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Neighborhood Sigma Algebras of Middle graph, Total graph and Join of Two Graphs

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Abstract

In this paper the neighborhood sigma algebras of middle graph, total graph and join of two graphs are examined and derived a necessary condition for a graph to be the middle graph or total graph of some other graphs.

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1. Introduction

A *graph* [3] G is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of vertices and a set $E(G)$, disjoint from $V(G)$, of edges, together with an incidence function ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G. If e is an edge and u and v are vertices such that $\psi_G(e) = \{u, v\}$, then e is said to join u and v, and the vertices u and v are called the ends of e. A graph, G is *finite* [3] if both its vertex set $V(G)$ and edge set $E(G)$ are finite. If uv is an edge of the graph G, then u and v are called *adjacent vertices* [4]. Two adjacent vertices are referred to as neighbors of each other. The set of neighbors of a vertex v is called the *open neighborhood* [4] of v and is denoted by $N(v)$. The set $N[v] = N(v) \cup \{v\}$ is called the *closed neighborhood* [4] of v. If uv is an edge of a graph G, then the vertex u and the edge uv are said to be *incident* [4] with each other. Similarly, v and uv are incident. A set of two or more edges of a graph G is called a set of *multiple edges* [1] if they have the same ends. An edge with identical ends is called a *loop* [1]. A graph is *simple* [1] if it has no loops and multiple edges. In this paper we consider only finite simple graphs. The *degree* [3] of a vertex v in a graph G, denoted by $d_G(v)$, is the number of neighbors of v in G. We denote by $\Delta(G)$ the maximum degrees of the vertices of G. Through out this paper we denote by $n(G)$, the number of vertices of the graph G. Two graphs are *disjoint* [3]if they have no vertex in common.

Another concept we used in this paper is that of a sigma algebra.

Definition 1.1. [8] A collection $\mathcal{R}$ of subsets of a set X is said to be a sigma algebra in X if $\mathcal{R}$ has the following properties:

(i) $X \in \mathcal{R}$.

(ii) If $A \in \mathcal{R}$, then $A^c \in \mathcal{R}$, where A^c is the complement of A relative to X.

(iii) If $A = \bigcup_{n=1}^{\infty} A_n$ and if $A_n \in \mathcal{R}$ for $n = 1, 2, 3, \ldots$, then $A \in \mathcal{R}$.

A set X together with a sigma algebra $\mathcal{R}$ of subsets of X is called a *measurable space* [8], and the members of $\mathcal{R}$ are called the *measurable sets* [8] in X. Let X be a measurable space and Y be a topological space. A mapping f from X into Y is said to be *measurable* [8] if $f^{-1}(V)$ is a measurable set in X for every open set V in Y.

Proposition 1.2. [8] If $\mathcal{F}$ is any family of subsets of a set X, there exists a smallest sigma algebra containing $\mathcal{F}$, called the *sigma-algebra generated by* $\mathcal{F}$.

Definition 1.3. [6] Let $G = (V(G), E(G))$ be a graph. The sigma algebra, $\mathcal{A}$ generated by the collection $\{N[v] : v \in V(G)\}$ on $V(G)$ is called the neighborhood sigma algebra of G and it is denoted by $\mathcal{A}$.

Throughout this paper by a graph G, we mean a graph with the neighborhood algebra $\mathcal{A}$ on the vertex set $V(G)$.

2. Smallest measurable set and measurable function

Let G be a graph with vertex set $V(G)$. For $v \in V(G)$, let E_v denotes the smallest measurable set containing v. That is the intersection of all measurable sets containing v. In [6], we proved the following theorem.

Theorem 2.1. [6] If G is a graph with vertex set $V(G)$ and $v \in V(G)$, then $E_v = \{u \in V(G) : N[u] = N[v]\}$.

In [6], we also proved that if G is a graph with vertex set $V(G)$ and if f is a measurable real valued function defined on $V(G)$, then for each $v \in V(G)$, f is constant on E_v. The converse of this result is also true.

Theorem 2.2. Let G be a graph and $f : V(G) \to [0, 1]$. Then f is measurable if f is constant on E_v for all $v \in V(G)$.

Proof. Assume that f is constant on E_v for all $v \in V(G)$. To prove f is measurable. Let U be an open subset of [0,1]. Suppose that $f(V(G)) \cap U = \{k_1, k_2, \ldots, k_m\}$. Then $f^{-1}(U) = f^{-1}(\{k_1\}) \cup f^{-1}(\{k_2\}) \cup \ldots \cup f^{-1}(\{k_m\})$. Let $1 \le i \le m$. Then $f^{-1}(\{k_i\}) = \cup_{f(v_j)=k_i} E_{v_j}$. This implies $f^{-1}(k_i)$ is measurable for all $1 \le i \le m$. Therefore $f^{-1}(U) = \cup_{i=1}^{m} \cup_{f(v_j)=k_i} E_{v_j}$ is measurable. Hence f is measurable. ■

Theorem 2.3. Let G be a graph and $v \in V(G)$ be such that $d_G(v) = n(G) - 1$. Then $E_v = \{u \in V(G) : d_G(u) = n(G) - 1\}$.

Proof. Let $u \in E_v$. Then $N[u] = N[v]$. This implies $d_G(u) = n(G) - 1$. Therefore $E_v \subseteq \{u \in V(G) : d_G(u) = n(G) - 1\}$. Let $u \in V(G)$ be such that $\deg(u) = n(G) - 1$. Then $N[u] = V(G) = N[v]$. This implies $u \in E_v$. Thus we get $E_v = \{u \in V(G) : d_G(u) = n(G) - 1\}$. ■

Corollary 2.4. Let G be a graph with $\Delta(G) = n(G) - 1$ and $f : V(G) \longrightarrow [0, 1]$ be measurable. Then f is constant on the set, $\{v \in V(G) : deg(v) = n(G) - 1\}$.

Note 2.5. The conclusion of Theorem 2.3 need not true for the vertices of degree $< n(G) - 1$.

For example consider the graph C_4.

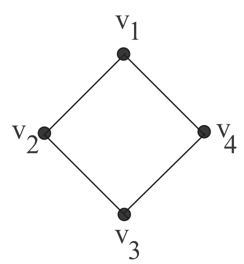

Figure 1: C_4

For the graph C_4 in Figure 1, $d(v_1) = d(v_2) = d(v_3) = d(v_4) = 2$.
But $E_{v_1} = \{v_1\}$, $E_{v_2} = \{v_2\}$, $E_{v_3} = \{v_3\}$, $E_{v_4} = \{v_4\}$.

3. Neighborhood Sigma algebra of Middle graph

Definition 3.1. [7] The middle graph $M(G)$ of a graph $G = (V(G), E(G))$ is the graph with vertex set is $V(G) \cup E(G)$ and in which two vertices u and v are adjacent if either (i) u and v are in $E(G)$ and u, v are adjacent in G or (ii) u is in $V(G)$, v is in $E(G)$ and u and v are incident in G.

Figure 2: Path P_3 and its middle graph $M(P_3)$.

Theorem 3.2. Let G be a graph and $M(G)$ be its middle graph. Then the neighborhood sigma algebra of $M(G)$ is $\mathcal{P}(V(M(G)))$. Hence all functions defined from $V(G)$ to [0, 1] are measurable.

Proof. For $x \in V(M(G))$, let $N_M[x]$ denotes $\{x\} \cup \{u \in V(M(G)) : u$ is adjacent to x in $M(G)\}$. To prove the neighborhood sigma algebra of $M(G)$ is $\mathcal{P}(V(M(G)))$, by theorem 2.1, it is enough to prove that $N_M[x] \neq N_M[y]$ for all $x,\ y \in V(M(G))$. Let $v, e \in V(M(G))$ be such that $v \in V(G)$ and $e \in E(G)$. Then $N_M[v] = \{v\} \cup \{f \in E(G) : f$ is incident with v in $G\}$. Also if $e = uw$, then $N_M[e] = \{e, u, w\} \cup \{f \in E(G) : f$ is adjacent to e in $G\}$. Then$N_M[v] \neq N_M[e]$, because $N_M[v]$ contains only one vertex of G but $N_M[e]$ contains two vertices of G. Let $v_1, v_2 \in V(M(G))$ be such that $v_1, v_2 \in V(G)$. If $N_M[v_1] = N_M[v_2]$, then $v_1 \in N_M[v_2]$. That is v_1 and v_2 are adjacent in $M(G)$. But this is not possible, because no two vertices of G are adjacent in $M(G)$. Let $e_1, e_2 \in V(M(G))$ be such that $e_1, e_2 \in E(G)$. Suppose that $N_M[e_1] = N_M[e_2]$. Let $e_1 = uv$. Then $u, v \in N_M[e_1]$. This implies $u, v \in N_M[e_2]$. Hence u and v are incident on e_2 also. This is not possible since G is simple. Hence $N_M[x] \neq N_M[y]$ for all $x, y \in V(M(G))$. Therefore the neighborhood sigma algebra of $M(G)$ is $\mathcal{P}(V(M(G)))$. Hence by theorem2.2, all functions defined from $V(G)$ to [0, 1] are measurable. ■

Theorem 3.2 implies that if the neighborhood sigma algebra of a graph G is not equal to $\mathcal{P}(V(G))$, then it cannot be the middle graph of some other graph.

Corollary 3.3. Let G be a graph with two vertices u and v such that $N[u] = N[v]$. Then G is not middle graph of any graph.

4. Neighborhood Sigma algebra of Total graph

Definition 4.1. [7] The total graph $T(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices u and v are adjacent if either (i) u, v are in $V(G)$ and u, v are adjacent in G or (ii) u, v are in $E(G)$ and u is adjacent to v in G or (iii) u is in $V(G)$, v is in $E(G)$ and u and v are incident in G.

Theorem 4.2. Let G be a graph such that every component of G is different from P_2 and $T(G)$ be its total graph. Then the neighborhood sigma algebra of $T(G)$ is $\mathcal{P}(V(T(G)))$. Hence all functions from $V(T(G))$ to [0,1] are measurable.

Figure 3: Path P_3 and its total graph $T(P_3)$

Proof. If $G = \overline{K_n}$, $n = 1, 2, \ldots$ then the result is obvious. Suppose that $G \neq \overline{K_n}$ for every n. Therefore $n(G) \geq 3$. For $x \in V(T(G))$, let $N_T[x]$ denotes $\{x\} \cup \{u \in V(T(G)) : u$ is adjacent to x in $T(G)\}$. Let $e_1, e_2 \in V(T(G))$ be such that $e_1, e_2 \in E(G)$. Suppose that $N_T[e_1] = N_T[e_2]$. Let $e_1 = uv$. Then $u, v \in N_T[e_1]$. So $u, v \in N_T[e_2]$. This implies u and v are incident on e_2 also. This is not possible since G is simple. If possible, let $v \in V(G)$ and $e \in E(G)$ be such that $N_T[v] = N_T[e]$. Hence $v \in N_T[e]$. Hence v is incident on e in G. Let e is the edge uv where $u \in V(G)$. If $w \neq u$ is adjacent to v in G, then $w \in N_T[v]$. But $w \notin N_T[e]$. Hence in G, v is adjacent to u only. That means v is an end vertex. Suppose e is adjacent to an edge $e^{'}$ in G. Then v is not incident on $e^{'}$ since v is an end vertex. Hence $e^{'} \in N_T[e]$ but $e^{'} \notin N_T[v]$. So there doesnot exist $e^{'} \in E(G)$ adjacent to e in G. Hence P_2 is a component of G. Let $v_1, v_2 \in V(G)$, be such that $N_T[v_1] = N_T[v_2]$. This implies v_1 and v_2 are adjacent in G. Suppose there exists $w \neq v_2$ in $V(G)$ which is adjacent to v_1 in G. Then $e = v_1 w$ is an edge in G and $e \in N_T[v_1]$. But then v_2 is not incident on e. So $e \notin N_T[v_2]$. This is not possible. Hence v_1 is an end vertex of G. Similarly we will get v_2 is an end vertex of G. Hence P_2 is a component of G. Thus we get if P_2 is not a component of G, $N_T[x] \neq N_T[y]$ for $x \neq y \in V(T(G))$. Hence the neighborhood sigma algebra of $T(G)$ is $\mathcal{P}(V(T(G)))$. Therefore all functions from $V(T(G))$ to [0,1] are measurable. ∎

The following corollary is an immediate consequence of Theorem 4.2.

Corollary 4.3. Let G be a graph such that no component of G is K_3. If there exists two vertices u and v in G such that $N[u] = N[v]$. Then G is not total graph of any graph.

5. Join of two graphs

Definition 5.1. [3] Join of the two graphs G and H is formed from disjoint copies of G and H by connecting each vertex of G to each vertex of H and it is denoted by $G \vee H$.

In this section, for a graph G and $v \in V(G)$, E_v^G denotes the smallest measurable set containing v in the neighborhood sigma algebra of G and $N_G[v] = \{v\} \cup \{u \in V(G) : u$ is adjacent to v in $G\}$. Next we observe how the neighborhood sigma algebra of the join is related to the neighborhood sigma algebra of its component graphs.

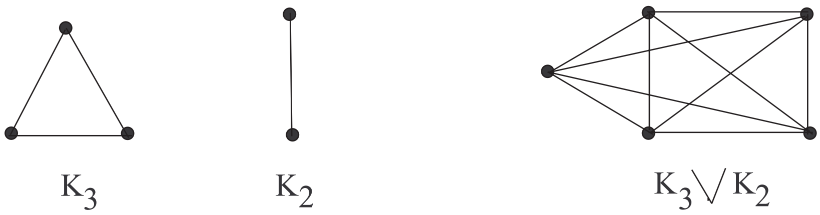

Figure 4: Join of the graphs K_3 and K_2.

Theorem 5.2. Let J be the join of two graphs G_1 and G_2. Then for $v \in V(G_i)$, $E_v^J = E_v^{G_i}$ if $d_{G_i}(v)$ is not equal to $n(G_i) - 1$, where $i = 1$ or 2.

Proof. Let $v \in V(G_1)$. If $x \in E_v^{G_1}$, then $E_x^{G_1} = E_v^{G_1}$. This implies $N_{G_1}[x] = N_{G_1}[v]$. Then $N_{G_1}[x] \cup V(G_2) = N_{G_1}[v] \cup V(G_2)$. That is $N_J[x] = N_J[v]$. This implies $E_v^J = E_x^J$. So, $x \in E_v^J$. Hence $E_v^{G_1} \subseteq E_v^J$. To get the opposite inclusion assume that $x \in E_v^J$. Then $x \in V(G_1)$ or $V(G_2)$. If $x \in V(G_1)$, then $N_J[x] = N_{G_1}[x] \cup V(G_2)$ and $N_J[v] = N_{G_1}[v] \cup V(G_2)$. Since $x \in E_v^J$, $N_J[x] = N_J[v]$. Therefore $N_{G_1}[x] = N_{G_1}[v]$. Hence $x \in E_v^{G_1}$. Therefore $E_v^J \subseteq E_v^{G_1}$. Suppose $x \in V(G_2)$. Since $x \in E_v^J$, $N_J[x] = N_J[v]$. That is $N_{G_2}[x] \cup V(G_1) = N_{G_1}[v] \cup V(G_2)$. This implies $N_{G_1}[v] = V(G_1)$ and $N_{G_2}[x] = V(G_2)$, since G_1 and G_2 are disjoint graphs. That is degree of v in G_1 is equal to $\mid V(G_1) \mid -1$. Hence, $E_v^J = E_v^{G_1}$ if degree of v in G_1 is not equal to $\mid V(G_1) \mid -1$. Similarly we can prove for $v \in V(G_2)$ also. ∎

Note 5.3. If $\Delta(G_i) < n(G_i) - 1$ for $i = 1, 2$ and $v \in V(G_i)$, then $E_v^J = E_v^{G_i}$.

Theorem 5.4. Let J denotes the join of two disjoint graphs G and H. Let $v \in V(G)$ be such that $d_G(v) = n(G) - 1$. Then $E_v^J = E_v^G \cup \{u \in V(H) : d_H(u) = n(H) - 1\}$.

Proof. Let $u \in E_v^G$. Then $N_G[u] = N_G[v]$. Also $N_J[u] = N_G[u] \cup V(H)$ and $N_J[v] = N_G[v] \cup V(H)$. Hence $N_J[u] = N_J[v]$. This implies $u \in E_v^J$. Hence $E_v^G \subseteq E_v^J$. Suppose $u \in V(H)$ and $d_H(u) = n(H) - 1$. Then

$$\begin{aligned} N_J[u] &= N_H[u] \cup V(G) \\ &= V(H) \cup V(G) \end{aligned}$$

and

$$\begin{aligned} N_J[v] &= N_G[v] \cup V(H) \\ &= V(G) \cup V(H) \end{aligned}$$

Thus we got $N_J[u] = N_J[v]$. This implies $u \in E_v^J$. Therefore $\{u \in V(H) : d_H(u) = n(H) - 1\} \subseteq E_v^J$. To prove the reverse inclusion. Let $u \in E_v^J$. Then $u \in V(G)$ or $V(H)$. Let $u \in V(G)$. Since $u \in E_v^J$, $N_J[u] = N_J[v]$. That is $N_G[u] \cup V(H) = N_G[v] \cup V(H)$. This implies that $N_G[u] = N_G[v]$. Hence $u \in E_v^G$. Suppose $u \in V(H)$. Then

$N_J[u] = N_J[v]$. This implies $N_H[u] \cup V(G) = N_G[v] \cup V(H)$. Which is equal to $V(G) \cup V(H)$. Hence $N_H[u] = V(H)$. That is $d_H(u) = n(H) - 1$. Therefore $E_v^J = E_v^G \cup \{u \in V(H) : d_H(u) = n(H) - 1\}$. ■

Suppose we are given two graphs G_1 and G_2 and measurable functions f_1 and f_2 defined on $V(G_1)$ and $V(G_2)$ respectively. Using these two measurable functions a new measurable function can be constructed on $V(G_1 \vee G_2)$.

Theorem 5.5. Let G_1 and G_2 be two disjoint graphs. Also let f_1 be a measurable function defined from $V(G_1)$ to $[0, 1]$ and f_2 be a measurable function defined from $V(G_2)$ to $[0, 1]$. Suppose $R = \{v \in V(G_1) : d_{G_1}(v) = n(G_1) - 1\}$ and $S = \{v \in V(G_2) : d_{G_2}(v) = n(G_2) - 1\}$.

(i) If $R = \phi$ or $S = \phi$, then the function $g : V(G_1 \vee G_2) \to [0, 1]$ defined by $g(v) = \begin{cases} f_1(v) & \text{if } v \in V(G_1) \\ f_2(v) & \text{if } v \in V(G_2) \end{cases}$ is a measurable function.

(ii) Suppose $R \neq \phi$ and $S \neq \phi$. Then the function $h : V(G_1 \vee G_2) \to [0, 1]$ defined by

$$h(v) = \begin{cases} f_1(v) & \text{if } v \in V(G_1) \setminus R \\ f_2(v) & \text{if } v \in V(G_2) \setminus S \\ rs & \text{if } v \in R \cup S \end{cases}$$

where r is the value of f_1 on R and s is the value of f_2 on S, is a measurable function.

Proof.

(i) Suppose $R = \phi$ or $S = \phi$. To prove g is measurable it is enough to prove that g is constant on E_v^J for all $v \in V(G_1 \vee G_2)$ by theorem 2.2. Assume that $R = \phi$. Let $v \in V(G_1)$. Then $E_v^J = E_v^{G_1}$ by theorem 5.2. Also $g \equiv f_1$ on $V(G_1)$. Since f_1 is measurable, f_1 is constant on $E_v^{G_1}$. This implies g is constant on E_v^J. Let $v \in V(G_2)$. If $d_{G_2}(v) \neq n(G_2) - 1$, $E_v^J = E_v^{G_2}$ by theorem 5.2. Also $g \equiv f_2$ on $V(G_2)$. Since f_2 is measurable, f_2 is constant on $E_v^{G_2}$. This implies g is constant on E_v^J. If $d_{G_2}(v) = n - 1$,

$$\begin{aligned} E_v^J &= E_v^{G_2} \cup \{u \in V(G_1) : d_{G_1}(u) = n(G_1) - 1\} \\ &= E_v^{G_2}, \text{ since } R = \phi \end{aligned}$$

Also $g \equiv f_2$ on G_2. Since f_2 is measurable, f_2 is constant on $E_v^{G_2}$. This implies g is constant on $E_v^{G_2}$. Therefore g is constant on E_v^J for all $v \in V(G_1 \vee G_2)$. Similarly, if $S = \phi$ also we can prove that g is constant on E_v^J for all $v \in V(G_1 \vee G_2)$.

(ii) Suppose $R \neq \phi$ and $S \neq \phi$. To prove h is measurable it is enough to prove that h is constant on E_v^J for all $v \in V(G_1 \vee G_2)$ by theorem 2.2. Let $v \in V(G_1)$. If

$d_{G_1}(v) \neq n(G_1) - 1$, $E_v^J = E_v^{G_1}$. Let $u \in E_v^{G_1}$. This implies $N_{G_1}[u] = N_{G_1}[v]$. Therefore $d_{G_1}(u) = d_{G_1}(v)$. Hence $u \notin R$. This implies $E_v^{G_1} \subseteq V(G_1) \setminus R$. Therefore $h \equiv f_1$ on $E_v^{G_1}$. Since f_1 is measurable, f_1 is constant on $E_v^{G_1}$. Hence h is constant on $E_v^{G_1}$. If $d_{G_1}(v) = n(G_1) - 1$, then $v \in R$ and $E_v^J = E_v^{G_1} \cup S$. Since $d_{G_1}(v) = n(G_1) - 1$, $E_v^{G_1} = R$ by theorem 2.3. Therefore $E_v^J = R \cup S$. Since $h(u) = rs$ for all $u \in E_v^J$, h is constant on E_v^J. Hence h is constant on E_v^J for all $v \in V(G_1)$. Similarly we can prove that h is constant on E_v^J for all $v \in V(G_2)$ also. Hence h is measurable. ■

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