Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 9 (2017), pp. 6407–6419

http://www.ripublication.com/gjpam.htm

On a Semi-Symmetric Metric Connection in Generalized Sasakian Space Forms

Abhishek Singh
Department of Mathematics and Astronomy,
University of Lucknow, Lucknow 226007, Uttar Pradesh, India.

Shyam Kishor
Department of Mathematics and Astronomy,
University of Lucknow, Lucknow 226007, Uttar Pradesh, India

Abstract

The purpose of this paper is to study generalized Sasakian space forms admitting a semi-symmetric metric connection with some properties.

AMS subject classification: 53C25, 53C15.
Keywords: Generalized Sasakian-space-forms, semi-symmetric metric connection, Recurrent, ξ-conharmonicly flat.

1. Introduction

In differential geometry, the curvature of a Riemannian manifold (M, g) plays a fundamental role as well known, the sectional curvature of a manifold determine the curvature tensor R-completely. A Riemannian manifold with constant sectional curvature c is called a real-space forms and its curvature tensor is given by the equation

$$R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\},$$

for any vector fields X, Y, Z on M. Models for these spaces are the Euclidean spaces $(c = 0)$, the sphere $(c > 0)$ and the Hyperbolic space $(c < 0)$.

A Sasakian manifold $M(\phi, \xi, \eta, g)$ is said to be a Sasakian space forms if all the ϕ-sectional curvatures $K(X \wedge \phi X)$ are equal to a constant c; where $K(X \wedge \phi X)$ denotes the sectional curvature of the section spanned by the unit vector field X, orthogonal to ξ and ϕX. In such a case, Riemannian curvature tensor of M is given by

$$
\begin{aligned}
R(X, Y)Z &= \frac{c+3}{4}\{g(Y, Z)X - g(X, Z)Y\} \\
&+ \frac{c-1}{4}\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} \\
&+ \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi \\
&- g(Y, Z)\eta(X)\xi\}.
\end{aligned}
$$

In 2004, P. Alegre, D. E. Blair and A. Carriazo [1] introduced the concept of generalized Sasakian space forms. The generalized Sasakian space forms is defined as follows:

A generalized Sasakian space forms is an almost contact metric manifold $M(\phi, \xi, \eta, g)$ whose curvature tensor is given by

$$
\begin{aligned}
R(X, Y)Z &= f_1\{g(Y, Z)X - g(X, Z)Y\} \\
&+ f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} \\
&+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi \\
&- g(Y, Z)\eta(X)\xi\},
\end{aligned}
$$

where f_1, f_2, f_3 are differentiable functions on M and X, Y, Z are vector fields on M Sasakian space forms appear as natural examples of generalized Sasakian space forms, with constant functions

$$f_1 = \frac{c+3}{4}, \; f_2 = \frac{c-1}{4} \text{ and } f_3 = \frac{c-1}{4}$$

where c denotes constant ϕ-sectional curvature.

The idea of a semi-symmetric linear connection on a differentiable manifold was first introduced by Friedmann and Schouten [6] in 1924. Hayden [7] introduced a semi-symmetric metric connection on a Riemannian manifold. Duggal and Sharma [5] studied a semi-symmetric metric connection in a semi-Riemannian manifold. They studied some properties of the Ricci tensor, affine conformal motions, geodesics and group manifolds with respect to the semi-symmetric metric connection. The generalized Sasakian space forms admitting semi-symmetric metric connections by using warped products studied by [11]. Motivated by [11], in this paper, we consider generalized Sasakian space forms admitting semi-symmetric metric connections.

The present paper is organized as follows. In Section 2, we recall some well known basic formulas and properties of generalized Sasakian space forms. In Section 3, we give a brief account of semi-symmetric metric connection. In Section 4, we find the curvature tensor, the Ricci tensor and the scalar curvature in generalized Sasakian space forms with respect to the semi-symmetric metric connection. A generalized Sasakian space forms (M, g) whose curvature tensor of manifold is covariant constant with respect to the semi-symmetric metric connection and M is recurrent with respect to the Levi-Civita connection is studied in Section 5. The last section gives ξ-conformally flat generalized Sasakian space forms with respect to the semi-symmetric metric connection.

2. Generalized Sasakian space forms

An odd dimensional manifold $M^{2n+1}(n \geq 1)$ is said to admit an almost contact structure, sometimes called a (ϕ, ξ, η)-structure, if it admits a tensor field ϕ of type $(1, 1)$, a vector field ξ and a 1-forms η satisfying

$$\eta(\xi) = 1, \tag{2.1}$$

$$\phi^2(X) = -X + \eta(X)\xi, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$(\nabla_X \eta)Y = g(\nabla_X \xi, Y), \tag{2.4}$$

$$g(X, \phi Y) = -g(\phi X, Y), \tag{2.5}$$

$$g(X, \xi) = \eta(X), \tag{2.6}$$

for any vector fields X, Y on M^{2n+1}. In particular, in an almost contact metric manifold we also have

$$\phi\xi = 0, \qquad \eta \circ \phi = 0. \tag{2.7}$$

Such a manifold is said to be a contact metric manifold if $d\eta = \Phi$, where

$$d\eta(X, Y) = \Phi(X, Y) = g(X, \phi Y), \tag{2.8}$$

is called the fundamental 2-forms of M. If, ξ is a Killing vector field, then M is said to be a K-contact manifold. It is well-known that a contact metric manifold is a K-contact manifold if and only if

$$\nabla_X \xi = -\phi X, \tag{2.9}$$

for any vector field X on M. On the other hand, an almost contact metric structure of M is said to be normal if

$$[\phi, \phi](X, Y) = -2d\eta(X, Y)\xi,$$

for any vector fields X, Y, where $[\phi, \phi]$ denotes the Nijenhuis torsion of ϕ, given by

$$[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y].$$

A normal contact metric manifold is called a Sasakian manifold. It can be proved that an almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.10}$$

for any vector fields X, Y.

Moreover, the curvature tensor R of a Sasakian manifold satisfies

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y. \tag{2.11}$$

An almost contact metric manifold M is called a trans-Sasakian manifold [10] if there exist two functions α and β on M such that

$$(\nabla_X \phi)Y = \alpha[g(X, Y)\xi - \eta(Y)X] + \beta[g(\phi X, Y)\xi - \eta(Y)\phi X], \tag{2.12}$$

for any vector fields X, Y on M. From above equation, it follows that

$$\nabla_X \xi = -\alpha\phi X + \beta[X - \eta(X)\xi]. \tag{2.13}$$

If $\beta = 0$ (resp. $\alpha = 0$), then M is said to be an α-Sasakian manifold (resp. β-Kenmotsu manifold). Sasakian manifolds (resp. Kenmotsu manifolds [8]) appear as examples of α-Sasakian manifolds (β-Kenmotsu manifolds), with $\alpha = 1$ (resp. $\beta = 1$). Another kind of trans-Sasakian manifolds is that of cosymplectic manifolds, obtained for $\alpha = \beta = 0$. From (2.13), for a cosymplectic manifold it follows that $\nabla_X \xi = 0$, which implies that ξ is a Killing vector field for a cosymplectic manifold [3].

Given an almost contact metric manifold $M(\phi, \xi, \eta, g)$, a ϕ-section of M at $p \in M$ is a section $\pi \subseteq T_pM$ spanned by a unit vector X_p orthogonal to ξ_p and ϕX_p. The ϕ-sectional curvature of β is defined by $K(X \wedge \phi X) = R(X, \phi X, \phi X, X)$. A Sasakian manifold with constant ϕ-sectional curvature c is called a Sasakian space forms. Similarly, a Kenmotsu manifold with constant ϕ-sectional curvature c is called a Kenmotsu space forms. A cosymplectic manifold with constant ϕ-sectional curvature c is called a cosymplectic space forms.

On the other hand, given an almost contact metric manifold $M(\phi, \xi, \eta, g)$, we say that M is a generalized Sasakian space forms if there exist three functions f_1, f_2, f_3 on M such that curvature tensor R is given by

$$\begin{aligned} R(X, Y)Z \;=\; & f_1\{g(Y, Z)X - g(X, Z)Y\} \\ & + f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} \\ & + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi \\ & - g(Y, Z)\eta(X)\xi\}, \end{aligned} \tag{2.14}$$

for any vector fields X, Y, Z on M [1]. Such a manifold is denoted by $M^{2n+1}(f_1, f_2, f_3)$. This kind of manifold appears as a generalization of the well known Sasakian space forms, which can be obtained as a particular case of generalized Sasakian space forms by taking

$$f_1 = \frac{C+3}{4}, \quad f_2 = f_3 = \frac{C-1}{4}.$$

In a $(2n+1)$-dimensional generalized Sasakian space forms $M^{2n+1}(f_1, f_2, f_3)$, we have the following relations [4]:

$$R(X, Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y], \tag{2.15}$$

$$R(\xi, X)Y = (f_1 - f_3)[g(X, Y)\xi - \eta(Y)X)], \tag{2.16}$$

$$R(\xi, X)\xi = (f_1 - f_3)[\eta(X)\xi - X)], \tag{2.17}$$

$$\eta(R(X, Y)Z) = (f_1 - f_3)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \tag{2.18}$$

$$\eta(R(X, Y)\xi) = 0, \tag{2.19}$$

$$S(X, Y) = (2nf_1 + 3f_2 - f_3)g(X, Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y), \tag{2.20}$$

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi, \tag{2.21}$$

$$S(X, \xi) = 2n(f_1 - f_3)\eta(X), \tag{2.22}$$

$$S(\phi X, \phi Y) = S(X, Y) - 2n(f_1 - f_3)\eta(X)\eta(Y), \tag{2.23}$$

$$S(\xi, \xi) = 2n(f_1 - f_3), \tag{2.24}$$

$$Q\xi = 2n(f_1 - f_3)\xi, \tag{2.25}$$

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3, \tag{2.26}$$

where R, S and r denote the curvature tensor, Ricci tensor of type (0, 2) and scalar curvature of the space-forms respectively and Q is the Ricci operator defined by $g(QX, Y) =$

$S(X, Y)$. We know that [1] the ϕ-sectional curvature of a generalized Sasakian space forms $M^{2n+1}(f_1, f_2, f_3)$ is $f_1 + 3f_2$.

In a Riemannian manifold of dimension $(2n+1)$ the Weyl conformal curvature tensor is given by

$$\begin{aligned} C(X, Y)Z &= R(X, Y)Z - \frac{1}{(2n-1)}[S(Y, Z)X - S(X, Z)Y \\ &\quad + g(Y, Z)QX - g(X, Z)QY] \\ &\quad + \frac{r}{2n(2n-1)}[g(Y, Z)X - g(X, Z)Y], \end{aligned} \tag{2.27}$$

for any vector fields X, Y. A generalized Sasakian space forms of dimension greater than three is said to be conformally flat if its Weyl conformal curvature tensor vanishes. It is known (see [9]) that a $(2n + 1)$-dimensional $(n > 1)$ generalized Sasakian space forms is conformally flat if and only if $f_2 = 0$.

3. Semi-symmetric metric connection

Hayden [7] introduced semi-symmetric linear connection on a Riemannian manifold. Let M be an $m = (2n + 1)$-dimensional Riemannian manifold of class C^∞ endowed with the Riemannian metric g and ∇ be the Levi-Civita connection on (M^m, g). A linear connection $\widetilde{\nabla}$ defined on (M^m, g) is said to be semi-symmetric [6], if its torsion tensor T is of the forms

$$T(X, Y) = \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X, Y]. \tag{3.1}$$

Satisfies

$$T(X, Y) = \eta(Y)X - \eta(X)Y, \tag{3.2}$$

where η is a 1-forms and is a vector field defined by

$$\eta(X) = g(X, \xi), \tag{3.3}$$

for all vector fields $X \in \chi(M^m)$, $\chi(M^m)$ is the set of all differentiable vector fields on M^m. A semi-symmetric connection $\widetilde{\nabla}$ is called a semi-symmetric metric connection [7] if it further satisfies $\widetilde{\nabla} g = 0$.

A relation between the semi-symmetric metric connection $\widetilde{\nabla}$ and the Levi-Civita connection ∇ on (M^m, g) has been obtained by Yano [12] which is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi. \tag{3.4}$$

Further, a relation between the curvature tensor R of the Levi-Civita connection ∇ and the curvature tensor $\widetilde{R}$ of the semi-symmetric metric connection $\widetilde{\nabla}$ is given by

$$\begin{aligned} \widetilde{R}(X, Y)Z &= R(X, Y)Z + \alpha(X, Z)Y - \alpha(Y, Z)X \\ &\quad + g(X, Z)AY - g(Y, Z)AX, \end{aligned} \tag{3.5}$$

for all vector fields X, Y, Z on M, where α is the $(0, 2)$-tensor field and A is a tensor field of type $(1, 1)$ defined by

$$\alpha(X, Y) = (\nabla_X \eta)Y - \eta(X)\eta(Y) + \frac{1}{2}\eta(\xi)g(X, Y), \tag{3.6}$$

and

$$\alpha(X, Y) = g(AX, Y), ([6]). \tag{3.7}$$

4. Curvature tensor of a generalized Sasakian space forms with respect to semi-symmetric metric connection

From (3.5), we have

$$\begin{aligned} g(\widetilde{R}(X, Y)Z, W) \;\; = \;\; & g(R(X, Y)Z, W) + \alpha(X, Z)g(Y, W) \\ & -\alpha(Y, Z)g(X, W) + g(X, Z)g(AY, W) \\ & -g(Y, Z)g(AX, W), \end{aligned}$$

using (3.6), (3.7) and (2.4) in the above equation, we obtain

$$\begin{aligned} \widetilde{R}(X, Y)Z \;\; = \;\; & R(X, Y)Z + \{g(\phi Y, Z)X - g(\phi X, Z)Y \\ & +g(Y, Z)\phi X - g(X, Z)\phi Y\} \\ & +\{\eta(Y)X - \eta(X)Y\}\eta(Z) \\ & +\{g(X, Z)Y - g(Y, Z)X\} \\ & +\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}\xi. \end{aligned} \tag{4.1}$$

Putting $Z = \xi$ in (4.1), we get

$$\widetilde{R}(X, Y)\xi = R(X, Y)\xi + \eta(Y)\phi X - \eta(X)\phi Y,$$

using (2.15) in the above equation, we have

$$\widetilde{R}(X, Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y] + \eta(Y)\phi X - \eta(X)\phi Y. \tag{4.2}$$

Putting $X = \xi$ in (4.1) and using (2.16), we get

$$\widetilde{R}(\xi, Y)Z = (f_1 - f_3)[g(Y, Z)\xi - \eta(Z)Y)] + g(\phi Y, Z)\xi - \eta(Z)\phi Y. \tag{4.3}$$

Putting $Y = \xi$ in (4.1), we get

$$\widetilde{R}(X, \xi)Z = R(X, \xi)Z + \eta(Z)\phi X - g(\phi X, Z)\xi. \tag{4.4}$$

Putting $X = Z = \xi$ in (4.1), we get

$$\widetilde{R}(\xi, Y)\xi = R(\xi, Y)\xi - \phi Y, \tag{4.5}$$

taking inner product of (4.1) with ξ and using (2.6) and (2.7), we have

$$\begin{aligned} g(\widetilde{R}(X,Y)Z,\xi) &= \eta(\widetilde{R}(X,Y)Z) \\ &= \eta(R(X,Y)Z) + g(\phi Y,Z)\eta(X) - g(\phi X,Z)\eta(Y). \end{aligned} \tag{4.6}$$

Now putting $Z=\xi$ in (4.6) and using (2.19), we get

$$\eta(\widetilde{R}(X,Y)\xi) = 0. \tag{4.7}$$

Again putting $X=\xi$ in (4.6) and using (2.7), we get

$$\eta(\widetilde{R}(\xi,Y)Z) = \eta(R(\xi,Y)Z) + g(\phi Y,Z). \tag{4.8}$$

Taking the inner product of (4.1) with W, it follows that

$$\begin{aligned} g(\widetilde{R}(X,Y)Z,W) &= g(R(X,Y)Z,W) + \{g(\phi Y,Z)g(X,W) \\ &\quad -g(\phi X,Z)g(Y,W) + g(Y,Z)g(\phi X,W) \\ &\quad -g(X,Z)g(\phi Y,W)\} + \{\eta(Y)g(X,W) \\ &\quad -\eta(X)g(Y,W)\}\eta(Z) + \{g(X,Z)g(Y,W) \\ &\quad -g(Y,Z)g(X,W)\} + \{g(Y,Z)\eta(X) \\ &\quad -g(X,Z)\eta(Y)\}\eta(W). \end{aligned} \tag{4.9}$$

Let $\{e_1,\ldots,e_n\}$ be a local orthonormal basis of vector fields in M. Then by putting $X=W=e_i$ in the above equation and taking summation over i, $1\le i\le n$, we get

$$\widetilde{S}(Y,Z) = S(Y,Z) + (2n-1)[g(\phi Y,Z) + \eta(Y)\eta(Z) - g(Y,Z)], \tag{4.10}$$

putting $Y=\phi Y$ and $Z=\phi Z$ in the above equation and using (2.22), we get

$$\begin{aligned} \widetilde{S}(\phi Y,\phi Z) &= S(Y,Z) - [2n(f_1-f_3)-(2n-1)]\eta(Y)\eta(Z) \\ &\quad +(2n-1)g(\phi Y,Z) - (2n-1)g(Y,Z), \end{aligned} \tag{4.11}$$

from (4.11) and (4.12), we get

$$\widetilde{S}(\phi Y,\phi Z) = \widetilde{S}(Y,Z) - 2n(f_1-f_3)\eta(Y)\eta(Z). \tag{4.12}$$

Again, contracting (4.10) over Y and Z, we get

$$\widetilde{r} = r - 2n(2n-1),$$

using (2.26) in the above equation, we find

$$\widetilde{r} = 2n(2n+1)f_1 + 6nf_2 - 4nf_3 - 2n(2n-1), \tag{4.13}$$

where $\widetilde{r}$ and r are the scalar curvatures with respect to the semi-symmetric metric connection and the Levi-Civita connection respectively. Putting $Z=\xi$ in (4.10), we get

$$\widetilde{S}(Y,\xi) = S(Y,\xi) = 2n(f_1-f_3)\eta(Y), \tag{4.14}$$

from (4.10), we get

$$\widetilde{Q}Y = QY + (2n-1)\phi Y + (2n-1)\eta(Y)\xi - (2n-1)Y. \tag{4.15}$$

Using (3.4), (2.7) and (2.10), it follows that

$$(\widetilde{\nabla}_X \phi)Y = g(X,Y)\xi - \eta(Y)X - g(X,\phi Y)\xi - \eta(Y)\phi X. \tag{4.16}$$

Using (3.4), (2.4) and (2.7), we get

$$(\widetilde{\nabla}_X \eta)Y = g(X,Y) + g(\phi X, Y) - \eta(X)\eta(Y). \tag{4.17}$$

Using (3.4), (2.4) and (2.9), implies that

$$\widetilde{\nabla}_X \xi = -\phi X + X - \eta(X)\xi. \tag{4.18}$$

Thus, we obtain the following:

Theorem 4.1. For a generalized Sasakian space forms M with respect to the semi-symmetric metric connection $\widetilde{\nabla}$

(i) The curvature tensor $\widetilde{R}$ is given by (4.1),

(ii) The Ricci tensor $\widetilde{S}$ is given by (4.10),

(iii) The scalar curvature $\widetilde{r}$ is given by (4.13),

(iv) $\widetilde{r} = r - 2n(2n-1)$

(v) $\widetilde{S}(Y,\xi) = S(Y,\xi) = 2n(f_1 - f_3)\eta(Y)$,

(vi) $(\widetilde{\nabla}_X \phi)Y = g(X,Y)\xi - \eta(Y)X - g(X,\phi Y)\xi - \eta(Y)\phi X$.

(vii) $(\widetilde{\nabla}_X \eta)Y = g(X,Y) + g(\phi X, Y) - \eta(X)\eta(Y)$.

(viii) $\widetilde{\nabla}_X \xi = -\phi X + X - \eta(X)\xi$.

5. A generalized Sasakian space forms (M, g) whose curvature tensor of manifold is covariant constant with respect to the semi-symmetric metric connection and M is recurrent with respect to the Levi-Civita connection

Definition 5.1. A generalized Sasakian space forms M with respect to the Levi-Civita connection is called recurrent if its curvature tensor R satisfies the condition

$$(\nabla_W R)(X,Y)Z = A(W)R(X,Y)Z, \tag{5.1}$$

where A is the 1-forms.

Definition 5.2. A generalized Sasakian space forms M is said to be an η-Einstein manifold if its Ricci tensor S of the Levi-Civita connection is of the forms

$$S(Z, W) = ag(Z, W) + b\eta(Z)\eta(W),$$

where a and b are smooth functions on the manifold.

Using (1.1), (2.7), (2.8) and (2.10), we obtain

$$\begin{aligned}(\widetilde{\nabla}_W R)(X, Y)Z &= \widetilde{\nabla}_W R(X, Y)Z - R(\widetilde{\nabla}_W X, Y)Z \\ &\quad -R(X, \widetilde{\nabla}_W Y)Z - R(X, Y)\widetilde{\nabla}_W Z,\end{aligned}$$

$$\begin{aligned}(\widetilde{\nabla}_W R)(X, Y)Z &= (\nabla_W R)(X, Y)Z - {}^\circ R(X, Y, Z, W)\xi \\ &\quad -\eta(X)R(W, Y)Z - \eta(Y)R(X, W)Z \\ &\quad -\eta(Z)R(X, Y)W + (f_1 - f_3)[\eta(X)g(Y, Z)W \\ &\quad -\eta(Y)g(X, Z)W + g(X, W)g(Y, Z)\xi \\ &\quad -\eta(Z)g(X, W)Y + \eta(Z)g(Y, W)X \\ &\quad -g(X, Z)g(Y, W)\xi + \eta(Y)g(Z, W)X \\ &\quad -\eta(X)g(Z, W)Y].\end{aligned} \tag{5.2}$$

where $g(R(X, Y)Z, W) = {}^\circ R(X, Y, Z, W)$.

Suppose $(\widetilde{\nabla}_W R)(X, Y)Z = 0$, then from (4.2), we get

$$\begin{aligned}&(\nabla_W R)(X, Y)Z - {}^\circ R(X, Y, Z, W)\xi - \eta(X)R(W, Y)Z \\ &-\eta(Y)R(X, W)Z - \eta(Z)R(X, Y)W + (f_1 - f_3) \\ &[\eta(X)g(Y, Z)W - \eta(Y)g(X, Z)W + g(X, W)g(Y, Z)\xi \\ &-\eta(Z)g(X, W)Y + \eta(Z)g(Y, W)X - g(X, Z)g(Y, W)\xi \\ &+\eta(Y)g(Z, W)X - \eta(X)g(Z, W)Y] = 0.\end{aligned} \tag{5.3}$$

Using (4.1) in (4.3), we have

$$\begin{aligned}&A(W)R(X, Y)Z - {}^\circ R(X, Y, Z, W)\xi - \eta(X)R(W, Y)Z \\ &-\eta(Y)R(X, W)Z - \eta(Z)R(X, Y)W + (f_1 - f_3) \\ &[\eta(X)g(Y, Z)W - \eta(Y)g(X, Z)W + g(X, W)g(Y, Z)\xi \\ &-\eta(Z)g(X, W)Y + \eta(Z)g(Y, W)X - g(X, Z)g(Y, W)\xi \\ &+\eta(Y)g(Z, W)X - \eta(X)g(Z, W)Y] = 0.\end{aligned} \tag{5.4}$$

Now, contracting X in (4.4) and using (2.1) and (2.7), it follows that

$$\begin{aligned}&A(W)S(Y, Z) - \eta(Y)S(Z, W) - \eta(Z)S(Y, W) \\ &+(f_1 - f_3)[g(Y, Z)W + (2n - 2)\eta(Z)g(Y, W) \\ &+(2n + 1)\eta(Y)g(Z, W)] \\ = \;& 0.\end{aligned} \tag{5.5}$$

Putting $Y = \xi$ in (5.5) and using (2.1) and (2.11), we obtain

$$\begin{aligned} S(Z,W) &= (2n+1)(f_1 - f_3)g(Z,W) \\ &\quad +2n(f_1 - f_3)\eta(Z)A(W) - (f_1 - f_3)\eta(Z)\eta(W). \end{aligned} \tag{5.6}$$

Suppose the associated 1-forms A is equal to the associated 1-forms, then from (4.6), we get

$$S(Z,W) = (2n+1)(f_1 - f_3)g(Z,W) + (2n-1)(f_1 - f_3)(W)(Z).$$

Therefore,

$$S(Z,W) = ag(Z,W) + b\eta(Z)\eta(W),$$

where $a = (2n+1)(f_1 - f_3)$ and $b = (2n-1)(f_1 - f_3)$.

Thus, we obtain the following:

Theorem 5.3. If an $(2n+1)$-dimensional generalized Sasakian space forms whose curvature tensor of manifold is covariant constant with respect to the semi-symmetric metric connection and the manifold is recurrent with respect to the Levi-Civita connection and the associated 1-forms A is equal to the associated 1-forms, then the manifold is an η-Einstein manifold.

6. ξ-Conformally flat generalized Sasakian space forms with respect to the semi-symmetric metric connection

The Weyl conformal curvature tensor with respect to the semi-symmetric metric connection is given by

$$\begin{aligned} \widetilde{C}(X,Y)Z &= \widetilde{R}(X,Y)Z - \frac{1}{(2n-1)}[\widetilde{S}(Y,Z)X - \widetilde{S}(X,Z)Y \\ &\quad +g(Y,Z)\widetilde{Q}X - g(X,Z)\widetilde{Q}Y] \\ &\quad +\frac{\widetilde{r}}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{6.1}$$

for any vector fields X, Y and Z. where $\widetilde{S}$ is the Ricci tensor with respect to the semi-symmetric metric connection.

Definition 6.1. A generalized Sasakian space forms with respect to the semi-symmetric metric connection is said to be ξ-conformally flat if

$$\widetilde{C}(X,Y)\xi = 0.$$

In the present section, we study ξ-conformally flat generalized Sasakian space forms. Let $M^{2n+1}(f_1, f_2, f_3)$ be a generalized Sasakian space forms. Putting $Z = \xi$ in (2.15), we obtain

$$\begin{aligned}\widetilde{C}(X,Y)\xi &= \widetilde{R}(X,Y)\xi - \frac{1}{(2n-1)}[\widetilde{S}(Y,\xi)X - \widetilde{S}(X,\xi)Y \\ &\quad +\eta(Y)\widetilde{Q}X - \eta(X)\widetilde{Q}Y] \\ &\quad +\frac{\widetilde{r}}{2n(2n-1)}[\eta(Y)X - \eta(X)Y].\end{aligned} \tag{6.2}$$

Using (4.2), (4.14) and (4.15) in (5.2), which yields

$$\begin{aligned}\widetilde{C}(X,Y)\xi &= [(f_1 - f_3) - \frac{2n(f_1 - f_3)}{(2n-1)} + \frac{\widetilde{r}}{2n(2n-1)} \\ &\quad -\frac{(2nf_1 + 3f_2 - f_3)}{(2n-1)} + 1][\eta(Y)X - \eta(X)Y].\end{aligned} \tag{6.3}$$

Putting the value of $\widetilde{r}$ from (4.13) in (5.3), we obtain the following:

$$\widetilde{C}(X,Y)\xi = 0. \tag{6.4}$$

Therefore, the generalized Sasakian space forms with respect to the semi-symmetric metric connection is ξ-conformally flat. Thus, we obtain the following:

Theorem 6.2. A $(2n+1)$-dimensional generalized Sasakian space forms $M(f_1, f_2, f_3)$ with respect to the semi-symmetric metric connection is always ξ-conformally flat.

References

[1] Alegre, P. Blair, D. E. and Carriazo, A., *Generalized Sasakian space forms*, Israel J. Math. 141(2004), 157–183.

[2] Barman, A., *On Para-Sasakian manifolds admitting semi-symmetric metric connection*, Publication De L'Insitut Mathematique 109(2014), 239–247.

[3] Blair, D. E., *The theory of quasi-Sasakian structures*, J. Differential Geometry, 1(1967), 331–345.

[4] De. U.C, and Sarkar. A., *Some curvature properties of generalized Sasakian space forms*, Lobachevskii journal of mathematics. 33(2012), no. 1, 22–27.

[5] Duggal, K. L. and Sharma, R., *Symmetries of spacetimes and Riemannian manifolds*, Springer Science & Business Media; 2013.

[6] Friedmann, A. and Schouten, J.A., *Über die Geometric der halbsymmetrischen Übertragung*, Math. Zs. 21(1924), 211–223.

[7] Hayden, H. A., *Subspaces of space with torsion*, Proc. London Math. Soc. 34(1932), 27–50.

[8] Kenmotsu, K., *A class of almost contact Riemannian manifolds*, Tôhoku Math. J., 24(1972), 93–103.

[9] Kim, U. K., *Conformally flat generalized Sasakian space forms and locally symmetric generalized Sasakian space forms,* Note di matematica 3.1(2006), 55–67.

[10] Oubiña, J. A., *New classes of almost contact metric structures*, Publ. Math. Debrecen, 32(1985), 187–193.

[11] Sular, S. and Özgür, C., *Generalized Sasakian space forms with semi-symmetric metric connections* Annals of the Alexandru Ioan Cuza University-Mathematics 60.1(2014), 145–156.

[12] Yano, K., *On semi-symmetric connection*, Rev. Roum. Math. Pure Appl. 15(1970), 1570–1586.