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On Einstein velocity addition under some Pseudo-fields

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Abstract

In this article we have introduced the notion of pseudo-field and pseudo force. We have also obtained the explicit formula of Einstein velocity addition under some continuous and discontinuous pseudo-fields.

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1. Introduction

Lal has proved that every right loop S with identity can be embedded as a right transversal to a subgroup G_S of a group $G_S S$ with some universal property [3]. It is also observed that any right loop (S, o) can be deformed to another right loop with the help of a map $g : S \to G_S,\ g(e) = 1$ [5]. Since the right loop $(\mathbb{R}^n_1, \oplus)$ [6], where $\oplus$ is the relativistic velocity addition, can be viewed as a right transversal to the subgroup $SO(n) \times SO(1)$ $(O(n) \times O(1))$ in the positive special Lorentz group $\mathbf{SO(n, 1)}^+$ [5] (in the positive Lorentz group $O(n, 1)^+$ [2]), therefore we can deform the Einstein velocity addition $\oplus$ with the help of a map $g : \mathbb{R}^n_1 \to SO(n) \times SO(1)$ with $g(\mathbf{0}) = I_{n+1}$.

In this article, first we give the notion of pseudo-field and pseudo force, and then we discuss Einstein velocity addition under some continuous and discontinuous pseudo-fields.

2. Preliminaries

Most of the results and discussions are taken from [5].

Let S be a right transversal to a subgroup H in a group G. *Throughout the article, a right transversal will be assumed to contain the identity.* Then (S, o) is a right loop, where $\{xoy\} = S \cap Hxy,\ x, y \in S$. Conversely every right loop can be embedded as a right transversal in a group with some universal property [3].

Let S be a fixed right transversal to a subgroup H in a group G. In [5], we have observe that every right transversal to H in G determines and is determined uniquely by a map $g : S \to H$ such that $g(e) = e$, the identity of G. The right transversal S_g determined by a map $g : S \to H$ is given by $S_g = \{g(x)x | x \in S\}$. S and S_g are right quasigroups with identities with respect to the operation o on S and o' on S_g given by

$$\{xoy\} = Hxy \cap S$$

and

$$\{g(x)xo'g(y)y\} = S_g \cap Hg(x)xg(y)y$$

respectively. Further H acts on S from right through an action θ given by $\{x\theta h\} = Hxh \cap S,\ \forall x \in S, h \in H$. As we shall be dealing with right transversals and right actions, it is convenient for us to follow the convention

$$(gf)(x) = x\theta(gf) = (x\theta g)\theta f = f(g(x))$$

for composition of maps.

By Proposition 4.1 [5], we have observed that the right loop (S_g, o') is isomorphic to the right loop (S, o_g) where o_g is the operation given by

$$xo_g y \quad = \quad x\theta g(y)oy. \tag{2.1}$$

If s is a section then the image $s(G/^r H)$ is a right transversal to H in G. Conversely every right transversal S determines a section s given by $\{s(Hx)\} = S \cap Hx$. Let H be a closed subgroup of a topological group G and S a right transversal to H in G. Suppose that the section s from the quotient space $G/^r H$ to G given by $\{s(Hg)\} = S \cap Hg$ is continuous. Then the binary operation o on S given by $\{xoy\} = S \cap Hxy$ and the map $\chi : S \times S \to S$ given by $\chi(x, y)\ ox = y$ are continuous (Here S is given the subspace topology) (Proposition 4.4, [5]).

It is also observed that if S is the image of a continuous section $s : G/^r H \to G$. Then the right transversal $S_g = \{g(x)x \mid x \in S\}$ corresponding to the map $g : S \to H$ with $g(e) = e$ is the image of a continuous section if and only if g is continuous from S (with subspace topology) to H (Proposition 4.5, [5]).

Note that the continuity of o and χ does not imply the continuity of the corresponding section. Consider, for example, the group $\mathbb{R} \times \mathbb{R}$ with usual addition. Give the product topology on G by taking the usual topology on the first factor and taking the discrete topology on the second factor. Then G is a topological group. Consider the subgroup $H = \{0\} \times \mathbb{R}$ and the right transversal $S = \mathbb{R} \times \{0\}$. The quotient space can be identified with the topological space $\mathbb{R}$ with usual topology. $S = \mathbb{R} \times \{0\}$ is the right transversal to H in G which is the image of a continuous section $s : \mathbb{R} \to G$ given by $s(a) = (a, o)$. Since the only continuous map g from S to H with $g((0, 0)) = (0, 0)$ is the constant map, it follows that S is the only transversal which is image of a continuous section. However $\Delta = \{(a, a) \mid a \in \mathbb{R}\}$ is also a right transversal to H in G and the transversal operation induced on Δ is the usual addition, with respect to which it is a topological subgroup of G.

3. Einstein velocity addition under Pseudo-fields

Consider open unit disc $\mathbb{R}_1^n$, where $\mathbb{R}_1^n = \{\mathbf{v} \in \mathbb{R}^n | \ ||\mathbf{v}|| < 1\}$.

Definition 3.1. A map $g : \mathbb{R}_1^n \to O(n)$ with $g(\mathbf{0}) = I_n$ will be called a Pseudo-field and the image $g(\mathbf{v})$ of $\mathbf{v} \in \mathbb{R}_1^n$ will be called the pseudo force corresponding to velocity $\mathbf{v}$.

The Einstein velocity addition $\oplus$ [5] is given by

$$\mathbf{v} \oplus \mathbf{w} = \frac{\mathbf{v}+\mathbf{w}}{1+\mathbf{v}\mathbf{w}^t} + \frac{\gamma_{\mathbf{w}}}{1+\gamma_{\mathbf{w}}} \frac{\mathbf{v}\mathbf{w}^t\mathbf{w} - \mathbf{w}\mathbf{w}^t\mathbf{v}}{1+\mathbf{v}\mathbf{w}^t}. \tag{3.1}$$

Indeed, we have

Proposition 3.2. [5] The Lorentz boost map $\mathbf{B}$ [2] from $\mathbb{R}_1^n$ to $\mathcal{S}$ given by

$$\mathbf{B}(\mathbf{v}) = \begin{pmatrix} I_n + \frac{\gamma_{\mathbf{v}}^2}{1+\gamma_{\mathbf{v}}}\mathbf{v}^{\mathbf{t}}\mathbf{v} & \gamma_{\mathbf{v}}\mathbf{v}^{\mathbf{t}} \\ \gamma_{\mathbf{v}}\mathbf{v} & \gamma_{\mathbf{v}} \end{pmatrix} \tag{3.2}$$

is an isomorphism from $(\mathbb{R}_1^n, \oplus)$ to $(\mathcal{S}, o)$ which is also an equivariant map from $(\mathbb{R}_1^n, *)$ to $(\mathcal{S}, \theta)$ in the sense that $\mathbf{B}(\mathbf{v} * U) = \mathbf{B}(\mathbf{v})\theta U,\ U \in SO(n) \times SO(1)$.

Consider the right transversal $\mathcal{S} = \mathbf{SO(n, 1)}^+ \cap \mathcal{H}^+(n+1)$ to the subgroup $SO(n) \times SO(1)$ in the group $\mathbf{SO(n, 1)}^+$. Let ϕ be a map from $\mathbb{R}_1^n$ to $SO(n) \times SO(1)$ with $\phi(\mathbf{0}) = I_{n+1}$. Then, we have a map $g = \phi o \mathbf{B}^{-1}$ from $\mathcal{S}$ to $SO(n) \times SO(1)$ with $g(I_{n+1}) = I_{n+1}$. The operation o_g on $\mathcal{S}$ is given by $x o_g y = x\theta g(y) o y$, where θ is given by $x\theta h = h^{-1}xh$ for $x \in \mathcal{S}$ and $h \in SO(n) \times SO(1)$. We call o_g as the the deformation of operation o under the map g. Thus, we have the following:

Theorem 3.3. Consider $\mathbb{R}_1^n$ as right transversal to the subgroup $SO(n) \times SO(1)$ in the group $\mathbf{SO(n, 1)}^+$ through the Lorentz boost map $\mathbf{B}$. Then any deformation of the Einstein addition $\oplus$ is $\oplus_g$ for a unique map $g : \mathbb{R}_1^n \to SO(n)$ with $g(\mathbf{0}) = I_n$ where $\oplus_g$ is given by $\mathbf{u} \oplus_g \mathbf{v} = \mathbf{u}g(\mathbf{v}) \oplus \mathbf{v},\ \mathbf{u}, \mathbf{v} \in \mathbb{R}_1^n$. The operation $\oplus_g$ is continuous provided that g is continuous.

Proof. Let $\phi : \mathbb{R}_1^n \to SO(n) \times SO(1)$ be a map given by $\phi(\mathbf{v}) = Diag\,(g(\mathbf{v}), 1)$, where g is a map from $\mathbb{R}_1^n$ to $SO(n)$ with $g(\mathbf{0}) = I_n$. Then ϕ determines a map $\psi = \phi o \mathbf{B}^{-1}$ from $\mathcal{S} = \mathbf{SO(n, 1)}^+ \cap \mathcal{H}^+(n+1)$ to $SO(n) \times SO(1)$. The deformation o_ψ of the operation o on $\mathcal{S}$ (as described above) is given by

$$\begin{aligned} \mathbf{B}(\mathbf{v}) o_\psi \mathbf{B}(\mathbf{w}) &= \mathbf{B}(\mathbf{v})\theta\psi\left(\mathbf{B}(\mathbf{w})\right) o\mathbf{B}(\mathbf{w}) \\ &= \mathbf{B}(\mathbf{v})\theta\phi(\mathbf{w}) o\mathbf{B}(\mathbf{w}) \\ &= \mathbf{B}\left(\mathbf{v} * \phi(\mathbf{w})\right) o\mathbf{B}(\mathbf{w}) \qquad \text{(by Proposition 3.2)} \\ &= \mathbf{B}(\mathbf{v}g(\mathbf{w})) o\mathbf{B}(\mathbf{w}) \\ &= \mathbf{B}\left(\mathbf{v}g(\mathbf{w}) \oplus \mathbf{w}\right) \qquad \text{(by Proposition 3.2)} \end{aligned}$$

The velocity addition $\oplus_g$ corresponding to pseudo-field g, thus obtained, is given by

$$\mathbf{v} \oplus_g \mathbf{w} = \mathbf{v}g(\mathbf{w}) \oplus \mathbf{w}. \tag{3.3}$$

The last assertion is evident because the Lorentz boost map is homeomorphism from $\mathbb{R}_1^n$ to $\mathcal{S}$ and the section s with image $\mathcal{S}$ is continuous. ∎

In other words, it asserts that if $g : \mathbb{R}_1^n \to SO(n) \hookrightarrow O(n)$ is a pseudo field on $\mathbb{R}_1^n$, then the sum of two velocities $\mathbf{v}, \mathbf{w}$ will be $\mathbf{v} \oplus_g \mathbf{w} = \mathbf{v}\theta g(\mathbf{w}) \oplus \mathbf{w}$ instead of $\mathbf{v} \oplus \mathbf{w}$. We call $\oplus_g$ as the **Einstein velocity addition under pseudo field g.**

4. Einstein addition under a Continuous pseudo-field

Since there are several embeddings of $\mathbb{R}_1^n$ in $\mathfrak{so}(n)$, the set of $n \times n$ skew-symmetric matrices, depending upon the choices of the n strictly upper diagonal places for the possible non-zero entries of the matrix. Therefore these composed with the exponential map $\exp : \mathfrak{so}(n) \to SO(n)$ [1] given by

$$\exp A = I_n + \frac{A}{1!} + \frac{A^2}{2!} + \ldots + .$$

will give several continuous maps (and also embeddings) g from $\mathbb{R}_1^n$ to $SO(n)$. Thus, it determines several pseudo-fields on $\mathbb{R}_1^n$.

In this section we obtain the explicit formula for Einstein addition corresponding to a pseudo-field $g : \mathbb{R}_1^n \to SO(n)$ given by $g(\mathbf{v}) = \exp M(\mathbf{v})$, where

$$M(\mathbf{v}) = M((v_1, v_2, v_3, \ldots, v_n))$$

$$= \begin{pmatrix} 0 & 0 & \ldots & 0 & v_1 & v_2 \\ 0 & 0 & \ldots & \ldots & 0 & v_3 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -v_1 & 0 & \ldots & \ldots & 0 & v_n \\ -v_2 & -v_3 & \ldots & \ldots & -v_n & 0 \end{pmatrix}. \tag{4.1}$$

The case $n = 3$ is special and needs special attention. It may also be noted that the subgroup generated by the image $g(\mathbb{R}_1^3)$ is $SO(3)$.

Proposition 4.1. Let M be a map from $\mathbb{R}_1^3$ to $\mathfrak{so}(3)$ given by

$$M((a, b, c)) = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}.$$

Then $\exp M((a,b,c))$ is

$$\begin{cases} I_3 \text{ if } (a,b,c)=(0,0,0) \\ \begin{pmatrix} \frac{c^2+(\lambda^2-c^2)\cos\lambda}{\lambda^2} & \frac{bc(\cos\lambda-1)+a\lambda\sin\lambda}{\lambda^2} & \frac{ac(1-\cos\lambda)+b\lambda\sin\lambda}{\lambda^2} \\ \frac{bc(\cos\lambda-1)-a\lambda\sin\lambda}{\lambda^2} & \frac{b^2+(\lambda^2-b^2)\cos\lambda}{\lambda^2} & \frac{ab(\cos\lambda-1)+c\lambda\sin\lambda}{\lambda^2} \\ \frac{ac(1-\cos\lambda)-b\lambda\sin\lambda}{\lambda^2} & \frac{ab(\cos\lambda-1)-c\lambda\sin\lambda}{\lambda^2} & \frac{a^2+(\lambda^2-a^2)\cos\lambda}{\lambda^2} \end{pmatrix} \text{ otherwise,} \end{cases}$$

where $\lambda=+\sqrt{a^2+b^2+c^2}$.

Proof. Let $\mathbf{v}=(a,b,c)\in\mathbb{R}^3_1$. If $\mathbf{v}=\mathbf{0}$ then $\exp M(\mathbf{v})=I_3$. Suppose that $\mathbf{v}\neq\mathbf{0}$ and $\lambda=+\sqrt{a^2+b^2+c^2}$. The eigen values of $M(\mathbf{v})$ are precisely 0, λi and $-\lambda i$. Using the unitary reduction, we obtain a unitary matrix $U_{\mathbf{v}}$ given by

$$U_{\mathbf{v}}=\begin{cases} \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 1 \\ 0 & i & -i \\ \sqrt{2} & 0 & 0 \end{pmatrix} & \text{if } \lambda=a \\ \frac{1}{\lambda\sqrt{2}}\begin{pmatrix} c\sqrt{2} & -\frac{ac+i\lambda b}{\sqrt{\lambda^2-a^2}} & \frac{-ac+i\lambda b}{\sqrt{\lambda^2-a^2}} \\ -b\sqrt{2} & \frac{ab-i\lambda c}{\sqrt{\lambda^2-a^2}} & \frac{ab+i\lambda c}{\sqrt{\lambda^2-a^2}} \\ a\sqrt{2} & \sqrt{\lambda^2-a^2} & \sqrt{\lambda^2-a^2} \end{pmatrix} & \text{otherwise} \end{cases}$$

such that

$$M(\mathbf{v})=U_{\mathbf{v}}Diag\,(0,\lambda i,-\lambda i)\,(\overline{U_{\mathbf{v}}})^t.$$

This in turn gives

$$\begin{aligned} \exp\,(M(\mathbf{v})) &= \exp\left(U_{\mathbf{v}}Diag\,(0,\lambda i,-\lambda i)\,(\overline{U_{\mathbf{v}}})^t\right) \\ &= U_{\mathbf{v}}Diag\left(1,e^{i\lambda},e^{-i\lambda}\right)(\overline{U_{\mathbf{v}}})^t \\ &= \begin{pmatrix} \frac{c^2+(\lambda^2-c^2)\cos\lambda}{\lambda^2} & \frac{bc(\cos\lambda-1)+a\lambda\sin\lambda}{\lambda^2} & \frac{ac(1-\cos\lambda)+b\lambda\sin\lambda}{\lambda^2} \\ \frac{bc(\cos\lambda-1)-a\lambda\sin\lambda}{\lambda^2} & \frac{b^2+(\lambda^2-b^2)\cos\lambda}{\lambda^2} & \frac{ab(\cos\lambda-1)+c\lambda\sin\lambda}{\lambda^2} \\ \frac{ac(1-\cos\lambda)-b\lambda\sin\lambda}{\lambda^2} & \frac{ab(\cos\lambda-1)-c\lambda\sin\lambda}{\lambda^2} & \frac{a^2+(\lambda^2-a^2)\cos\lambda}{\lambda^2} \end{pmatrix} \end{aligned} \tag{4.2}$$

■

Corollary 4.2. Let $(a,b,c)\in\mathbb{R}^3_1\setminus\{\mathbf{0}\}$. Then $\exp M((a,b,c))$ is a rotation about the axis $\mathbf{r}=t[c,-b,a],\ t\in\mathbb{R}$; through an angle $+\sqrt{a^2+b^2+c^2}$.

Proof. Let $\mathbf{v} = (a, b, c)$. Since $[c, -b, a]$ is an eigen vector of $M(\mathbf{v})$ corresponding to an eigen value 0, it is also an eigen vector of $\exp M(\mathbf{v})$ corresponding to an eigen value 1. This means that $\exp M(\mathbf{v})$ is a rotation about the axis $\mathbf{r} = t[c, -b, a],\ t \in \mathbb{R}$. Further, since trace of $\exp M(\mathbf{v})$ is $1 + 2\cos\lambda$, it is a rotation with an angle λ. ■

Corollary 4.3. Let $g : \mathbb{R}^3_1 \to SO(3)$ be a continuous pseudo-field given by $g(\mathbf{v}) = \exp M(\mathbf{v})$. Then Einstein addition $\oplus_g$ on $\mathbb{R}^3_1$ under pseudo-field g is given by

$$\mathbf{u} \oplus_g \mathbf{v} = \begin{cases} \mathbf{u} & \text{if } \mathbf{v} = \mathbf{0} \\ \mathbf{u} \exp M(\mathbf{v}) \oplus \mathbf{v} & \text{otherwise.} \end{cases}$$

More concretely, $\mathbf{u} \exp M(\mathbf{v}) \oplus \mathbf{v}$ is given by

$$\frac{1}{1 + aJ_1 + bJ_2 + cJ_3} \begin{pmatrix} J_1 + a + \frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\{(a^2 - \lambda^2)J_1 + abJ_2 + acJ_3\} \\ J_2 + b + \frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\{baJ_1 + (b^2 - \lambda^2)J_2 + bcJ_3\} \\ J_3 + c + \frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\{caJ_1 + cbJ_2 + (c^2 - \lambda^2)J_3\} \end{pmatrix}^t$$

where $\mathbf{v} = (a, b, c)$, $\lambda = +\sqrt{a^2 + b^2 + c^2}$,

$$J_1 = \frac{c^2 + (\lambda^2 - c^2)\cos\lambda}{\lambda^2} u_1 + \frac{bc(\cos\lambda - 1) - a\lambda\sin\lambda}{\lambda^2} u_2 + \frac{ac(1 - \cos\lambda) - b\lambda\sin\lambda}{\lambda^2} u_3,$$

$$J_2 = \frac{bc(\cos\lambda - 1) + a\lambda\sin\lambda}{\lambda^2} u_1 + \frac{b^2 + (\lambda^2 - b^2)\cos\lambda}{\lambda^2} u_2 + \frac{ab(\cos\lambda - 1) - c\lambda\sin\lambda}{\lambda^2} u_3,$$

and

$$J_3 = \frac{ac(1 - \cos\lambda) + b\lambda\sin\lambda}{\lambda^2} u_1 + \frac{ab(\cos\lambda - 1) + c\lambda\sin\lambda}{\lambda^2} u_2 + \frac{a^2 + (\lambda^2 - a^2)\cos\lambda}{\lambda^2} u_3.$$

Proof. Follows from Eq. 4.2 and Eq. 3.1. ■

Proposition 4.4. Let $\mathbf{v} = (v_1, v_2, v_3, v_4)$ in $\mathbb{R}^4_1$. Then $\exp M(\mathbf{v})$ is either a rotation in the plane W leaving $W^\perp$ fixed or composition of two rotations, one in the two dimensional plane W_1 through an angle α and the other in the two dimensional plane W_2 through an angle β respectively, where W_1, W_2, α and β are given by

$$W_1 = \left\langle \left\{ \left(\frac{v_1 v_4}{v_1^2 - \alpha^2}, 0, -\frac{v_1 v_2}{v_1^2 - \alpha^2}, 1 \right), \left(\frac{\alpha v_2}{v_1^2 - \alpha^2}, \frac{-v_3}{\alpha}, \frac{\alpha v_4}{v_1^2 - \alpha^2}, 0 \right) \right\} \right\rangle$$

$$W_2 = \left\langle \left\{ \left(\frac{v_1 v_4}{v_1^2 - \beta^2}, 0, -\frac{v_1 v_2}{v_1^2 - \beta^2}, 1 \right), \left(\frac{\beta v_2}{v_1^2 - \beta^2}, \frac{-v_3}{\beta}, \frac{\beta v_4}{v_1^2 - \beta^2}, 0 \right) \right\} \right\rangle.$$

and

$$\alpha^2 = \frac{\|\mathbf{v}\|^2 - \sqrt{\|\mathbf{v}\|^4 - 4v_1^2v_3^2}}{2}, \qquad \beta^2 = \frac{\|\mathbf{v}\|^2 + \sqrt{\|\mathbf{v}\|^4 - 4v_1^2v_3^2}}{2} \tag{4.3}$$

Proof. If $\mathbf{v} = \mathbf{0}$ then $\exp M(\mathbf{v}) = I_4$. Suppose that $\mathbf{v} \neq \mathbf{0}$. The eigen values of $M(\mathbf{v})$ are $\pm\alpha i$, $\pm\beta i$, where α and β are given by the Eq. 4.3.

Clearly $\alpha = 0$ if and only if either $v_1 = 0$ or $v_3 = 0$. If $v_1 = v_3 = 0$ then eigen vectors of $M(\mathbf{v})$ corresponding to eigen values βi and $-\beta i$ are

$$\mathbf{u}_1 = \left(v_2, 0, v_4, \sqrt{v_2^2 + v_4^2}\, i\right) \text{ and } \mathbf{u}_2 = \left(v_2, 0, v_4, -\sqrt{v_2^2 + v_4^2}\, i\right).$$

This in turn shows that $\exp M(\mathbf{v})$ is a rotation in the plane

$$W = \langle\{(v_2, 0, v_4, 0), (0, 0, 0, 1)\}\rangle$$

through an angle $\beta = \sqrt{v_2^2 + v_4^2}$ leaving $W^\perp$ fixed.

If $v_1 = 0, v_3 \neq 0$. Then eigen vectors of $M(\mathbf{v})$ corresponding to eigen values βi and $-\beta i$ are $\mathbf{u}_1 = (v_2, v_3, v_4, \beta i)$ and $\mathbf{u}_2 = (v_2, v_3, v_4, -\beta i)$. This in turn shows that $\exp M(\mathbf{v})$ is a rotation in the plane

$$W = \langle\{(v_2, v_3, v_4, 0), (0, 0, 0, 1)\}\rangle$$

through an angle $\beta = \sqrt{v_2^2 + v_3^2 + v_4^2}$ leaving $W^\perp$ fixed.

Similarly, if $v_1 \neq 0, v_3 = 0$ then $\exp M(\mathbf{v})$ is a rotation in the plane W determined by vectors

$$\left(\frac{v_1v_4}{v_1^2 - \beta^2}, 0, -\frac{v_1v_2}{v_1^2 - \beta^2}, 1\right) \text{ and } \left(\frac{\beta v_2}{v_1^2 - \beta^2}, 0, -\frac{\beta v_4}{v_1^2 - \beta^2}, 1\right)$$

through an angle $\beta = \sqrt{v_1^2 + v_2^2 + v_4^2}$ leaving $W^\perp$ fixed.

Suppose that $v_1, v_3 \neq 0$. Then $\alpha, \beta \neq 0$. Observe that $\alpha, \beta \neq v_1$. The eigen vectors of $M(\mathbf{v})$ corresponding to eigen values αi, $-\alpha i$, βi and $-\beta i$ are

$$\mathbf{u}_1 = \left(\frac{v_1v_4 + \alpha v_2 i}{v_1^2 - \alpha^2}, \frac{-v_3 i}{\alpha}, -\frac{v_1v_2 - \alpha v_4 i}{v_1^2 - \alpha^2}, 1\right),$$

$$\mathbf{u}_2 = \left(\frac{v_1v_4 - \alpha v_2 i}{v_1^2 - \alpha^2}, \frac{v_3 i}{\alpha}, -\frac{v_1v_2 + \alpha v_4 i}{v_1^2 - \alpha^2}, 1\right),$$

$$\mathbf{u}_3 = \left(\frac{v_1v_4 + \beta v_2 i}{v_1^2 - \beta^2}, \frac{-v_3 i}{\beta}, -\frac{v_1v_2 - \beta v_4 i}{v_1^2 - \beta^2}, 1\right)$$

and

$$\mathbf{u}_4 = \left(\frac{v_1 v_4 - \beta v_2 i}{v_1^2 - \beta^2},\ \frac{v_3 i}{\beta},\ -\frac{v_1 v_2 + \beta v_4 i}{v_1^2 - \beta^2},\ 1 \right)$$

respectively. The mutually orthogonal real planes determined by $\mathbf{u}_1, \mathbf{u}_2$ and $\mathbf{u}_3, \mathbf{u}_4$ are given by

$$W_1 = \left\langle \left\{ \left(\frac{v_1 v_4}{v_1^2 - \alpha^2}, 0, -\frac{v_1 v_2}{v_1^2 - \alpha^2}, 1 \right), \left(\frac{\alpha v_2}{v_1^2 - \alpha^2}, \frac{-v_3}{\alpha}, \frac{\alpha v_4}{v_1^2 - \alpha^2}, 0 \right) \right\} \right\rangle$$

and

$$W_2 = \left\langle \left\{ \left(\frac{v_1 v_4}{v_1^2 - \beta^2}, 0, -\frac{v_1 v_2}{v_1^2 - \beta^2}, 1 \right), \left(\frac{\beta v_2}{v_1^2 - \beta^2}, \frac{-v_3}{\beta}, \frac{\beta v_4}{v_1^2 - \beta^2}, 0 \right) \right\} \right\rangle.$$

■

Thus, *the Einstein addition* $\oplus_g$ *on* $\mathbb{R}^4_1$ *under pseudo-field g is given by*

$$\mathbf{u} \oplus_g \mathbf{v} = \begin{cases} \mathbf{u} & \text{if } \mathbf{v} = \mathbf{0} \\ \mathbf{u} \exp M(\mathbf{v}) \oplus \mathbf{v} & \text{otherwise} \end{cases}$$

where $exp\ M(\mathbf{v})$ *is described as above.*

More generally, for $n \geq 4$ we have the following:

Proposition 4.5. If $\mathbf{v} \neq \mathbf{0}$, then $\exp M(\mathbf{v})$ represents a rotation in a two dimensional plane through an angle ϕ in case $v_1 = 0$ or $\|\mathbf{v}\|^2 = v_1^2 + v_2^2 + v_n^2$ and otherwise it represents composition of two rotations in two orthogonal planes through angles θ and ϕ, where θ and ϕ are given by

$$\theta^2 = \frac{\|\mathbf{v}\|^2 - \sqrt{\|\mathbf{v}\|^4 - 4v_1^2 \left[\|\mathbf{v}\|^2 - v_1^2 - v_2^2 - v_n^2 \right]}}{2} \tag{4.4}$$

and

$$\phi^2 = \frac{\|\mathbf{v}\|^2 + \sqrt{\|\mathbf{v}\|^4 - 4v_1^2 \left[\|\mathbf{v}\|^2 - v_1^2 - v_2^2 - v_n^2 \right]}}{2} \tag{4.5}$$

respectively.

This prompts that the only four dimension is needed for group of motions; that is, the science has only four legs.

5. Einstein addition under a discontinuous Pseudo-field

Here we consider Einstein addition under a pseudo-field which is not continuous but still canonical and have geometric flavour. The usual transitive action of $SO(n)$ on $\mathbb{S}^{n-1}$ gives a natural map $g_0 : \mathbb{S}^{n-1} \to SO(n)$ (only discontinuity at $-\mathbf{e}_n$) given by

$$g_0(\mathbf{v}) = \tag{5.1}$$
$$\begin{cases} \begin{pmatrix} \mathbf{1} - \dfrac{\langle \mathbf{v}, \mathbf{e_1} \rangle^{\mathbf{2}}}{\mathbf{1} + \langle \mathbf{v}, \mathbf{e_n} \rangle} & -\dfrac{\langle \mathbf{v}, \mathbf{e_1} \rangle \langle \mathbf{v}, \mathbf{e_2} \rangle}{\mathbf{1} + \langle \mathbf{v}, \mathbf{e_n} \rangle} & \dots & \langle \mathbf{v}, \mathbf{e_1} \rangle \\ -\dfrac{\langle \mathbf{v}, \mathbf{e_1} \rangle \langle \mathbf{v}, \mathbf{e_2} \rangle}{\mathbf{1} + \langle \mathbf{v}, \mathbf{e_n} \rangle} & \mathbf{1} - \dfrac{\langle \mathbf{v}, \mathbf{e_2} \rangle^{\mathbf{2}}}{\mathbf{1} + \langle \mathbf{v}, \mathbf{e_n} \rangle} & \dots & \langle \mathbf{v}, \mathbf{e_2} \rangle \\ \dots & \dots & \dots & \dots \\ -\langle \mathbf{v}, \mathbf{e_1} \rangle & -\langle \mathbf{v}, \mathbf{e_2} \rangle & \dots & \langle \mathbf{v}, \mathbf{e_n} \rangle \end{pmatrix} & \text{if } \mathbf{v} \neq -\mathbf{e_n} \\ I_n & \text{if } \mathbf{v} = -\mathbf{e_n}. \end{cases}$$

This, in turn, determines a pseudo-field $g : \mathbb{R}^n_1 \to SO(n)$ given by

$$g(\mathbf{v}) = \begin{cases} I_n & \text{if } \mathbf{v} = \mathbf{0} \\ g_0\left(\dfrac{\mathbf{v}}{\|\mathbf{v}\|}\right) & \text{otherwise} \end{cases}. \tag{5.2}$$

Theorem 5.1. Let $g : \mathbb{R}^n_1 \to SO(n)$ be a pseudo-field given by Eq.(5.2). Then the Einstein addition $\oplus_g$ under pseudo-field g is given by

$$\mathbf{u} \oplus_g \mathbf{v} = \begin{cases} \mathbf{u}^* \oplus \mathbf{v} & \text{if } \mathbf{v} \neq \mathbf{0} \text{ and } \dfrac{\mathbf{v}}{||\mathbf{v}||} \neq -\mathbf{e_n} \\ \mathbf{u} \oplus \mathbf{v} & \text{if } \mathbf{v} \neq \mathbf{0} \text{ and } \dfrac{\mathbf{v}}{||\mathbf{v}||} = -\mathbf{e_n} \\ \mathbf{u} & \text{if } \quad \mathbf{v} = \mathbf{0} \end{cases} \tag{5.3}$$

where i^{th} component u_i^* of $\mathbf{u}^*$ is given by

$$u_i^* = \tag{5.4}$$
$$\begin{cases} \langle \mathbf{u}, \mathbf{e_i} \rangle - \dfrac{\left\langle \frac{\mathbf{v}}{||\mathbf{v}||}, \mathbf{e_i} \right\rangle}{\mathbf{1} + \left\langle \frac{\mathbf{v}}{||\mathbf{v}||}, \mathbf{e_n} \right\rangle} \left[\left\langle \mathbf{u}, \dfrac{\mathbf{v}}{||\mathbf{v}||} \right\rangle + \langle \mathbf{u}, \mathbf{e_n} \rangle \right] & \text{for } i = 1, 2, \dots, n-1 \\ \left\langle \mathbf{u}, \dfrac{\mathbf{v}}{||\mathbf{v}||} \right\rangle & \text{for } i = n. \end{cases}$$

The operation $\oplus_g$ is continuous everywhere except on the line segment $\{-t\mathbf{e_n} \mid 0 \leq t < 1\}$.

Proof. The Einstein addition $\oplus_g$ is given by $\mathbf{u} \oplus \mathbf{v} = \mathbf{u}g(\mathbf{v}) \oplus \mathbf{v}$. The case $\mathbf{v} = \mathbf{0}$ is trivial. Suppose that $\mathbf{v} \neq \mathbf{0}$. If $\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = -\mathbf{e}_n$ then $g_0\left(\dfrac{\mathbf{v}}{\|\mathbf{v}\|}\right) = I_n$ and so $\mathbf{u} \oplus_g \mathbf{v} = \mathbf{u} \oplus \mathbf{v}$.

Suppose that $\frac{\mathbf{v}}{\|\mathbf{v}\|} \neq -\mathbf{e}_n$. Using Eq. (5.1) and Eq. (5.2), the i^{th} component u_i^* of $\mathbf{u}^* = \mathbf{u}g_0\left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right)$ is given by Eq. (5.4). The discontinuity of g_0 at $-\mathbf{e_n}$ implies the discontinuity of g on the line segment $\{-t\mathbf{e_n} \mid 0 \leq t < 1\}$. Thus the operation $\oplus_g$ is continuous everywhere except on the line segment $\{-t\mathbf{e_n} \mid 0 \leq t < 1\}$. ■

References

[1] M. L. Curtis, *Matrix Groups*, Springer-Verlag, 1984.

[2] H. Kiechle, *Theory of K-loops*, Lecture notes, A.M.S., **1778**(2002).

[3] R. Lal, *Transversals in groups*, Journal of algebra, **181**, 70–81 (1996).

[4] S. Lang, *Algebra*, Rev. Third Edit., second Indian reprint, (2006).

[5] R. Lal and A. C. Yadav, *Topological right gyrogroups and gyrotransversals*, Communications in Algebra **41:9**, 3559–3575 (2013).

[6] A. A. Ungar, *Weakly associative groups*, Result. Math. **17**, 149–168 (1990).