alpha > -1$ and n is even. Then from Lemma 2.5 we get for all sufficiently large values of r and any ε $(0 < \varepsilon < \min\{\lambda_f, \lambda_g\})$

$$\begin{aligned}\log^{[n]} M(r, f_n) &\geq (\lambda_f - \varepsilon) \log M\left(\frac{r}{2^{n-1}}, g\right) + O(1) \\ &\geq (\lambda_f - \varepsilon)\left(\frac{r}{2^{n-1}}\right)^{\lambda_g - \varepsilon} + O(1). \end{aligned} \tag{3.2}$$

Again from Definition 1.1 it follows that for any $\varepsilon > 0$ and for all large values of r,

$$\{\log\log M(r^A, f)\}^{1+\alpha} < (\rho_f + \varepsilon)^{1+\alpha} A^{1+\alpha} (\log r)^{1+\alpha}. \tag{3.3}$$

From (3.2) and (3.3) we have for all large values of r and any ε $(0 < \varepsilon < \min\{\lambda_f, \lambda_g\})$

$$\begin{aligned}\frac{\log^{[n]} M(r, f_n)}{\{\log\log M(r^A, f)\}^{1+\alpha}} &\geq \frac{(\lambda_f - \varepsilon)\left(\frac{r}{2^{n-1}}\right)^{\lambda_g - \varepsilon} + O(1)}{(\rho_f + \varepsilon)^{1+\alpha} A^{1+\alpha} (\log r)^{1+\alpha}} \\ &\geq \frac{(\lambda_f - \varepsilon)\left(\frac{1}{2^{n-1}}\right)^{\lambda_g - \varepsilon}}{(\rho_f + \varepsilon)^{1+\alpha} A^{1+\alpha}} \frac{r^{\lambda_g - \varepsilon}}{(\log r)^{1+\alpha}} + o(1).\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary,

$$\therefore \quad \lim_{r\to\infty} \frac{\log^{[n]} M(r, f_n)}{\{\log\log M(r^A, f)\}^{1+\alpha}} = \infty. \tag{3.4}$$

Similarly for odd n we get

$$\log^{[n]} M(r, f_n) \geq (\lambda_g - \varepsilon)\left(\frac{r}{2^{n-1}}\right)^{\lambda_f - \varepsilon} + O(1). \tag{3.5}$$

So from (3.3) and (3.5) we have equation (3.4) for odd n.
Therefore for all n (even or odd) statement (i) follows.

Second part of this theorem follows similarly by using the following inequality instead of (3.3)

$$\{\log\log M(r^A, g)\}^{1+\alpha} < (\rho_g + \varepsilon)^{1+\alpha} A^{1+\alpha} (\log r)^{1+\alpha}$$

for all large values of r and arbitrary $\varepsilon > 0$.
This proves the theorem. ■

Note. If we consider $n = 2$ in Theorem 3.1 then we get Theorem A.

Theorem 3.2. Let f and g be two entire functions of finite orders and $\lambda_f, \lambda_g > 0$. Then for $p > 0$ and each $\alpha \in (-\infty, \infty)$

$$\text{(i)} \quad \lim_{r\to\infty} \frac{\{\log^{[n]} M(r, f_n)\}^{1+\alpha}}{\log\log M(\exp(r^p), f)} = 0 \text{ if } p > (1+\alpha)\rho_g \text{ and } n \text{ is even,}$$

$$\text{(ii)} \quad \lim_{r\to\infty} \frac{\{\log^{[n]} M(r, f_n)\}^{1+\alpha}}{\log\log M(\exp(r^p), f)} = 0 \text{ if } p > (1+\alpha)\rho_f \text{ and } n \text{ is odd.}$$

Proof. If $\alpha \leq -1$ then the theorem is trivial. So we suppose that $\alpha > -1$ and n is even. Then from Lemma 2.5 we get for all sufficiently large values of r and any $\varepsilon > 0$

$$\begin{aligned}\log^{[n]} M(r, f_n) &\leq (\rho_f + \varepsilon)\log M(r, g) + O(1)\\ &\leq (\rho_f + \varepsilon) r^{\rho_g + \varepsilon} + O(1). \qquad (3.6)\end{aligned}$$

Again from Definition 1.1 it follows that for any $0 < \varepsilon < \lambda_f$ and for all large values of r,

$$\log\log M(\exp(r^p), f) > (\lambda_f - \varepsilon) r^p. \qquad (3.7)$$

So from (3.6) and (3.7) we have for all large vales of r and any ε $(0 < \varepsilon < \lambda_f)$

$$\frac{\{\log^{[n]} M(r, f_n)\}^{1+\alpha}}{\log\log M(\exp(r^p), f)} \leq \frac{(\rho_f + \varepsilon)^{1+\alpha} r^{(1+\alpha)(\rho_g+\varepsilon)}}{(\lambda_f - \varepsilon) r^p} + o(1).$$

Since $\varepsilon > 0$ is arbitrary, we can choose ε such that $0 < \varepsilon < \min\{\lambda_f, \frac{p}{1+\alpha} - \rho_g\}$,

$$\therefore \quad \lim_{r\to\infty} \frac{\{\log^{[n]} M(r, f_n)\}^{1+\alpha}}{\log\log M(\exp(r^p), f)} = 0.$$

Similarly when n is odd then we get the second part of this theorem. This proves the theorem. ■

Theorem 3.3. Let f and g be two entire functions of finite orders and $\lambda_f, \lambda_g > 0$. Then for $p > 0$ and each $\alpha \in (-\infty, \infty)$

$$\text{(i)} \quad \lim_{r\to\infty} \frac{\{\log^{[n]} M(r, f_n)\}^{1+\alpha}}{\log\log M(\exp(r^p), g)} = 0 \text{ if } p > (1+\alpha)\rho_g \text{ and } n \text{ is even,}$$

$$\text{(ii)} \quad \lim_{r\to\infty} \frac{\{\log^{[n]} M(r, f_n)\}^{1+\alpha}}{\log\log M(\exp(r^p), g)} = 0 \text{ if } p > (1+\alpha)\rho_f \text{ and } n \text{ is odd.}$$

The proof of this theorem is similar to the proof of the Theorem 3.2.

Theorem 3.4. Let f and g be two entire functions such that $0 < \lambda_f \leq \rho_f < \infty$ and $0 < \lambda_g \leq \rho_g < \infty$. Then for $p > 0$ and each $\alpha \in (-\infty, \infty)$

$$\text{(i)} \quad \lim_{r\to\infty} \frac{\{\log^{[n-1]} T(r, f_n) + \log\log M(r, g)\}^{1+\alpha}}{\log T(\exp(r^p), f)} = 0 \text{ if } p > (1+\alpha)\rho_g \text{ and } n \text{ is even,}$$

$$\text{(ii)} \quad \lim_{r\to\infty} \frac{\{\log^{[n-1]} T(r, f_n) + \log\log M(r, f)\}^{1+\alpha}}{\log T(\exp(r^p), f)} = 0 \text{ if } p > (1+\alpha)\rho_f \text{ and } n \text{ is odd.}$$

Proof. If $\alpha \leq -1$ then the theorem is trivial. So we suppose that $\alpha > -1$ and n is even. Now from Lemma 2.6 we get for all sufficiently large values of r and any $\varepsilon > 0$

$$\begin{aligned}\log^{[n-2]} T(r, f_n) &\leq (\rho_g + \varepsilon)\log M(r, f(g)) + O(1)\\ &\leq (\rho_g + \varepsilon) 3T(2r, f(g)) + O(1)\\ &\leq [3(\rho_g + \varepsilon) + o(1)] T(2r, f(g)). \qquad (3.8)\end{aligned}$$

From Lemma 2.3 we have for all large values of r

$$\begin{aligned} T(2r, f_o g) &\leq \{1+o(1)\}\frac{T(2r,g)}{\log M(2r,g)} T(M(2r,g), f) \\ &\leq \{1+o(1)\}\frac{T(2r,g)}{\log M(r,g)} T(M(2r,g), f), \end{aligned}$$

$$\begin{aligned} \text{i.e. } logT(2r, f_o g) + \log\log M(r,g) &\leq \log\{1+o(1)\} + \log T(2r,g) \\ &\quad + \log T(M(2r,g), f) \\ &\leq (\rho_g+\varepsilon)\log(2r) + (\rho_f+\varepsilon)\log M(2r,g) + o(1) \\ &\leq (\rho_g+\varepsilon)\log(2r) + (\rho_f+\varepsilon)(2r)^{\rho_g+\varepsilon} + o(1). \end{aligned} \tag{3.9}$$

From (3.8) and (3.9) we have for all large values of r and any $\varepsilon > 0$,

$$\begin{aligned} \log^{[n-1]} T(r, f_n) + \log\log M(r,g) &\leq (\rho_g+\varepsilon)\log(2r) + (\rho_f+\varepsilon)(2r)^{\rho_g+\varepsilon} + O(1) \\ &\leq (\rho_f+\varepsilon+o(1))r^{\rho_g+\varepsilon}. \end{aligned} \tag{3.10}$$

Again from Definition 1.1 it follows that for any $0 < \varepsilon < \lambda_f$ and for all large values of r,

$$\log\log M(\exp(r^p), f) > (\lambda_f - \varepsilon)r^p. \tag{3.11}$$

From (3.10) and (3.11) we have for all large values of r and any $\varepsilon(0 < \varepsilon < \lambda_f)$,

$$\frac{\{\log^{[n-1]} T(r, f_n) + \log\log M(r,g)\}^{1+\alpha}}{\log T(\exp(r^p), f)} \leq \frac{(\rho_f+\varepsilon+o(1))^{1+\alpha} r^{(1+\alpha)(\rho_g+\varepsilon)}}{(\lambda_f-\varepsilon)r^p}.$$

Since $\varepsilon > 0$ is arbitrary, we can choose ε such that $0 < \varepsilon <\min\{\lambda_f, \frac{p}{1+\alpha} - \rho_g\}$,

$$\therefore \quad \lim_{r\to\infty} \frac{\{\log^{[n-1]} T(r, f_n) + \log\log M(r,g)\}^{1+\alpha}}{\log T(\exp(r^p), f)} = 0.$$

Similarly for odd n we get the second part of this theorem. This proves the theorem. ■

Note. If we consider $n = 2$ in Theorem 3.2 and Theorem 3.4 then we get Theorem B and Theorem C respectively.

Theorem 3.5. Let f and g be two entire functions such that $0 < \lambda_f \leq \rho_f < \infty$ and $0 < \lambda_g \leq \rho_g < \infty$. Then for $p > 0$ and each $\alpha \in (-\infty, \infty)$

(i) $\lim_{r\to\infty} \frac{\{\log^{[n-1]} T(r, f_n) + \log\log M(r,g)\}^{1+\alpha}}{\log T(\exp(r^p), g)} = 0$ if $p > (1+\alpha)\rho_g$ and n is even,

(ii) $\lim_{r\to\infty} \frac{\{\log^{[n-1]} T(r, f_n) + \log\log M(r,f)\}^{1+\alpha}}{\log T(\exp(r^p), g)} = 0$ if $p > (1+\alpha)\rho_f$ and n is odd.

The proof of the theorem is similar to the proof of Theorem 3.4.

Theorem 3.6. Let f and g be two entire functions such that $0 < \lambda_f \leq \rho_f < \infty$ and $0 < \lambda_g \leq \rho_g < \infty$. Then

$$\text{(i)}\ \limsup_{r\to\infty} \frac{\log^{[n-1]} T(r, f_n)}{\log T(\exp(r^p), g)} = \infty,\ \text{when } 0 < p < \lambda_g \text{ and } n \text{ is enen,}$$

$$\text{(ii)}\ \limsup_{r\to\infty} \frac{\log^{[n-1]} T(r, f_n)}{\log T(\exp(r^p), g)} = \infty,\ \text{when } 0 < p < \lambda_f \text{ and } n \text{ is odd.}$$

Proof. First we consider n is even then from Lemma 2.7 we get for all sufficiently large values of r and any $\varepsilon(0 < \varepsilon < \min\{\lambda_f, \lambda_g\})$

$$\begin{aligned}\log^{[n-1]}\ T(r, f_n) &\geq (\lambda_f - \varepsilon)\log M\left(\frac{r}{4^{n-1}}, g\right) + O(1)\\ &\geq (\lambda_f - \varepsilon)\left(\frac{r}{4^{n-1}}\right)^{\lambda_g-\varepsilon} + O(1).\end{aligned} \tag{3.12}$$

Again from Definition 1.1 it follows that for all large values of r and $\varepsilon > 0$,

$$\log T(\exp(r^p), g) < (\rho_g + \varepsilon)r^p. \tag{3.13}$$

From (3.12) and (3.13) we have for all large vales of r and any $\varepsilon(0 < \varepsilon < \lambda_f)$,

$$\frac{\log^{[n-1]} T(r, f_n)}{\log T(\exp(r^p), g)} \geq \frac{(\lambda_f - \varepsilon)\left(\frac{r}{4^{n-1}}\right)^{\lambda_g-\varepsilon} + O(1)}{(\rho_g + \varepsilon)r^p}.$$

Since $\varepsilon > 0$ is arbitrary,

$$\therefore\ \limsup_{r\to\infty} \frac{\log^{[n-1]} T(r, f_n)}{\log T(\exp(r^p), g)} = \infty.$$

Similarly when n is odd then we get statement (ii).
This proves the theorem. ■

Remark 3.7. The condition non zero lower order is essential for Theorem 3.6, which are illustrated by the following example.

Example 3.8. Let $f(z) = z$, $g(z) = \exp z$. Then $\lambda_f = \rho_f = 0$ and $0 < \lambda_g = \rho_g = 1 < \infty$. Now when n is even then

$$f_n = \exp^{[\frac{n}{2}]} z.$$

Therefore,

$$T(r, f_n) \leq \log M(r, f_n) = \exp^{[\frac{n}{2}-1]} r.$$

So,

$$\begin{aligned}\log^{[n-1]} T(r, f_n) &\le \log^{[n-1]}(\exp^{[\frac{n}{2}-1]} r) \\ &= \log^{[n-1-\frac{n}{2}+1]} r \\ &= \log^{[\frac{n}{2}]} r.\end{aligned}$$

Also when n is odd

$$f_n = \exp^{[\frac{n-1}{2}]} z.$$

Therefore,

$$T(r, f_n) \le \log M(r, f_n) = \exp^{[\frac{n-1}{2}-1]} r.$$

So,

$$\begin{aligned}\log^{[n-1]} T(r, f_n) &\le \log^{[n-1]}(\exp^{[\frac{n-1}{2}-1]} r) \\ &= \log^{[n-1-\frac{n-1}{2}+1]} r \\ &= \log^{[\frac{n+1}{2}]} r\end{aligned}$$

Now

$$\log T(\exp(r^p), g) = r^p - \log \pi.$$

Therefore when n is even

$$\frac{\log^{[n-1]} T(r, f_n)}{\log T(\exp(r^p), g)} \le \frac{\log^{[\frac{n}{2}]} r}{r^p - \log \pi} \to 0 \text{ as } r \to \infty,$$

and when n is odd

$$\frac{\log^{[n-1]} T(r, f_n)}{\log T(\exp(r^p), g)} \le \frac{\log^{[\frac{n+1}{2}]} r}{r^p - \log \pi} \to 0 \text{ as } r \to \infty.$$

Theorem 3.9. If f, g and h are three entire functions with non zero lower order and finite order also $\rho_h < \lambda_g$ then

$$\lim_{r\to\infty} \frac{\log^{[n-1]} T(r, f_n)}{\log^{[n-1]} T(r, f_n^{'})} = \infty$$

for n is even and $f_n^{'}(z) = f(h(f........(f(\ h(z))........)).$

Proof. When n is even then from Lemma 2.6 we have for $r > r_0$,

$$\log^{[n-1]} T(r, f_n^{'}) \le (\rho_f + \varepsilon) r^{\rho_h+\varepsilon}. \tag{3.14}$$

Hence from (3.12) and (3.14) we have for sufficiently large values of r and $0 < \varepsilon < \min\{\lambda_f, \lambda_h\}$,

$$\begin{aligned}\frac{\log^{[n-1]} T(r, f_n)}{\log^{[n-1]} T(r, f_n')} &\geq \frac{(\lambda_f - \varepsilon)\left(\frac{r}{4^{n-1}}\right)^{\lambda_g - \varepsilon} + O(1)}{(\rho_f + \varepsilon) r^{\rho_h + \varepsilon} + O(1)} \\ &= \frac{(\lambda_f - \varepsilon) r^{\lambda_g - \varepsilon} + O(1)}{\left(4^{n-1}\right)^{\lambda_g - \varepsilon}(\rho_f + \varepsilon) r^{\rho_h + \varepsilon} + O(1)}.\end{aligned}$$

Since $\rho_h < \lambda_g$, we can choose $\varepsilon > 0$ such that $\rho_h + \varepsilon < \lambda_g - \varepsilon$,

$$\therefore \quad \lim_{r \to \infty} \frac{\log^{[n-1]} T(r, f_n)}{\log^{[n-1]} T(r, f_n')} = \infty.$$

This proves the theorem. ■

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