Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 9 (2017), pp. 6539-6552

http://www.ripublication.com

Fractional Order Theory of Thermal Stresses to A 2 D Problem for a Thin Hollow Circular Disk

J. J. Tripathi[1], S. D. Warbhe[2], K. C. Deshmukh[3] and J. Verma[4]

.

[1]Department of Mathematics, Dr. Ambedkar College, Deekshabhoomi, Nagpurr-440010, Maharastra, India.

[2]Department of Mathematics, Laxminarayan Institute of Technology, Nagpur-440033, Maharastra, India.

[3]Departmentof Mathematics, R.T.M. Nagpur University, Nagpur-440033, Maharashtra, India

[4]Department of Applied Mathematics, Pillai HOC College of Engg & Tech , Rasayani, University of Mumbai 410207, Maharashtra, India.

Abstract

A quasi static uncoupled theory of thermoelasticity based on the heat conduction equation with the time fractional derivative of order alpha subjected to a time dependent heat flux at the outer boundary whereas the inner circular boundary is insulated in a 2 D problem of a thin hollow circular disk. The integral transform technique is used to find the temperature distribution in a physical domain with the help of Caputo type fractional derivative. The corresponding stresses are determined by using the displacement function. A mathematical model is constructed for a copper material. The thermoelastic stresses, temperature distribution and the displacement due to a time dependent heat flux is shown graphically and the effect of the fractional order parameter are discussed for the different values of alpha.

Keywords: Quasi-static, thermoelasticity, fractional order, Integral transform, Mittag-Leffler function.

INTRODUCTION

During the second half of the twentieth century, considerable amount of research in fractional calculus was published in engineering literature. Indeed, recent advances of fractional calculus are dominated by modern examples of applications in differential and integral equations, physics, signal processing, fluid mechanics, viscoelasticity, mathematical biology, and electrochemistry. There is no doubt that fractional calculus has become an exciting new mathematical method of solution of diverse problems in mathematics, science, and engineering. It is generally known that integer-order derivatives and integrals have clear physical and geometric interpretations. However, in case of fractional-order integration and differentiation, it is not so. Since the appearance of the idea of differentiation and integration of arbitrary (not necessary integer) order there was not any acceptable geometric and physical interpretation of these operations for more than 300 year. In [1], it is shown that geometric interpretation of fractional integration is "'Shadows on the walls' and its Physical interpretation is 'Shadows of the past' ".

The classical theory of thermoelasticity has aroused much interest in recent times due to its numerous applications in engineering discipline such as nuclear reactor design, high energy particle accelerators, geothermal engineering, advanced aircraft structure design, etc. The heat conduction of classical coupled theory of thermoelasticity is parabolic in nature and hence predicts infinite speed of propagation of heat waves. Clearly, this contradicts the physical observations. Hence, several non-classical theories such as, Lord-Shulman theory [2], Green Lindsay theory [3] have been proposed, in which the Fourier law and the parabolic heat conduction equation are replaced by more complicated equations, which are hyperbolic in nature predicting finite wave propagation. Green and Naghdi [4] developed the theory of thermoelasticity without energy dissipation. Chandrasekaraiah [5] gave review of thermoelasticity with second sound. Tripathi et al. [6-8] studied problems in generalized thermoelastic theories. Povstenko [9-14] studied problems of quasi static fractional thermoelasticity based on time fractional heat conduction equation in cylindrical coordinate for a long cylinder, an infinite solid with a long cylindrical cavity and a half space under various prescribed boundary conditions. The solutions are obtained employing integral transform technique. Recently, Tripathi et al. [15] studied a dynamic problem in fractional order thermoelasticity with finite wave speeds. In the last decade, study on Quasi-static thermoelasticity incorporating the time fractional derivative has gained momentum. Kulkarni et al. [16] discussed a quasi-static uncoupled thermoelastic problem in a thin hollow circular disk and discussed thermal stresses due to heat generation. Deshmukh et al. [17] studied quasi-static thermal deflection of a thin clamped hollow circular disk due to heat generation. A brief note on heat flow with arbitrary heating rates in a hollow cylinder was studied by Deshmukh et al. [18]. Raslan [19] studied the application of fractional order theory of thermoelasticity in a thick plate under axisymmetric temperature distribution. The fractional order theory of thermoelasticity was developed by Sherief et al. [20].Youssef and Abbas [21] solved a one dimensional problem of a elastic half space in context of fractional order generalized thermoelasticity. Chunbao Xiong and Ying Guo [22], studied effect of variable properties and moving heat source on magneto thermoelastic problem under fractional order thermoelasticity. Recently, Warbhe et al. [23] solved a fractional heat conduction problem in a thin circular plate with constant temperature distribution and associated thermal stresses within the context of quasi-static theory.

In this paper the work of Warbhe et.al.[23] is modified and prepare the mathematical model of fractional order thermoelastic problem for a finite thin hollow circular disk under the time dependent heat flux at the outer boundary by quasi-static approach . No one has discussed the fractional order thermoelasticity in a thin hollow circular disk in a finite domain by quasi-static approach so far. Hence, this is the new and novel contribution in this field. Copper material is chosen for numerical purposes and the results for temperature, displacement and stresses are discussed and illustrated graphically for different values of alpha i.e. for weak, normal and strong conductivity.

FORMULATION OF THE PROBLEM

Consider a thin hollow disk of thickness h occupying space D defined by $b_1 \le r \le b_2$, $0 \le z \le h$.. The inner circular boundary $(r = b_1)$ is at zero heat flux whereas the heat flux $Q(z)\delta(t)$ is applied on the outer circular boundary $(r = b_2)$. Also the upper surface $(z = h)$ and the lower surface $(z = 0)$ of the thin hollow disk are at zero temperature. A mathematical model is prepared considering non-local Caputo type time fractional heat conduction equation of order α for a thin hollow disk.

The definition of Caputo type fractional derivative is given by [10]

$$\frac{\partial^\alpha f(t)}{\partial t^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n f(\tau)}{d\tau^n} d\tau , \quad n-1 < \alpha < n \tag{1}$$

For finding the Laplace transform, the Caputo derivative requires knowledge of the initial values of the function $f(t)$ and its integer derivatives of the order

$k = 1,2,....,n-1$

$$L\left\{\frac{\partial^{\alpha} f(t)}{\partial t^{\alpha}}\right\} = s^{\alpha} f^{*}(s) - \sum_{k=0}^{n-1} f^{(k)}(0^{+}) s^{\alpha-1-k}, \quad n-1 < \alpha < n \tag{2}$$

Where the asterisk denotes the Laplace transform with respect to time, s is the Laplace transform parameter.

The temperature of the thin hollow circular disk $T(r,z,t)$ is satisfying time fractional order differential equation,

$$a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) = \frac{\partial^{\alpha} T}{\partial t^{\alpha}}, \qquad b_1 \le r \le b_2, 0 \le z \le h \tag{3}$$

with boundary conditions

$$\frac{\partial T}{\partial r} = 0 \qquad \text{at } r = b_1 \tag{4}$$

$$\frac{\partial T}{\partial r} = Q(z)\,\delta(t) \qquad \text{at } r = b_2 \tag{5}$$

$$T = 0 \qquad \text{at } z = 0, \quad t > 0 \tag{6}$$

$$T = 0 \qquad \text{at } z = h, \quad t > 0 \tag{7}$$

and under zero initial condition

$$T = 0 \qquad \text{at } t = 0, \quad 0 < \alpha < 2 \tag{8}$$

$$\frac{\partial T}{\partial t} = 0 \qquad \text{at } t = 0, \quad 1 < \alpha < 2 \tag{9}$$

Following, Kulkarni et al. [16], we assume that a hollow disk of small thickness *h* is in a plane state of stress. In fact, “the smaller the thickness of the hollow disk compared to its diameter, the nearer to a plane state of stress is the actual state”. The displacement equations of thermoelasticity have the form,

$$U_{i,kk} + \left(\frac{1+\nu}{1-\nu}\right) e,_{i} = 2\left(\frac{1+\nu}{1-\nu}\right) \cdot a_t \cdot T,_{i}$$

$$e = U_{k,k}; \quad k,i = 1,2, \tag{10}$$

where U_i – Displacement component,

e – Dilatation,

T – Temperature,

and ν and a_t are respectively, the Poisson's ratio and the linear coefficient of thermal expansion of the thin hollow disk material.

Introducing $U_i = \psi_{,i}$, $i = 1,2,$ we have

$$\nabla_1^{\ 2}\psi = (1+\nu)a_t T$$

$$\nabla_1^{\ 2} = \frac{\partial^2}{\partial x_1^{\ 2}} + \frac{\partial^2}{\partial x_2^{\ 2}}$$

$$\sigma_{ij} = 2\mu\left(\psi_{,ij} - \delta_{ij}\psi_{,kk}\right), \qquad i, j, k = 1,2, \tag{11}$$

where μ is the Lamé constant and δ_{ij} is the Kronecker symbol.

In the axially-symmetric case $\psi = \psi(r,t)$, $T = T(r,t)$

and the differential equation governing the displacement potential function $\psi(r,z,t)$ is as

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} = (1+\nu)a_t T \tag{12}$$

with $\frac{\partial\psi}{\partial r} = 0$ at $r = b_1$ and $r = b_2$ for all time t.

The stress function σ_{rr} and $\sigma_{\theta\theta}$ are given by

$$\sigma_{rr} = \frac{-2\mu}{r}\frac{\partial\psi}{\partial r} \tag{13}$$

$$\sigma_{\theta\theta} = -2\mu\frac{\partial^2\psi}{\partial r^2} \tag{14}$$

In the plane state of stress within the disk

$$\sigma_{rz} = \sigma_{zz} = \sigma_{\theta z} = 0$$

Equations (3) to (14) constitute the mathematical formulation of the problem.

Solution

To obtain the expression for temperature function $T(r,z,t)$; we first define the finite Fourier transform and its inverse transform over the variable z in the range $0 \le z \le h$ defined as

$$\overline{T}\left(r,\eta_p,t\right) = \int_{z'=0}^{h} K\left(\eta_p,z'\right) T\left(r,z',t\right) dz' \tag{15}$$

$$T(r,z,t) = \sum_{p=1}^{\infty} K(\eta_p,z).\overline{T}(r,\eta_p,t) \tag{16}$$

where

$K(\eta_p,z) = \sqrt{\dfrac{2}{h}}\sin(\eta_p z)$ and $\eta_1,\eta_2,.....$ are the positive roots of the transcendental equation $\sin\eta_p h = 0$, $p = 1,2,....$

Secondly, we define the finite Hankel transform and its inverse transform over the variable r in the range $b_1 \le r \le b_2$ as,

$$\overline{\overline{T}}\left(\beta_m,\eta_p,t\right) = \int_{b_1}^{b_2} r'.K_0\left(\beta_m,r'\right).\overline{T}\left(r',\eta_p,t\right) dr' \tag{17}$$

$$\overline{T}(r,\eta_p,t) = \sum_{m=1}^{\infty} K_0\left(\beta_m,r\right) \overline{\overline{T}}(\beta_m,\eta_p,t) \tag{18}$$

where

$$K_0(\beta_m,r) = \frac{\pi}{\sqrt{2}} \frac{\beta_m.J_0'(\beta_m b_2).Y_0'(\beta_m b_2)}{\left[1-\dfrac{J_0'^2(\beta_m b_2)}{J_0'^2(\beta_m b_1)}\right]^{\frac{1}{2}}} \left[\frac{J_0(\beta_m r)}{J_0'(\beta_m b_2)} - \frac{Y_0(\beta_m r)}{Y_0'(\beta_m b_2)}\right]$$

and $\beta_1,\beta_2,.....$ are the positive roots of the transcendental equation $\dfrac{J_0'(\beta b_1)}{J_0'(\beta b_2)} - \dfrac{Y_0'(\beta b_1)}{Y_0'(\beta b_2)} = 0$.

Applying Fourier, Hankel and Laplace transform and their inversions to equation (3) and making use of the transformed boundary and initial conditions (4)-(9), one obtains temperature distribution function expressed as follows,

$$T(r,z,t)=\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}K(\eta_p,z)K_0(\beta_m,r)\,a\,b_2\,K_0(\beta_m,b_2)$$
$$\times\left(\int_{z=0}^{h}K(\eta_p,z)Q(z)\,dz\right)\left[E_\alpha\left(-a\,(\beta_m^2+\eta_p^2)\,t^\alpha\right)\right]. \tag{19}$$

where $L^{-1}\left[\dfrac{1}{s^\alpha+a(\beta_m^2+\eta_p^2)}\right]=E_\alpha\left(-a\,(\beta_m^2+\eta_p^2)\,t^\alpha\right)$

Here $E_\alpha(.)$ represents the Mittag-Leffler function.

Displacement potential and thermal stresses

Using equation (12) and (19), we get displacement potential function as follows

$$\frac{\psi(r,z,t)}{X}=\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}K(\eta_p,z)K_0(\beta_m,r)\frac{1}{\beta_m^2}\,a\,b_2\,K_0(\beta_m,b_2)$$
$$\times\left(\int_{z=0}^{h}K(\eta_p,z)Q(z)\,dz\right)\left[E_\alpha\left(-a\,(\beta_m^2+\eta_p^2)\,t^\alpha\right)\right]. \tag{20}$$

Using equations (13), (14) and (20), we obtain radial and angular stresses as follows

$$\frac{\sigma_{rr}}{Y}=\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}K(\eta_p,z)K_1(\beta_m,r)\frac{1}{r\beta_m^2}\,ab_2\,K_0(\beta_m,b_2)$$
$$\times\left(\int_{z=0}^{h}K(\eta_p,z)Q(z)\,dz\right)\left[E_\alpha\left(-a\,(\beta_m^2+\eta_p^2)\,t^\alpha\right)\right] \tag{21}$$

$$\frac{\sigma_{\theta\theta}}{Y}=\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}K(\eta_p,z)K_2(\beta_m,r)\frac{1}{\beta_m^2}\,a\,b_2\,K_0(\beta_m,b_2)$$
$$\times\left(\int_{z=0}^{h}K(\eta_p,z)Q(z)\,dz\right)\left[E_\alpha\left(-a\,(\beta_m^2+\eta_p^2)\,t^\alpha\right)\right] \tag{22}$$

where

$$K_0(\beta_m,b_2)=\frac{\pi}{\sqrt{2}}\,\frac{\beta_m.J_0'(\beta_m b_2).Y_0'(\beta_m b_2)}{\left[1-\dfrac{J_0'^2(\beta_m b_2)}{J_0'^2(\beta_m b_1)}\right]^{\frac{1}{2}}}\left[\frac{J_0(\beta_m b_2)}{J_0'(\beta_m b_2)}-\frac{Y_0(\beta_m b_2)}{Y_0'(\beta_m b_2)}\right]$$

$$K_1(\beta_m, r) = -\frac{\pi}{\sqrt{2}} \frac{\beta_m^2 . J_0'(\beta_m b_2).Y_0'(\beta_m b_2)}{\left[1 - \frac{J_0'^2(\beta_m b_2)}{J_0'^2(\beta_m b_1)}\right]^{\frac{1}{2}}} \left[\frac{J_1(\beta_m r)}{J_0'(\beta_m b_2)} - \frac{Y_1(\beta_m r)}{Y_0'(\beta_m b_2)}\right]$$

$$K_2(\beta_m, r) = -\frac{\pi}{\sqrt{2}} \frac{\beta_m^3 . J_0'(\beta_m b_2).Y_0'(\beta_m b_2)}{\left[1 - \frac{J_0'^2(\beta_m b_2)}{J_0'^2(\beta_m b_1)}\right]^{\frac{1}{2}}}$$

$$\times \left\{\frac{1}{J_0'(\beta_m b_2)} \times \left[J_0(\beta_m r) - \frac{J_1(\beta_m r)}{\beta_m r}\right] - \frac{1}{Y_0'(\beta_m b_2)} \times \left[Y_0(\beta_m r) - \frac{Y_1(\beta_m r)}{\beta_m r}\right]\right\}$$

$$\frac{d}{dr} J_0(\beta_m r) = -\beta_m J_1(\beta_m r)$$

$$\frac{\partial^2}{\partial r^2}(J_0(\beta_m r)) = -\beta_m^2 \left(J_0(\beta_m r) - \frac{J_1(\beta_m r)}{\beta_m r}\right)$$

$$\frac{\partial^2}{\partial r^2}(Y_0(\beta_m r)) = -\beta_m^2 \left(Y_0(\beta_m r) - \frac{Y_1(\beta_m r)}{\beta_m r}\right)$$

RESULTS AND DISCUSSION

Setting: $Q(z) = z^2 \times \left(z^2 - h^2\right)^2$

Dimensions:

Inner radius of a thin hollow circular disk $b_1 = 1m$

Outer radius of a thin hollow circular disk $b_2 = 2m$

Thickness of hollow circular disk $z = 0.2m$

Material properties:

The numerical calculation has been carried out for a Copper (Pure) thin hollow disk with the material properties as,

Thermal diffusivity $a = 112.34 \times 10^{-6} (m^2 s^{-1})$.

Thermal conductivity $k = 386(W/mk)$.

Density $\rho = 8954\,kg/m^3$.

Specific heat $c_p = 383\,J/kgK.$

Poisson ratio $\nu = 0.35$.

Coefficient of linear thermal expansion $a_t = 16.5\times10^{-6}\,\frac{1}{K}$.

Lamé constant $\mu = 26.67$.

We set for convenience, $X = (1+\nu)a_t$ and $Y = 2(1+\nu)a_t\mu$.

The graphs are plotted for fractional order parameter $\alpha = 0.5, 1, 1.5, 2$ depicting weak, normal and strong conductivity and fixed time $t = 0.5$. Figures 1, 2 and 3 depict the distributions of temperature, radial stress and angular stress along the radial direction for various values of fractional order parameter α. The numerical calculation has been carried out in Matlab 2013a programming environment. The Mittag-Leffler functions used in the paper were evaluated following Podlubny [19].

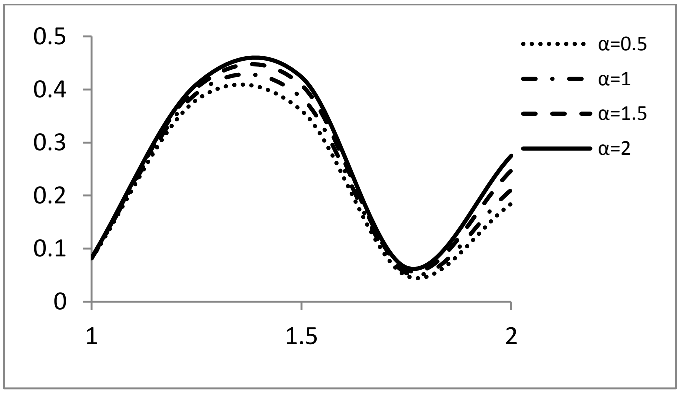

Figure 1: Temperature Distribution Function

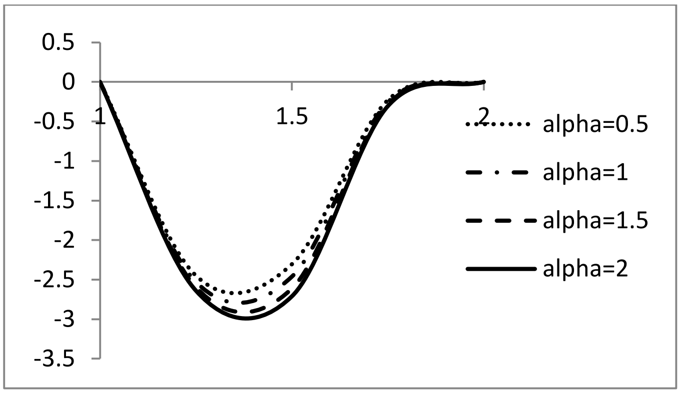

Figure 2: Radial Stress Function $\dfrac{\sigma_{rr}}{Y}$

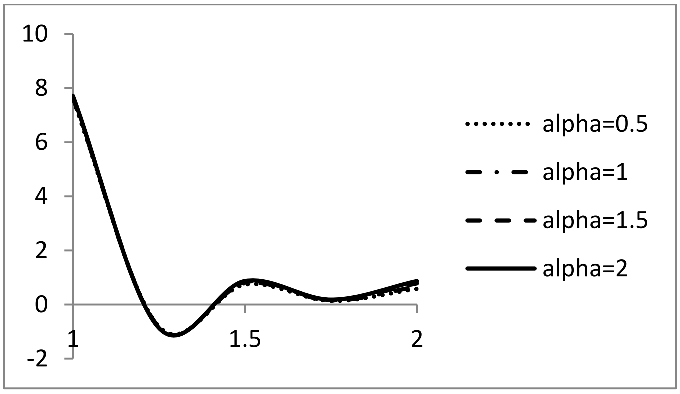

Figure 3: Angular Stress Function $\dfrac{\sigma_{\theta\theta}}{Y}$

Figure 1 represents the temperature distributions along the radial direction. For the different values of the fractional order parameter $\alpha = 0.5, 1, 1.5, 2$, the values of the temperature follow a non-uniform pattern with respect to radius. It is observed that temperature assumes a non-zero vale at both the inner and outer radii $r = 1$ and $r = 2$ respectively. It is seen that the speed of propagation of the thermal signals is directly proportional to the values of the fractional order parameter α. The temperature increases in the range $1 \le r \le 1.4$ and $1.75 \le r \le 2$ and it decreases in the range $1.4 \le r \le 1.75$.

Figure 2 represents the radial stress distributions along the radial direction. It is observed that the radial stresses are compressive in nature throughout the range $1 \le r \le 2$. The radial stress values initially decrese in the range $1 \le r \le 1.4$ and then increase in the range $1.4 \le r \le 2$ and finally converges to zero.

Figure 3, represents the angular stress distributions along the radial direction. For different values of fractional parameter α, angular stresses are tensile in the range $1 \le r \le 1.2$ and $1.4 \le r \le 2$ and compressive in the range $1.2 \le r \le 1.4$.

The case $\alpha = 1$, depicts classical thermoelasticity and fractional thermoelasticity for the cases $\alpha = 0.5, 1.5$. The case $\alpha = 2$ coincides with Green and Naghdi theory and the heat conduction equation becomes wave type, admitting the propagation of second sound (see, Chandrasekaraiah [5]). It is noted from the graphs that changing values of fractional order parameter α, the speed of wave propagation is affected. Hence, it can be an important factor for designing new materials applicable to real life situations.

CONCLUSION

The fractional order thermoelasticity is used to control the speed of wave propogation in terms of heat waves and the elastic waves for the weak, normal and strong conductivity.

The theory of thermoelasticity based on time fractional heat wave equation proposed by Povstenko [9] is used to study a problem of thin hollow circular disk. The cases $0 < \alpha < 1$ and $1 < \alpha < 2$ correspond to weak and strong conductivity respectively. While $\alpha = 1$ corresponds to normal conductivity.

We restrict ourselves to the quasi-static uncoupled theory neglecting the inertia term in the equations of motion and the coupling term. Neglecting the mechanical term implies that no account has been taken for mechanical heat waves. Here we observed that the quasi static thermoelastic problem is possible only when the relaxation time of mechanical oscillation is considerably less than the relaxation time of heat conduction process. Also it is observed that the time fractional differential operator describes memory effect with the long range interactions. The heat conduction equation is parabolic in nature which predicts infinite wave propagation in terms of heat waves. Figures 1-3 shows the characteristic features of temperature and stress distributions and represents the whole spectrum of order of fractional operators. The motivation behind the consideration of the fractional theory is that it predicts retarded response to physical stimuli, as seen in nature.

In real life situations, the problems dealing with finite domains are important, unfortunately due to the complexity involved in modeling a finite domain, the literature is limited to problems on infinite domains. Hence, this problem was developed for a finite thin hollow circular disk.

In the case $1<\alpha<2$, the time fractional heat conduction equation interpolates the standard parabolic heat conduction equation and the hyperbolic wave equations. Likewise, the thermoelasticity interpolates the classical theory of thermal stresses and that without energy dissipation introduced by Green and Naghdi [4].

REFERENCES

[1] Podlubny I., 2002, "Geometric and physical interpretation of fractional integration and fractional differentiation", Fractional Calculus and Applied Analysis, 5(4), pp. 367-386.

[2] Lord, H., and Shulman, Y., 1967, "A Generalized Dynamical theory of thermoelasticity", *Journal of the Mechanics and Physics of solids*, 15(5), pp. 299-307.

[3] Green, A.E., and Lindsay, K.A., 1972, "Thermoelasticity", *J. Elasticity*, 2(1), pp. 1-7, 1972. Doi: 10.1007/BF00045689

[4] Green, A.E., and Naghdi, P.M., 1993, "Thermoelasticity without energy dissipation", *J. Elasticity*, 31(3), pp. 189-208. Doi: 10.1007/BF00044969

[5] Chandrasekharaiah, D.S., 1986, "Thermoelasticity with Second sound: A review", *Appl. Mech. Rev.*, 39(3) , pp. 355-376. Doi:10.1115/1.3143705

[6] Tripathi, J. J., Kedar, G. D., and Deshmukh, K. C., 2015, "Generalized thermoelastic diffusion problem in a thick circular plate with axisymmetric heat supply", *Acta Mech.*, 226(7),2121- 2134. doi: 10.1007/s00707-015-1305-7.

[7] Tripathi, J. J., Kedar, G. D., and Deshmukh, K. C., 2015, "Two dimensional generalized thermoelastic diffusion in a half space under axisymmetric distributions", *Acta Mech.,* 226(10), pp. 3263-3274. DOI: 10.1007/s00707-015-1383-6.

[8] Tripathi, J. J., Kedar, G. D., and Deshmukh, K. C., 2016, "A Brief Note on Generalized Thermoelastic Response in a Half Space due to a Periodically Varying Heat Source under Axisymmetric Distribution", *International Journal of Thermodynamics*, 19(1), pp. 1-6. doi: 10.5541/ijot.5000145489

[9] Povstenko, Y., 2005, "Fractional heat conduction equation and associated thermal stresses", *J. Thermal Stresses,* 28(1) , pp. 83–102.doi: 10.1080/014957390523741

[10] Povstenko, Y., 2009, "Thermoelasticity which uses fractional heat conduction equation", *J. Math. Sci.,* 162(2), 296–305. Doi: 10.1007/s10958-009-9636-3

[11] Povstenko, Y., 2009, "Theory of thermoelasticity based on the space-time-

fractional heat conduction equation", *Phys. Scr., T*136 014017, pp. 1-6. doi:10.1088/0031- 8949/2009/T136/014017

[12] Povstenko, Y., 2010, "Signaling problem for time-fractional diffusion-wave equation in a half-plane in the case of angular symmetry", *Nonlinear Dyn.*, 59(4), pp. 593–605. DOI: 10.1007/s11071-009-9566-0

[13] Povstenko, Y., 2011, "Fractional Cattaneo-type equations and generalized thermoelasticity", *J. Thermal Stresses*, 34(2), pp. 97–114. Doi: 10.1080/01495739.2010.511931

[14] Povstenko, Y., 2012, "Theories of thermal stresses based on space-time-fractional telegraph equations", *Comp. Math. Appl.*, 64(10), pp. 3321–3328. Doi: 10.1016/j.camwa.2012.01.066.

[15] Tripathi, J. J., Kedar, G. D., and Deshmukh, K. C., 2016, "Dynamic problem of fractional order thermoelasticity for a thick circular plate with finite wave speeds", *J. Thermal Stresses,* 39(2), pp. 220-230. Doi: 10.1080/01495739.2015.1124646

[16] Kulkarni, V.S., Deshmukh, K.C. and Warbhe, S.D., Quasi-static thermal stresses due to heat generation in a thin hollow circular disk, *J. Thermal Stresses*, 31(8), pp 698-705, 2008. Doi: 10.1080/01495730802194078

[17] Deshmukh, K.C., S.D.Warbhe and V.S.Kulkarni, "Quasi-static thermal deflection of a thin clamped hollow circular disk due to heat generation", *Int J Thermophys*, Vol. 31(2010), 2337–2347.

[18] K.C.Deshmukh, Warbhe, S.D. and Kulkarni, V.S., Brief note on heat flow with arbitrary heating rates in a hollow cylinder, *J. Thermal Stresses*, Vol 15(1), pp 275-280, 2011.

[19] Podlubny, I., 1999, "*Fractional Differential Equations.* An Introduction to Fractional Derivatives, Fractional Differential Equations, Some Methods of Their Solution and Some of Their Applications" Academic Press, San Diego-Boston-New York-London-Tokyo-Toronto, 1999, 368 pages, ISBN 0125588402.

[20] Raslan, W. E., 2015, "Application of Fractional Order Theory of Thermoelasticity in a Thick Plate Under Axisymmetric Temperature Distribution", J. Thermal Stresses, 38(7),pp.733-743, DOI: 10.1080/01495739.2015.1040307

[21] Sherief, H., El-Sayed, A., and Abd El-Latief, A.M., 2010, "Fractional Order Theory Of Thermoelasticity", Int. J. Solids Struct., 47(2), pp. 269–275. Doi:10.1016/j.ijsolstr.2009.09.034

[22] Chunbao Xiong and Ying Guo, Effect of Variable Properties and Moving Heat

source on Magnetothermoelastic problem under fractional order thermoelasticity, Advances in Materials Science and Engineering, 2016 (2016), pp 1-12, doi: 10.1155/2016/5341569

[23] Warbhe, S.D., Tripathi, J. J., Deshmukh, K. C. and Verma, J., Fractional Heat Conduction in a Thin Circular Plate with Constant Temperature Distribution and Associated Thermal Stresses, *J. Heat Transfer*, (2016); doi: 10.1115/1.4035442