Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 9 (2017), pp. 6421–6428

http://www.ripublication.com/gjpam.htm

On Certain properties of Convolution conditions with q-derivative operator

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Abstract

By applying the concept of fractional q-caluclus, we investigate convolution proprties and coefficients estimates.

AMS subject classification: 30C45, 30C55.
Keywords: Coefficient inequalities, q-derivative operators, Convolution condition, Janowski class.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions in the open unit disc $\mathcal{U} = \{z; |z| < 1\}$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

For $-1 \leq B < A \leq 1$, let $\mathcal{P}(A, B)$ [4] denote the class of functions which are of the form

$$p(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)},$$

where ω is a bounded analytic function satisfying the conditions $\omega(0) = 0$ and $|\omega(z)| < 1$.

The q-shifted factorial is defined for $\alpha,\ q \in \mathbb{C}$ as a product of n factors by

$$(\alpha, q)_n = \begin{cases} 1, & \text{n=0;} \\ (1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{n-1}), & \text{n} \in \mathbb{N}, \end{cases} \tag{1.2}$$

and in terms of the basic analogue of the gamma function

$$(q^{\alpha};q)_n = \frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)}, \quad (n>0), \tag{1.3}$$

where the q-gamma functions [2, 3] is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty (1-q)^{1-x}}{(q^x;q)_\infty} \quad (0<q<1). \tag{1.4}$$

Note that, if $|q|<1$, the q-shifted factorial (1.3) remains meaningful for $n=\infty$ as a convergent infinite product

$$(\alpha;q)_\infty = \prod_{m=0}^{\infty}(1-\alpha q^m).$$

Now recall the following q-analogue definitions given by Gasper and Rahman [2]. The recurrence relation for q-gamma function is given by

$$\Gamma_p(x+1) = [x]_q \Gamma_p(x), \text{ where, } [x]_q = \frac{(1-q^x)}{(1-q)}, \tag{1.5}$$

and called q-analogue of x.

Jackson's q-derivative and q-integral of a function f defined on a subset of $\mathbb{C}$ are, respectively, given by (see Gasper and Rahman [2])

$$D_q f(z) = \frac{f(z)-f(zq)}{z(1-q)}, \quad (z \neq 0,\ q \neq 0). \tag{1.6}$$

$$\int_0^z f(t)d_q(t) = z(1-q)\sum_{m=0}^{\infty} q^m f(zq^m). \tag{1.7}$$

In view of the relation

$$\lim_{q\to 1^-} \frac{(q^{\alpha};q)_n}{(1-q)^n} = (\alpha)_n, \tag{1.8}$$

we observe that the q-shifted fractional (1.2) reduces to the familiar Pochhammer symbol $(\alpha)_n$, where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n+1)$.

The binomial theorem $(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}$ has a q-analogue of the form is defined by (Gasper and Rahman [2])

$$(ab;q)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q b^k (a;q)_k (b;q)_{n-k}, \text{ where } \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_q(A,\ B)$ if

$$\frac{zD_q f(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \qquad z \in \mathcal{U},\ -1 \le B < A \le 1,\ 0 < q < 1.$$

Also we define a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{K}_q(A,\ B)$ if

$$\frac{D_q(zD_q f(z))}{D_q f(z)} \prec \frac{1+Az}{1+Bz} \qquad z \in \mathcal{U},\ -1 \le B < A \le 1,\ 0 < q < 1.$$

Further function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_q^{\lambda}(A,\ B)$ if

$$\frac{e^{i\lambda} zD_q f(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, \qquad z \in \mathcal{U},\ -1 \le B < A \le 1,\ \frac{-\pi}{2} < \lambda < \frac{\pi}{2},\ 0 < q < 1.$$

Note that when $q \to 1^-$ we get the classes studied by Ganesan [1], Hayami, Owa and Srivastava [6], Latha [5] and others.

Note that $f \in \mathcal{K}_q(A,\ B) \Leftrightarrow zD_q f \in \mathcal{S}_q(A,\ B)$.

2. Main results

Theorem 2.1. A function $f \in \mathcal{A}$ is in the class $\mathcal{S}_q(A,\ B)$ if and only if

$$1 + \sum_{k=2}^{\infty} A_k z^{k-1} \neq 0, \tag{2.9}$$

where,

$$A_k = \frac{\left[([k]_q - 1) + (B[k]_q - A)\zeta\right]}{(B-A)\zeta} a_k \text{ and } |\zeta| = 1.$$

Proof. A function $f \in \mathcal{A}$ is in the class $\mathcal{S}_q(A,\ B)$ if and only if

$$\frac{zD_q f(z)}{f(z)} \prec \frac{1+Az}{1+Bz}.$$

But the function

$$p(z) = \frac{1+A\omega(z)}{1+B\omega(z)}, \qquad -1 \le B < A \le 1,$$

are subordinate to $\dfrac{1+Az}{1+Bz}$. They map the unit circle $|z| = 1$ onto the boundary of the circle on the line joining $\dfrac{1-A}{1-B}$ and $\dfrac{1+A}{1+B}$ as diameter.

When $B = -1$, the image of the unit circle is the line

$$\Re\{p(z)\} = \frac{1-A}{2},\ -1 < A < 1.$$

Further $p(z) = 1$ at $z = 0$ and $(1, 0)$ lies inside the image of the unit circle. The functions $\frac{1 + A\omega(z)}{1 + B\omega(z)}$ are analytic and hence map regions onto regions. Therefore, every point in the interior of the unit disc goes over to an interior point of the image disc. Thus, $f \in \mathcal{S}_q(A, B)$ is equivalent to

$$\frac{zD_q f(z)}{f(z)} \neq \frac{1 + A\zeta}{1 + B\zeta}, \ |\zeta| = 1.$$

That is,

$$\begin{aligned}
&(1 + B\zeta)zD_q f(z) - (1 + A\zeta)f(z)) \neq 0\\
&(1 + B\zeta)\left(z + \sum_{k=2}^{\infty}[k]_q a_k z^k\right) - (1 + A\zeta)\left(z + \sum_{k=2}^{\infty} a_k z^k\right) \neq 0\\
&(B - A)\zeta z + \sum_{k=2}^{\infty}\left[([k]_q - 1) + (B[k]_q - A)\zeta\right] a_k z^k \neq 0\\
&(B - A)\zeta z\left(1 + \sum_{k=2}^{\infty}\frac{\left[([k]_q - 1) + (B[k]_q - A)\zeta\right]}{(B - A)\zeta} a_k z^{k-1}\right) \neq 0.
\end{aligned}$$

That is,

$$1 + \sum_{k=2}^{\infty}\frac{\left[([k]_q - 1) + (B[k]_q - A)\zeta\right])}{(B - A)\zeta} a_k z^{k-1} \neq 0.$$

Hence the proof. ■

Remark 2.2. It follows from the normalization conditions
$a_0 = 0$ and $a_1 = 1$ that

$$A_o = 0 \text{ and } A_1 = \frac{\zeta(B - A)}{\zeta(B - A)} a_1 = 1.$$

Theorem 2.3. A function $f \in \mathcal{A}$ is in the class $\mathcal{S}_q^{\lambda}(A, B)$ if and only if

$$1 + \sum_{k=2}^{\infty} B_k z^{k-1} \neq 0,$$

where,

$$B_k = \frac{([k]_q - 1)e^{i\lambda} + \zeta([k]_q Be^{i\lambda} + iB\sin\lambda - A\cos\lambda)}{\cos\lambda(B - A)\zeta} a_k$$

and $|\zeta| = 1$.

Proof. Since, $f \in \mathcal{S}_q^{\lambda}(A, B)$,

$$e^{i\lambda}\frac{\left(\frac{zD_q f(z)}{f(z)}\right) - i\sin\lambda}{\cos\lambda} \neq \frac{1 + A\zeta}{1 + B\zeta}.$$

That is,
$(1+B\zeta)e^{i\lambda}zD_q f(z)-(1+B\zeta)i\sin\lambda f(z)$

$$-(1+A\zeta)\cos\lambda f(z)\neq 0,$$

which implies

$$(1+B\zeta)e^{i\lambda}\left(z+\sum_{k=2}^{\infty}[k]_q a_k z^k\right)-(1+B\zeta)i\sin\lambda\left(z+\sum_{k=2}^{\infty}a_k z^k\right)$$

$$-(1+A\zeta)\cos\lambda\left(z+\sum_{k=2}^{\infty}a_k z^k\right)\neq 0$$

$[(1+B\zeta)e^{i\lambda}-(1+B\zeta)i\ sin\lambda-(1+A\zeta)\cos\lambda]z$

$$+\sum_{k=2}^{\infty}[(1+B\zeta)e^{i\lambda}[k]_q-(1+B\zeta)i\sin\lambda-(1+A\zeta)\cos\lambda]a_k z^k\neq 0$$

$\cos\lambda\zeta(B-A)z$

$$+\sum_{k=2}^{\infty}[([k]_q-1)e^{i\lambda}+\zeta([k]_q Be^{i\lambda}+iB\sin\lambda-A\cos\lambda)]a_k z^k\neq 0$$

$(\cos\lambda\zeta(B-A)z)$

$$\left\{1+\sum_{k=2}^{\infty}\frac{([k]_q-1)e^{i\lambda}+\zeta([k]_q Be^{i\lambda}+iB\sin\lambda-A\cos\lambda)}{\cos\lambda(B-A)\zeta}\right\}a_k z^{k-1}$$

$\neq 0$

$$1+\sum_{k=2}^{\infty}\frac{[([k]_q-1)e^{i\lambda}+\zeta([k]_q Be^{i\lambda}+iB\sin\lambda-A\cos\lambda)]}{\cos\lambda(B-A)\zeta}a_k z^{k-1}$$

$\neq 0.$ ■

3. Coefficient Conditions

Theorem 3.1. If $f\in\mathcal{A}$ satisfies the following condition

$$\sum_{k=2}^{\infty}\left[\left|\sum_{i=1}^{k}\left(\sum_{j=1}^{i}(-1)^{i-j}([j]_q-1)\begin{bmatrix}n\\i-j\end{bmatrix}_q q^{\frac{(i-j)(i-j-1)}{2}}a_j\right)\begin{bmatrix}m\\k-i\end{bmatrix}_q q^{\frac{k-i(k-i-1)}{2}}\right|+\right.$$

$$\left|\sum_{i=1}^{k}\left(\sum_{j=1}^{i}(-1)^{i-j}([j]_q B-A)\begin{bmatrix} n \\ i-j \end{bmatrix}_q q^{\frac{(i-j)(i-j-1)}{2}} a_j\right)\begin{bmatrix} m \\ k-i \end{bmatrix}_q q^{\frac{k-i(k-i-1)}{2}}\right|\Bigg] \leq B-A, \tag{3.10}$$

where, $n,\ m \in \mathbb{N}$ and $-1 \leq B < A \leq 1,\ 0 < q < 1$, then, $f \in \mathcal{S}_q(A,\ B)$.

Proof. We note that for $(-z; q)_n \neq 0$ and $(z; q)_m \neq 0$ $(n,\ m \in \mathbb{N})$

$$\left(1+\sum_{k=2}^{\infty} A_k z^{k-1}\right)(-z; q)_n (z; q)_m \neq 0,\ (z \in \mathcal{E},\ n,\ m \in \mathbb{N}). \tag{3.11}$$

If (3.11) holds true, then we have

$$1+\sum_{k=2}^{\infty} A_k z^{k-1} \neq 0,$$

which is the relation (2.9) of the Theorem 2.1.

Equation (3.11) is equivalent to

$$\left(1+\sum_{k=2}^{\infty} A_k z^{k-1}\right)\left(\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} z^k\right)\left(\sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} z^k\right) \neq 0, \tag{3.12}$$

Using the Cauchy product of the first two factors, we can write the expression (3.12), as

$$\left(1+\sum_{k=2}^{\infty} C_k z^{k-1}\right)\left(\sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} z^k\right) \neq 0, \tag{3.13}$$

where,

$$C_k=\sum_{j=1}^{k}(-1)^{k-j} A_j \begin{bmatrix} n \\ k-j \end{bmatrix}_q q^{\frac{k-j(k-j-1)}{2}}.$$

Further, by applying the Cauchy product again in (3.13), we find that

$$1+\sum_{k=2}^{\infty}\left(\sum_{i=1}^{k} C_i \begin{bmatrix} m \\ k-i \end{bmatrix}_q q^{\frac{k-i(k-i-1)}{2}}\right) z^{k-1} \neq 0.$$

Equivalently, we have,

$$1+\sum_{k=2}^{\infty}\left[\sum_{i=1}^{k}\left(\sum_{j=1}^{i}(-1)^{i-j} A_j \begin{bmatrix} n \\ i-j \end{bmatrix}_q q^{\frac{i-j(i-j-1)}{2}}\right)\begin{bmatrix} m \\ k-i \end{bmatrix}_q q^{\frac{k-i(k-i-1)}{2}}\right] z^{k-1} \neq 0, \quad z \in \mathcal{U}.$$

If $f \in \mathcal{A}$ satisfies the following inequality

$$\sum_{k=2}^{\infty}\left|\sum_{i=1}^{k}\left(\sum_{j=1}^{i}(-1)^{i-j} A_j \begin{bmatrix} n \\ i-j \end{bmatrix}_q q^{\frac{i-j(i-j-1)}{2}}\right)\begin{bmatrix} m \\ k-i \end{bmatrix}_q q^{\frac{k-i(k-i-1)}{2}}\right| \leq 1, \quad z \in \mathcal{U},$$

that is, if

$$\frac{1}{(B-A)}\sum_{k=2}^{\infty}\left|\sum_{i=1}^{k}\left(\sum_{j=1}^{i}(-1)^{i-j}[([j]_q-1)+(B[j]_q-A)\zeta]\begin{bmatrix} n \\ i-j\end{bmatrix}_q q^{\frac{i-j(i-j-1)}{2}}\right)\begin{bmatrix} m \\ k-i\end{bmatrix}_q q^{\frac{k-i(k-i-1)}{2}}\right|$$
$$\leq \frac{1}{(B-A)}\sum_{k=2}^{\infty}\left[\left|\sum_{i=1}^{k}\left(\sum_{j=1}^{i}(-1)^{i-j}([j]_q-1)a_j\begin{bmatrix} n \\ i-j\end{bmatrix}_q q^{\frac{i-j(i-j-1)}{2}}\right)\begin{bmatrix} m \\ k-i\end{bmatrix}_q q^{\frac{k-i(k-i-1)}{2}}\right|\right.$$
$$\left.+|\zeta|\left|\sum_{i=1}^{k}\left(\sum_{j=1}^{i}(-1)^{i-j}(B[j]_q-A)a_j\begin{bmatrix} n \\ i-j\end{bmatrix}_q q^{\frac{i-j(i-j-1)}{2}}\right)\begin{bmatrix} m \\ k-i\end{bmatrix}_q q^{\frac{k-i(k-i-1)}{2}}\right|\right]\leq 1,$$

for $-1\leq B<A\leq 1,\ 0<q<1,\ \zeta\in\mathbb{C}, |\zeta|=1,$ then $f\in\mathcal{S}_q(A,\ B)$ which establishes the result. ■

Theorem 3.2. If $f\in\mathcal{A}$ satisfies the following condition

$$\sum_{k=2}^{\infty}\left[\left|\sum_{i=1}^{k}\left(\sum_{j=1}^{i}(-1)^{i-j}[j_q([j]_q-1)\begin{bmatrix} n \\ i-j\end{bmatrix}_q q^{\frac{(i-j)(i-j-1)}{2}}a_j\right)\begin{bmatrix} m \\ k-i\end{bmatrix}_q q^{\frac{k-i(k-i-1)}{2}}\right|+\right.$$
$$\left.\left|\sum_{i=1}^{k}\left(\sum_{j=1}^{i}(-1)^{i-j}[j]_q([j]_qB-A)\begin{bmatrix} n \\ i-j\end{bmatrix}_q q^{\frac{(i-j)(i-j-1)}{2}}a_j\right)\begin{bmatrix} m \\ k-i\end{bmatrix}_q q^{\frac{k-i(k-i-1)}{2}}\right|\right]\leq B-A,$$

where, $n,\ m\in\mathbb{N}$ and $-1\leq B<A\leq 1,\ 0<q<1$, then, $f\in\mathcal{K}_q(A,\ B)$.

Theorem 3.3. If $f\in\ \mathcal{A}$ satisfies the following condition

$$\sum_{k=2}^{\infty}\left[\left|\sum_{i=1}^{k}\left(\sum_{j=1}^{i}(-1)^{i-j}e^{i\lambda}([j]_q-1)\begin{bmatrix} n \\ i-j\end{bmatrix}_q q^{\frac{i-j(i-j-1)}{2}}\right)\begin{bmatrix} m \\ k-i\end{bmatrix}_q q^{\frac{k-i(k-i-1)}{2}}\right|+\right.$$
$$\left.\left|\sum_{i=1}^{k}\left(\sum_{j=1}^{i}(-1)^{i-j}[Be^{i\lambda}[j]_q-A\cos\lambda+Bi\sin\lambda]\begin{bmatrix} n \\ i-j\end{bmatrix}_q q^{\frac{i-j(i-j-1)}{2}}\right)\begin{bmatrix} m \\ k-i\end{bmatrix}_q q^{\frac{k-i(k-i-1)}{2}}\right|\right] \tag{3.14}$$

$\leq\cos\lambda(A-B)$, where,

$$m,\ n\in\mathbb{N}\text{ and }-1\leq B<A\leq 1,\ 0<q<1,$$

then $f\in\mathcal{S}_q^{\lambda}(A,\ B)$.

Proof. Applying the same method as in the proof of the Theorem 3.1, we see that f is in the class $\mathcal{S}_q^{\lambda}(A,\ B)$ if

$$\sum_{k=2}^{\infty}\left|\sum_{i=1}^{k}\left(\sum_{j=1}^{i}(-1)^{i-j}B_j\begin{bmatrix} n \\ i-j\end{bmatrix}_q q^{\frac{i-j(i-j-1)}{2}}\right)\begin{bmatrix} m \\ k-i\end{bmatrix}_q q^{\frac{k-i(k-i-1)}{2}}\right|\leq 1. \tag{3.15}$$

By using the Theorem 2.3, it follows from the inequality (3.15) that

$$\frac{1}{\cos\lambda(B-A)}\sum_{k=2}^{\infty}\left[\left|\sum_{i=1}^{k}\left(\sum_{j=1}^{i}(-1)^{i-j}[e^{i\lambda}([j]_q-1)]a_j\begin{bmatrix}n\\ i-j\end{bmatrix}_q q^{\frac{i-j(i-j-1)}{2}}\right)\begin{bmatrix}m\\ k-i\end{bmatrix}_q q^{\frac{k-i(k-i-1)}{2}}\right|\right.$$
$$\left.+|\zeta|\left|\sum_{i=1}^{k}\left(\sum_{j=1}^{i}(-1)^{i-j}[Be^{i\lambda}[j]_q-A\cos\lambda+Bi\sin\lambda]a_j\begin{bmatrix}n\\ i-j\end{bmatrix}_q q^{\frac{i-j(i-j-1)}{2}}\right)\begin{bmatrix}m\\ k-i\end{bmatrix}_q q^{\frac{k-i(k-i-1)}{2}}\right|\right]$$
$$\leq 1,\ \ ; -\frac{\pi}{2}<\lambda<\frac{\pi}{2}; |\zeta|=1),$$

which implies that, if f satisfies the (3.14), then, $f \in \mathcal{S}_q^{\lambda}(A,\ B)$. ■

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