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An approximation algorithm to find a Dom-chromatic set and the Dom-chromatic number of a graph

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Abstract

A set $S \subseteq V$ is said to be a dominating set of G if for each $u \in V - S$, there exists a vertex $v \in S$ such that u is adjacent to v. The minimum cardinality of a dominating set in G is called the domination number of G and is denoted by $\gamma(G)$. A proper coloring of G is an assignment of colors to the vertices of G such that no two adjacent vertices receive the same color. G is said to be k-colorable if it is has proper coloring using k colors. The minimum k such that G is k-colorable is called the chromatic number of G, denoted by $\chi(G)$. A subset S of V is called a dom-chromatic set if S is a dominating set and $\chi(< S >) = \chi(G)$. In this paper, it is shown that the problem of finding the dom-chromatic set of a graph is NP-Complete and also an approximation algorithm is proposed to find the Dom-chromatic set and the Dom-chromatic number of a graph.

AMS subject classification:
Keywords: Graph Domination, Graph Coloring, Dom-chromatic set, Dom-chromatic number.

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1. Introduction

A graph G consists of a finite nonempty set V of vertices together with a set E of pairs of distinct elements of $V(G)$, called edges. A set $S \subset V(G)$ is said to be a dominating set of G if for each $u \in V - S$, there exists a vertex $v \in S$ such that u is adjacent to v. The minimum cardinality of a dominating set in G is called the domination number of G, denoted by $\gamma(G)$. A proper coloring of G is an assignment of colors to the vertices of G such that no two adjacent vertices receive the same color. G is said to be k-colorable if it is has proper coloring using k colors. The minimum k such that G is k-colorable is called the chromatic number of G, denoted by $\chi(G)$. A subset S of V is called a dom-chromatic set if S is a dominating set and $\chi(< S >) = \chi(G)$.

For more than a century, though researchers have obtained many good results in graph coloring, there are still many interesting problems left unsolved. The book by Jensen and Toft [10] lists the unsolved graph coloring problems. One of the most interesting problems is on critical graphs. The importance of the notion of criticality is that problems for k-chromatic graphs in general may often be reduced to problems of k-critical graphs. These are more restricted than k-chromatic graphs. Critical graphs were defined and used by Dirac in 1952 [3]. He called a graph G critical if $\chi(G - v) < \chi(G)$, for all vertices v. In general any element t of the set $V \cup E$ is critical if $\chi(G - t) < \chi(G)$. By this terminology, Dirac defines a critical graph as one where all vertices are critical. This aspect led to the definition of color-critical graphs, in which each vertex and each edge are critical. It is to be noted that no k-critical graph can be infinite and the only k-criticalgraphs for $k = 1, 2$ and 3 are K_1, K_2 and odd cycles, respectively. For $k \geq 4$, the k-critical graphs have not been characterized. Ordinarily, it is extremely difficult to determine whether a given graph is critical; however every k-chromatic graph for $k \geq 2$ contains a k-critical subgraph. In fact, if H is any smallest (in terms of number of vertices) induced subgraph of G such that $\chi(G) = \chi(H)$, then H is critical. Also, there is not much work on the procedure to find the smallest critical subgraph of a non-critical graph. In 2000, Herrmann and Hertz [8] made an attempt to propose exact algorithms for finding the chromatic number of a graph G.

A number of domination parameters have been defined in the literature by combining the domination property and another graph property. [6] defined the conditional domination number $\gamma(G : P)$ as the smallest cardinality of a dominating set $S \subseteq V$ such that the subgraph $< S >$, inducted by S satisfies the property P. Total dominating set, connected dominating set, clique dominating set, path dominating set etc. are a few of the many conditional dominating sets [7]. But, at the same time many conditional domination parameters do not exist for all graphs.

The nature of conditional parameters provides a wide opening to researchers by considering different properties for P. In the same way, the conditional parameter namely, dom-chromatic number was introduced and studied by Poobalaranjani and Janakiraman [12, 9]. It can be observed that this conditional domination parameter exists for all graphs. Since the theory of graph coloring plays an equally important role in graph theory, the combination of the two widely used graph theoretic concepts leads to many applications. To find a dominating set having ghe same chromatic number as G, it is necessary to find a set of smallest cardinality having the same chromatic number as that of the graph. As both the problems of finding the chromatic number of an arbitrary graph and finding the domination number of a graph are NP-Complete [7], it is a natural intuition that the problem of finding a dom-chromatic set of an arbitrary graph is NP-Complete. In this paper, we show that the above problem is NP-Complete and an attempt is made to give an approximation algorithm to find a dc-set and hence the dc-number of a given graph.

For all basic definitions and terminolgies not defined in this paper, reader may refer to [5, 14, 7]. All graphs considered in this paper are finite and undirected, unless otherwise specified.

2. Defintions and Terminologies

2.1. Chromatic Preserving sets in graphs

A set $S \subseteq V$ is said to be a *Chromatic preserving set or a cp-set* of a graph G, is $\chi(< S >) = \chi(G)$, where $< S >$ denotes the subgraph of G induced by S. The minimum cardinality of a cp-set in G is called the *Chromatic preserving number or cp-number* of G and is denoted by $cpn(G)$. A cp-set of cardinality $cpn(G)$ is called a *cpn-set* of G.

Example 2.1. The graph G shown in Figure 1 has $cpn(G) = 5$.

2.1.1 Properties of a minimal cp-set

1. If $\chi(G) \geq 3$, then $cpn - set$ induces a 2-connected vertex-color-critical graph.

2. If G is connected of order p, then $cpn(G) = p$, if and only if G is vertex-color-critical.

3. There does not exist any disconnected graph G of order p such that $cpn(G) = p$.

4. $cpn(G) = 1$ if and only if $G = nK_1$, for $n \geq 1$.

5. G is a bipartite graph with $|E(G)| \geq 1$ if and only if $cpn(G) = 2$.

6. If G is 3-chromatic, then $cpn(G) = g_0(G)$, where $g_0(G)$ denotes the odd girth of G.

7. If $cpn(G) = 3$, then G is 3-chromatic.

8. For any non-trivial graph, G of order p which is neither vertex-color-critical nor totally disconnected, $2 \leq cpn(G) \leq p - 1$.

9. For a disconnected graph G of order p with $k < p$ components, $cpn(G) \leq p - k + 1$.

10. If $\chi(G-u) < \chi(G)$ for a vertex $u \in V(G)$, then u is in every minimal *cp*-set of G and conversely. Similarly, if $\chi(G-e) < \chi(G)$ for an edge $e \in E(G)$, then e is in every minimal *cp*-set of G and conversely.

11. If $\chi(G-u) = \chi(G)$, then $cpn(G-u) \geq cpn(G)$.

2.1.2 cp number of some well known graphs

The following results give the *cp*-number of some standard graphs.

1. $cpn(K_n) = n$

2. $cpn(nK_1) = 1$

3. $cpn(C_n) = \begin{cases} 2; & \text{if } n \text{ is even} \\ n; & \text{if } n \text{ is odd} \end{cases}$

4. $cpn(W_n) = \begin{cases} 3; & \text{if } n \text{ is even} \\ n; & \text{if } n \text{ is odd} \end{cases}$

5. $cpn(K_{m,n} = 2)$

2.2. Dom-Chromatic sets in graphs

A set $S \subseteq V(G)$ is said to be a *dom-chormatic set or dc-set* of G if S is a dominating set and $\chi(< S >) = \chi(G)$. That is, if S is a dominating set as well as a cp-set of G. The minimum cardinality of a dom-chromatic set in a graph G is called the *dom-chromatic number or dc-number* of G and is denoted by $\gamma_{ch}(G)$.

A dom-chromatic set S is *minimal* if and only if for each $u \in S$, one of the following conditions hold.

1. $\chi(< S - u >) < \chi(G)$.

2. $S - u$ is not a dominating set.

2.2.1 Properties of a dom-chromatic set

1. Dom-chromatic set exists for all graphs.

2. The vertex set $V(G)$ is a trivial dom-chromatic set for any graph G.

3. For a vertex-color-critical graph, V is the only dom-chromatic set.

4. For any graph G, $cpn(G) \leq \gamma_{ch}(G)$.

5. If S is a dom-chromatic set of G, then for each vertex $u \in V - S$, there exists atleast one vertex $v \in S$ such that u is not adjacent to v.

6. A dc-set of a graph is a global dominating set.
7. $\gamma(\bar{G}) \leq \gamma_g(G) \leq \gamma_{ch}(G)$.

2.2.2 Dom-chromatic number of some well known graphs

The following results give the dc-number of some standard graphs.

1. $\gamma_{ch}(K_n) = n$.
2. $\gamma_{ch}(nK_1) = n$.
3. $\gamma_{ch}(K_{m,n}) = 2$.
4. $\gamma_{ch}(P_n) = \begin{cases} \frac{n+3}{3}; & \text{if } n \equiv 0 \pmod 3 \\ \frac{n+2}{3}; & \text{if } n \equiv 1 \pmod 3 \\ \frac{n+4}{3}; & \text{if } n \equiv 2 \pmod 3 \end{cases}$
5. If n is odd, then $\gamma_{ch}(C_n) = n$.
6. If n is even, then $\gamma_{ch}(C_n) = \begin{cases} \frac{n+3}{3}; & \text{if } n \equiv 0 \pmod 3 \\ \frac{n+2}{3}; & \text{if } n \equiv 1 \pmod 3 \\ \frac{n+4}{3}; & \text{if } n \equiv 2 \pmod 3 \end{cases}$
7. $\gamma_{ch}(W_n) = \begin{cases} 3; & \text{if } n \text{ is odd.} \\ n; & \text{if } n \text{ is even.} \end{cases}$

3. NP-Completeness of finding a dom-chromatic set

In this section, we discuss how difficult is it to find the dom-chromatic number of an arbitrary graph G. We will show that the problem of finding the dom-chromatic number of an arbitrary graph G is NP-Complete. The problems of Graph coloring and Graph domination are the well known and well studied problems in Graph Theory and it has been proved that both the decision version of a dominating set problem is NP-Complete and similarly the problem of finding the chromatic number of an arbitrary graph is NP-Hard, as well [7, 1].

In the problem of finding a minimum dc-set of a graph, it follows from the definition that for a set $S \subseteq V$ to be a dc-set, it should satisfy the two properties: *(i) S must be a dominating set and (ii) The subgraph* $< S >$ *should have the same chromatic number as that of* G. Thus, to find a dc-set of a given graph G, either we need to initially search for a dominating set S in G and then find the chromatic number of the subgraph induced by S and finally verify whether $\chi(< S >) = \chi(G)$ or we need to search for a cp-set and verify whether it is dominating or not. But, the former procedure is very difficult and tedious. Instead, as *a dc-set is basically a dominating cp-set*, to find the minimum dominating set whose induced subgraph has the same chromatic number as that of G, it suffices to identify a minimum cp-set and then add minimum number of vertices so as to make the set dominate $V(G)$ and this results in the required minimum dc-set. Equivalently, finding a minimum dc-set is equivalent to finding a minimum cp-set and then verifying whether it dominates V. If not, minimum number of vertices can be added to make it a dominating set. Following this strategy, though it can be verified in polynomial time whether a given set is dominating or not [7, 1], the problem of finding the chromatic number of a graph is still NP-Hard [1]. Thus, it can be seen that the decision version of a dc-set problem takes the following form and is NP-Complete. Alternately, the problem of finding a dc-set and hence the dc-number of an arbitrary graph is NP-Complete, which is stated in the following theorem.

DOM-CHROMATIC SET (DC-SET) Problem
INSTANCE: A graph $G = (V, E)$ and a positive integer k.
QUESTION: Does G have a dc-set of size atmost k?

Theorem 3.1. The DC-set problem is NP-Complete.

If H is any smallest (in terms of number of vertices) induced subgraph of G such that $\chi(G) = \chi(H)$, then H is said to be critical. There is not much result on how to find the smallest critical subgraph of a non-critical graph. Herrmann and Hertz in 2000 [8] made an attempt to propose new exact algorithms for finding the chromatic number of a graph G. The algorithm attempts to determine the smallest possible induced subgraph H of G, which has the same chromatic number as G. The algorithm works on three phases: *Intialization phase, Descending phase and Augmenting phase*. In the initialization phase, it assumes the existence of a good HEURISTIC to find an upper bound on the chromatic number of G and computes $\chi(G)$. Then, starts the descending phase, in which it finds a critical subgraph H of G such that $\chi(H) = \chi(G)$ as follows: Starting with $H = G$, the algorithm scans each vertex $v \in V(H)$ and removes v in an arbitrary order. Initially determines an upper bound on $\chi(H - v)$ using the HEURISTIC. If the upper bound on $\chi(H - v)$ coincides with that of G, then the procedure is repeated recursively by assuming $H = H - v$, as long as such removals do not alter $\chi(H)$. Then applies an EXACT algorithm to find the chromatic number of the new subgraph H. The algorithm terminates if $\chi(H)$ coincides with the determined upper bound on $\chi(G)$. In doing so, the necessity to compare $\chi(H)$ and $\chi(H - v)$, for each $v \in V(H)$, is avoided by using upper bounds on $\chi(H)$ and $\chi(H - v)$ and using the property that "If k is an upper bound on $\chi(G)$ and H is any subgraph of G and if $\chi(H) = k$ then $\chi(G) = \chi(H)$, where each such bound in obtained using HEURISTIC. Thus, if $\chi(H)$ equals the obtained upper bound on $\chi(G)$, the algorithm eventually terminates. Otherwise, enters into the next phase, namely augmenting phase, in which vertices are added to H, until either $\chi(H)$ coincides with the upper bound on $\chi(G)$ or $H = G$.

In the same way of finding $\chi(G)$ using critical graphs as explained above, there exist different procedures to determine the chromatic number of a given graph G, like determining $\chi(G)$ by contraction of non-adjacent vertices [13], branch and bound techniques to find $\chi(G)$ using chromatic polynomial [11] and so on.

Thus, with the understanding that *a dc-set is basically a dominating cp-set* and following the strategy that "*to find the minimum dominating set whose induced subgraph has the same chromatic number as that of G, it suffices to identify a minimum cp-set and then add minimum number of vertices so as to make the set dominate $V(G)$*", we propose an approximation algorithm to find a minimum dc-set of an arbitrary graph G in the following section.

Result 3.2. **[8]** If k is an upper bound on $\chi(G)$ and H is any subgraph of G and if $\chi(H) = k$ then $\chi(G) = \chi(H)$.

4. An approximation algorithm to find a dom-chromatic set and dom-chromatic number of a graph G

The proposed algorithm is a modified version of the exact algorithm proposed by Hermann and Hertz [8], to find the smallest critical subgraph of G. A graph G by itself is *critical* if all its induced proper subgraphs have chromatic number strictly less than that of G and every non-critical graph G has atleast one induced proper subgraph H such that $\chi(H) = \chi(G)$. To find a minimum dc-set of G, initially we find the smallest subgraph H of G such that $\chi(H) = \chi(G)$. Let $S = V(H)$ and verify whether S dominates $V(G)$. If yes, then S is the required minimum dc-set of G. If not, then we suitably add minimum number of vertices so as to extend S to dominate G. Thus, the resultant set will be a minimum dc-set of G. The above procedure is given in the following pseudocode. The execution of the algorithm involves three steps as listed below:

1. Pre-processing step [Finding an upper bound on $\chi(G)$ using some better Heuristic.]
2. Finding a minimum cp-set S of G.
3. Extending S to form a dc-set of G.

There exist several exact algorithms in the literature to compute an upper bound on the chromatic number of a graph, whereas most of the algorithms work best depending on the instance. A survey of the most efficient heuristic coloring approaches can be found in [2, 4, 11]. In the proposed algorithm it is assumed that an upper bound k of the chromatic number of G is determined by using some efficient heuristic or metaheuristic method, referred to as HEURISTIC and is given as input to the algorithm.

Theorem 4.1. The algorithm DC-SET eventually terminates and computes a minimum dc-set of G and dc-number of G.

Proof. It is evident from the definition of cp-set and from Result 3.2, that the set S obtained in Step 11 is a minimum cp-set, as the graph H induced by S has the same

Algorithm 1: DC-SET

Input: G, k

Output: Minimum dc-set of G

1 **function DCSET(**G**)** ;
2 Set $H = G$;
3 Choose a vertex v from $V(H)$ of minimum degree in H. In case of tie, choose one arbitrarily ;
4 Find an upper bound k' on $\chi(H - v)$ using HEURISTIC ;
5 **if** $k' = k$ **then**
6   Set $H = H - v$;
7   Goto Step 3 ;
8 **end**
9 Find $\chi(H)$ using HEURISTIC ;
10 **if** $\chi(H) = k$ **then**
11   Set $S = V(H)$;
12 **end**
13 **if** $\cup_{u \in S} N[u] = V(G)$ **then**
14   Return S ;
15 **end**
16 **else**
17   From $V - S$, choose a vertex x of maximum degree in G. In case of tie, choose one arbitrarily ;
18     $S = S \cup x$;
19   Goto Step 13 ;
20 **end**

chromatic number as G. Then as a dc-set is a dominating cp-set, in Step 13, it is verified the cp-set S dominates $V(G)$. If so, then S itself is a minimum dc-set and its cardinality gives the dc-number of G. Otherwise, S is extended to form a dominating set by adding vertices from $V - S$ and thereby generating a dc-set. While choosing vertices from $V - S$, to generate a minimum dc-set, in Step 17, from the remaining pool of vertices, those of maximum degree are added to S and this process is repeated until the new set S dominates $V(G)$. The set S so obtained at the end is a minimum dc-set and its cardinality is the dc-number of G. As every graph has a dc-set, $V - S$ becomes empty and in the worst case, $V(G)$ itself becomes a dc-set. In other words, the algorithm eventually terminates. ■

5. Conclusion

As the two well studied problems in Graph Theroy namely, Graph Coloring and Graph domination play a predominant role in many applications, interest is shown in this paper to study the combined effect of the two parameters. The parameter dom-chromatic number, defined by combining the two concepts is studied. It is shown the decision version of the dom-chromatic problem is NP-Complete and an attempt is made to propose an approximation algorithm for finding the dc-set and dc-number of an arbitrary graph.

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References

[1] Cormen, T., Leiserson, C., Rivest, R., and Stein, C. *Introduction to Algorithms*, 2nd ed. Prentice-Hall of India, Pvt. Ltd., New Delhi, 2008.

[2] DeWarra, D. Heuristics for graph coloring. *Computing 7* (1990), 191–208.

[3] Dirac, G. A property of 4-chromatic graphs and some remarks on critical graphs. *J. London Math. Soc. 27* (1952), 85–92.

[4] Formanowicz, P., and Tanas, K. A survey of graph coloring - its types, methods and applications. *Found. Computing and Decision Sci. 37* (2012), 223–238.

[5] Harary, F. *Graph Theory*. Narosa Publishing House, 1988.

[6] Harary, F., and Haynes, T. Conditional graph theory iv: Dominating sets. *Utilitas Math. 48* (1995), 179–192.

[7] Haynes, T. W., Hedetniemi, S., and Slater, P. *Fundamentals of domination in graphs*. New York: Marcel Dekker, Inc., 1998.

[8] Herrmann, F., and Hertz, A. Finding the chromatic number by means of critical graphs. *Electronic Notes in Discrete Mathematics 5* (2000), 174–176. 6th International Conference on Graph Theory.

[9] Janakiraman, T., and Poobalaranjani, M. Dom-chromatic sets in bipartite graphs. *Int. J. Engineering Sicences, Advanced Computing and Bio-technology 1* (2010), 80–95.

[10] Jensen, T., and Toft, B. *Graph coloring problems*. John Wiley, 1995.

[11] McDiarmid, C. Determining the chromatic number of a graph. *Siam J. Comput. 8* (1979).

[12] Poobalaranjani, M. *On some coloring and domination parameters in graphs*. PhD thesis, Bharathidasan University, Tiruchirappalli, February 2005.

[13] Ram, A., and Rama, R. An alternative method to find the chromatic number of a finite, connected graph. doi: https://arxiv.org/ftp/arxiv/papers/1309/1309.3642.pdf.

[14] West, D. *Introduction to graph theory*, vol. 2. Prentice hall Upper Saddle River, 2001.