Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 9 (2017), pp. 6827–6853

http://www.ripublication.com/gjpam.htm

A Report On GRADED Rings and Graded MODULES

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Abstract

The investigation of the ring-theoretic property of graded rings started with a question of Nagata. If G is the group of integers, then is Cohen-Macaulay property of the G-graded ring determined by their local data at graded prime ideals? Matijevic-Roberts and Hochster-Ratliff gave an affirmative answer to the conjecture as above. Graded rings play a central role in algebraic geometry and commutative algebra. The objective of this paper is to study rings graded by any finitely generated abelian group, graded modules and their applications.

AMS subject classification: 13A02, 16W50.
Keywords:

1. Introduction

Dedekind first introduced the notion of an ideal in 1870s. For it was realized that only when prime ideals are used in place of prime numbers do we obtain the natural generalization of the number theory of $\mathbb{Z}$. Commutative algebra first known as ideal theory. Later David Hilbert introduced the term ring (see [73]). Commutative algebra evolved from problems arising in number theory and algebraic geometry. Much of the modern development of the commutative algebra emphasizes graded rings. A bird's eye view of the theory of graded modules over a graded ring might give the impression that it is nothing but ordinary module theory with all its statements decorated with the adjective âŁœgradedâŁž. Once the grading is considered to be trivial, the graded theory reduces to the usual module theory. So from this perspective, the theory of graded modules can be considered as an extension of module theory. Graded rings play a central role in algebraic geometry and commutative algebra. Gradings appear in many circumstances, both in elementary and advanced level. Here we present two examples on gradings. Following both examples show the applications of gradings in commutative algebra and algebraic geometry as well as in real life:

1. In the elementary school when we distribute 10 apples giving 2 apples to each person, we have 10 Apple : 2 Apple = 5 People. The psychological problem caused to many kids of how the word per People appears in the equation can be justified by correcting 10 Apple: 2 Apple/People = 5 People. This shows that already at the level of elementary school arithmetic, children are working in a much more sophisticated structure, i.e., a graded ring $\mathbb{Z}[x_1, x_1^{-1}, x_2, x_2^{-1} \dots, x_n, x_n^{-1}]$ of Laurent polynomial rings (See [24] and [98]).

2. If R is a commutative ring which is generated by a finite number of elements of degree 1, then by the celebrated work of Serre [58], the category of quasi coherent sheaves on the scheme $Pr - R$ is equivalent to $QGr - R \cong Gr - R/Fdim - R$, where $Gr - R$ is the category of graded modules over R and $Fdim - R$ is the Serre subcategory of (direct limit of) infinite dimensional submodules. In particular when $R = K[x_0, x_1, \dots, x_n]$, where K is a field, then $QCoh\mathbb{P}^n$ is equivalent to $QGr - R[x_0, x_1, \dots, x_n]$ (see [14], [98], [58],and [109] for more precise statements and relations with noncommutative algebraic geometry).

The study of graded rings arises naturally out of the study of affine schemes and allows them to formalize (and unify) arguments by induction [102]. However, this is not just an algebraic trick. The concept of grading in algebra, in particular graded modules are essential in the study of homological aspect of rings. In recent years, rings with a group-graded structure have become increasingly important and consequently, the graded analogues of different concept are widely studied (see [31], [34], [54], [66] - [79], [83] - [95]). As a result, graded analogue of different concepts are being developed in recent research. The objective of this paper is to study rings graded by any finitely generated abelian group, graded modules and their applications.

2. Preliminaries

In this section, we define some terms used in this paper and provide some examples without proof. We hope that this will improve the readability and understanding of this article.

Definition 2.1. Let G be an abelian group (written additively) and R a commutative ring. A G-grading for R is a family $\{R_g\}_{g\in G}$ of abelian groups of $(R,+)$ such that $R = \bigoplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of R_g are called the homogeneous elements of R of degree g. If $r \in R_g$, we write the degree of r as deg $r = g$ or $|r| = g$.

Example 2.2. Examples on graded rings are as follows:

- Consider $k[x] = \oplus_{n\in\mathbb{Z}} kx^n$ where $kx^n = 0$ if $n < 0$. Then $k[x] = \ldots 0 \oplus \cdots \oplus 0 \oplus k \oplus kx \oplus \ldots$.

- Let $R = T[x]$ and G be any abelian group. Set $|x| = g$ for some $g \in G$. For $h \in G$, we see $R_h = \oplus_{ig=h} Tx^i$, where $T \subseteq R_0$.
  If $G = \mathbb{Z}$ and $|x| = 1$, then for $n \in \mathbb{Z}$,
  $$R_h = \begin{cases} Tx^h & \text{if } n \geq 0; \\ 0 & \text{if } n < 0. \end{cases}$$
  This is N-grading.

- Let $R = T[x_1, \ldots, x_d]$ and G be any abelian group. Set $|x_i| = g_i$. For $h \in G$, we have $R_h = \oplus_{\alpha_1 g_1+\cdots+\alpha_d g_d = h} T{x_1}^{\alpha_1} \ldots {x_d}^{\alpha_d}$.
  1. If $R = T[x, y]$, $G = \mathbb{Z}$, and $|x| = |y| = 1$, then for $n \in \mathbb{Z}$, $R_h = \oplus_{i+j=n, i,j\geq 0}, Tx^i y^j$.
  2. If $R = T[x, y]$, $G = \mathbb{Z}$, and $|x| = 2$, $|y| = 3$, then $R_m = \oplus_{2i+3j=m} Tx^i y^j$.

Definition 2.3. Let $R = \oplus R_n$ be a graded ring. A subring S of R is called a graded subring of R if $S = \sum_n (R_n \bigcap S)$. Equivalently, S is graded if for every element $f \in S$ all the homogeneous components of f (as an element of R) are in S.

Example 2.4. We can construct several examples on graded subrings which are mentioned here, e.g.:

- Let $R = \oplus R_n$ be a graded ring and $f_1, \ldots, f_d$ homogeneous elements of R of degrees $\alpha_1, \ldots, \alpha_d$ respectively. Then $S = R_0[f_1, \ldots, f_d]$ is a graded subring of R, where
  $$S_n = \{ \sum_{m\in N^d} r_m f_1^{m_1} \ldots f_d^{m_d} | r_m \in R_0 \; and \; \alpha_1 m_1 + \cdots + \alpha_d m_d = n\}.$$

- $k[x^2, xy, y^2]$ is a graded subring of $k[x, y]$.
- $k[x^3, x^4, x^5]$ is a graded subring of $k[x]$.
- $\mathbb{Z}[x^3, x^2 + y^3]$ is a graded subring of $\mathbb{Z}[x, y]$, where $deg(x) = 3$ and $deg(y) = 2$.

Definition 2.5. Let R be a graded ring and M an R-module. We say that M is a graded R-module (or has an R-grading) if there exists a family of subgroups $\{M_g\}_{g\in\mathbb{G}}$ of M such that

1. $M = \bigoplus_{g\in G} M_g$ and
2. $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$.

If $a \in M \setminus \{0\}$ and $a = a_{i_1} + \cdots + a_{i_k}$ where $a_{i_j} \in R_{i_j} \setminus \{0\}$ then $a_{i_1}, \ldots, a_{i_k}$ are called the homogeneous components of a.

Example 2.6. Examples on graded modules are as follows:

- If R is a graded ring, then R is a graded module over itself.
- Let $\{M_\lambda\}$ be a family of graded R- modules then $\oplus_\lambda M_\lambda$ is a graded R-module. Thus $R_n = R \oplus \cdots \oplus R$ (n times) is a graded R-module for any $n \geq 1$.
- Given any graded R-module M, we can form a new graded R-module by twisting the grading on M as follows: if n is any integer, define $M(n)$ (read M twisted by n) to be equal to M as an R-module, but with it's grading defined by $M(n)_k = M_{n+k}$. (For if $M = R(-3)$ $then$ $1 \in M_3$.) then $M(n)$ is a graded R-module.
  Thus, if $n_1, \ldots, n_k$ are any integers then $R(n_1) \ldots R(n_k)$ is a graded R-module. Such modules are called free.
- Let R be a graded ring and S a multiplicatively closed set of homogeneous elements of R. Then R_S is a graded ring, where

  $$(R_S)_n = \{\frac{r}{s} \in R_S | \ r \text{ and } s \text{ are homogeneous and } \deg r - \deg s = n\}$$

  Similiarly, if M is a graded R-module then M_S is graded both as an R-module and as an R_S-module.

3. A Report on graded rings and graded modules

In this section, we give a report on graded rings, graded modules and their applications.

Let K be a field, $X = [x_{ij}]$ be an $n \times (n+m)$ matrix whose elements are algebraically independent over K. Yoshino [122] studied the canonical module of the graded ring R, which is a quotient ring of the polynomial ring $S = K[X]$ by the ideal $a_n(X)$ generated by all the $n \times n$ minors of X.

An alternative construction of the duality between finite group actions and group gradings on rings which was shown by Cohen and Montgomery in [30]. This duality is then used to extend known results on skew group rings to corresponding results for large classes of group-graded rings. Quinn [90] modified the construction slightly to handle infinite groups.

One can find the definition of smash product $R \neq G^*$ for a graded ring R, graded by a finite group G. Apply the results obtained from [30] to achieve new results on group graded rings and on fixed point rings for group acting on rings as well as to achieve new and simpler proofs for known results concerning skew group rings and fixed points rings (see [55]). Jensen and Jondrup [55] proved following:

(1) An $R \neq G^*$ module M is flat (projective or injective) if and only if M_R is flat. This means that if R has a certain "homological" property so has $R \neq G^*$. In general properties from $R \neq G^*$ are not inherited by R, but for “separably” graded rings R and $R \neq G^*$ are alike.

(2) A ring is perfect if and only if R_1, the rings of constants, is perfect.

Let R be a ring graded by a group G. Haefner [43] concerned with describing those G-graded rings that are graded equivalent to G-crossed products. He [43] gave necessary and suficient conditions for when a strongly graded ring is graded equivalent to a crossed product, provided that the 1-component is either Azumaya or semiperfect. His result was used the torsion product theorem of Bass and Guralnick (see [43]).

Let G be a multiplicative group with identity e, and R an associative G-graded ring with unity 1. Let R_e be the identity component of R, and R_e-gr the category of all graded R_e-modules and their graded R_e-maps. Then the concepts and properties of augmented graded rings and augmented graded modules have been studied some [92] and [96]. The study of augmented graded Noetherian modules, generalization of augmented Noetherian modules has been given in [93]. Some of the materials in [93] are related to the work done by C. Nastasescu and F.V. Oystaeyen [78], [77], [75], and [76]. M. Refai [93] introduced some relationships between the Noetherian modules in the category R-A_{gr} and the Noetherian modules in the category R_e-gr.

Let R be a Dedekind domain with global quotient field K. The purpose of [44] is to provide a characterization of when a strongly graded R-order with semiprime 1-component is hereditary. This generalized previous work by Haefner and G.

A construction of toric varieties which have enough invariant cartier divisor as the spectrum of homogenous prime ideals of graded ring, along of the proj-construction has been found in [81].

Based on generalized algorithm for the division of polynomials in several variables, a method for the construction of standard bases for polynomial ideals with respect to arbitrary grading structure is derived. In the case of ideals with finite co-dimension, which can be viewed upon as a polynomial interpolation problem, an explicit representation for the interpolation space of reduced polynomials can be found in [101].

The weighted projective spaces(wps) $\mathbb{P} = \mathbb{P}(a_0, \ldots, a_n)$ and the projective correspondence has been studied in [97]

$$projectivevariety(X \subset P) \longleftrightarrow gradedring\left(R = \frac{K[x_0 \ldots, x_n]}{I}\right). \tag{1}$$

The correspondence (1) is generalization of the usual idea of varieties in straight projective space $\mathbb{P}^n = \mathbb{P}(1, \ldots, 1)$. The study of graded rings and varieties in weighted projective space has been given in [97]. Perling [82] derived a formalism for describing equivariant sheaves over toric varieties. He [82] constructed the theory from the point of view of graded ring theory and also connected the formalism to the theory of find graded modules over 'cox' homogeneous coordinate ring of a toric variety. The purpose of [106] paper is to generalize Northcott's inequality on Hilbert coeficients of I given in Northcott without assuming that A is a Cohen–Macaulay ring. They have investigated when their inequality turns into an equality. It is related to the Buchsbaumness of the associated graded ring of I.

In [3] it has contained a number of practical remarks on Hilbert series that authors expected to be useful in various contexts. Then authors worked with graded ring $R = \oplus_{n\geq 0} R_n$ that are finitely generated over an algebraically closed field k of characteristic 0 and satisfy $R_0 = k$. The Hilbert function of R is the numerical function $P_n = dim R_n$ for $n \geq 0$. Using the fractional Riemann-Roch formula of Fletcher and Reid to write out explicit formulas for the Hilbert series $P(t)$ in a number of cases of interest for singular surfaces and 3-folds in [3]. The concept of graded primary ideal and graded primary decomposition have been introduced in [95].

Definition 3.1. [95] Let I be a graded ideal of (R, G). Then:

- I is a graded prime ideal (in abbreviation, "G-prime ideal") if $I \neq R$; and whenever $rs \in I$, $r \in I$ or $s \in I$,where $r, s \in h(R)$.
- I is a graded maximal ideal (in abbreviation, "G-maximal ideal") if $I \neq R$ and there is no graded J of (R, G) such that $I \subset J \subset R$.
- The graded radical of I (in abbreviation "Gr(I)") is the set of all $x \in R$ such that for each $g \in G$ there exist $n_g > 0$ with $x_g^{n_g} \in I$. Note that, if r is a homogenous element of (R, G), then $r \in Gr(I)$ iff $r^n \in I$ for some $n \in N$.

Definition 3.2. [95] Let I be a graded ideal of (R, G). Then say that I is a graded primary ideal of (R, G) (in abbreviation, " G-primary ideal") if $I \neq R$; and whenever $a, b \in h(R)$ with $ab \in I$ then $a \in I$ or $b \in I$ or $b \in Gr(I)$.

Example 3.3. [95] Let $R = \mathbb{Z}[i]$ (The Gaussian integers) and let $G = \mathbb{Z}_2$. Then R is a G-graded ring with $R_0 = \mathbb{Z}$, $R_1 = i\mathbb{Z}$. Let $I = 2R$ be a graded prime ideal. Then I is a graded primary ideal. But I is not a primary ideal because 2 is not irreducible element of $R = \mathbb{Z}[i]$.

Definition 3.4. [95] Let I be a proper graded ideal of (R, G). A graded primary G-decomposition of I is an intersection of finitely many graded primary ideals of (R, G). Such a graded primary G-decomposition $I = Q_1 \bigcap Q_2 \bigcap \cdots \bigcap Q_n$ with $Gr(Q_i) = P_i$ for $i = 1, 2, \ldots n$ of I is said to be minimal graded primary G- decomposition of I precisely when

- $P_1, \ldots, P_n$ are different graded prime ideals of R, and
- $Q_j \not\supseteq \bigcap_{(i=1, j\neq i)}^{n} Q_i$ for all $j = 1, \ldots, n$.

Say I is G-decomposable graded ideal of (R, G) precisely when it has a graded primary G-decomposition.

A new direction in the study of graded ideals as well as an integration study of that done for the graded ideals had given in [95]. They believed that this work will lead to constructive ideals which introduce good tools for solving open problems of primary ideals and primary decomposition by turning them over into graded prime ideals and graded primary ideals and graded primary G-decomposition. Where G is non finitely generated abelian group. This work given in [95] be the primary ground to initiate more useful studies concerning the graded primary ideal and graded G-decomposition. They [95] had defined the graded primary G-decomposition of graded ideal and studied the uniqueness of this decomposition.

Several characterizations for the linearity property for a maximal Cohen-Macaulay module over a local or graded ring, as well as proofs of existence in some new cases have been given in [45] and also the proof of the existence of such modules is preserved when taking segre product, as well as when passing to veronese subring in low dimensions has been formed in [45]. One can study the radical theory of graded rings ([37], [120]). Two graded radical α^* and α^- of graded rings introduced in [100] Which can be associated with a given radical α of ordinary associative rings, and proved some result relating to them. A special graded class also defined in [100].

Let G be a finite group and let $\Lambda = \oplus_{g\in G}\Lambda_g$ be a strongly G-graded R-algebra, where R is a commutative ring with unity. Authors [13] proved that if R is a Dedekind domain with quotient field K, Λ is an R-order in a separable K-algebra such that the algebra Λ_1 is a Gorenstein R-order, then Λ is also a Gorenstein R-order. Further, proved that the induction functor $ind : Mod\Lambda_H \to Mod\Lambda$ in [13], for a subgroup H of G, commutes with the standard duality functor. Aoki [12] have shown that the graded ring of Siegel modular forms of $\Gamma_0(N) \subset S_p(2, \mathbb{Z})$ has a very simple unified structure for $N = 1, 2, 3, 4$, taking Neben-type case (the case with character) for $N = 3$ and 4. All are generated by 5 generators, and all the fifth generators are obtained by using the other four by means of differential operators, and it is also obtained as Borcherds products. The necessary and sufficient condition on a graded ring of finite support to be semiprime is given in [28]. Let G be a monoid with identity e, and let R be a G-graded commutative ring. Here Shahabaddin, Ching Mai studied the graded prime submodules of a G-graded

R-module. A number of results concerning of these class of submodules are given in [15]. While the bulk of this work is devoted to investigate the graded primary avoidance theorem for modules in [17].

In the context of algebraic geometry, the introduction of homogenous coordinate rings for toric varities (see [31]) gave new motivation for studying rings which have grading by general finitely generated abelian group in [34]. In [85], global primary decomposition of coherent sheaves over a toric variety is compared with graded primary decomposition of graded modules. The case of grading by torsion free abelian groups have been covered in [25]. However, for the homogenous coordinate rings one also has to consider grading by groups with torsion. Authors [86] claimed that a lot of work have been done on the theory of graded rings (see [22],[30], [64], [79]) a generlization of the treatment of [25] to the case of grading by finitely generated abelian groups were not available in their literature review. The aim of their work [86] is to fill this gap. An analogue of primary decomposition which works when G has torsion has been discussed in [86]. More precisely, they [86] discussed whether for some G-graded modules M and N, where N is a submodule of M, there exists a decomposition $N = \bigcap_{i \in I} Q_i$, where the Q_i are G-graded and $ann(M/Q_i)$ are irreducible in a suitable sense. The support of the ideal J is reducible, i.e. It is the union of two distinct closed proper subset in the Zariski topology of A^1_k. However, it is G-invariantly irreducible, i.e. It is not the union of two distinct G-invariant closed subset. The right notion for describing G-invariant irreducible subset in commutative algebra is that of G-prime ideals. A graded ideal $I \subset A$ is G-prime if and only if for every two G-graded ideals J, K, $JK \subset I$ implies $J \subset I$ or $K \subset I$. Note that G-prime ideals behave quite naturally, and essentially all elementary lemmas which hold for the usual prime ideals have a graded analouge for G-prime ideal. Relative to this idea, the following definition can be found in [86].

Definition 3.5. [86] Let M be a finitely generated G-graded A-module.

- An ideal $I \subset A$ is G-associated if and only if I is G-prime and $I = ann(x)$ for some element $x \in M$.
- $ASS^G M$ denotes the set of all G-associated ideals of M.
- M is G-coprimary if $ASS^G M = \{p\}$ for some G-prime ideal p.
- A G-graded submodule N of M is said to be G-primary if the quotient module M/N is G-coprimary.
- Let N be a G-graded submodule of M, then they call an expression $N = \bigcap_{i \in I} Q_i$ a G-primary decomposition of N in M if and only if the Q_i are G-primary submodules of M with $Ass^G M/Q_i = \{p_i\}$ and p_i are G-associated to M/N.

- A G-primary decomposition $N = \bigcap_{i \in I} Q_i$ of N in M is called reduced if all the p_i are distinct and there exists no $i \in I$ such that $\bigcap_{j \neq i} Q_j \subset Q_i$.

In [86] after introducing G-primary decomposition as a natural analogue to primary decomposition for G-graded R-modules and above notion they proved the following them.

Theorem 3.6. [86] Let G be finitely generated abelian group, R a G-graded commutative Noetherian ring. $N \subset M$ finitely generated, G-graded R-modules, and $N = \bigcap_{i \in I} Q_i$ a primary decomposition of N in M. Then:

- Let $Q_i^{'}$ be the largest submodule of M contained in Q_i. Then $Q_i^{'}$ is G-primary for every $i \in I$ and $N = \bigcap_{i \in J} Q_i^{'}$.
- There exists a subset J of I such that $N = \bigcap_{i \in J} Q_i$ is a reduced G-primary decomposition.
- If some $Q_i^{'}$ corresponds to a G-prime ideal p_i which is a minimal element of $Ass^G M/N$, then Q_i is grade.

Theorem 3.7. [65] Let K be any field and A Noetherian K-algebra. Let M be G-coprimary with respect to some G-prime ideal p. Then:

$$AssM = AssA/p$$

$\mathbb{Z}$-graded rings A and B. One can ask when the graded module categories gr-A and gr-B are equivalent. Using Z-algebra, [107] related the morita type results of Ahn-Marki and Del Rio to the twitting system introduced by Zhang, and proved for example.

Theorem 3.8. [107] If A and B are $\mathbb{Z}$-graded rings, then:

- If A is isomorphic to Zhang twist of B if and only if the $\mathbb{Z}$-algebras $A = L_{i,j \in \mathbb{Z}} A_{j-i}$ an $B = L_{i,j \in \mathbb{Z}} B_{j-i}$ are isomorphic.
- If A and B are connected graded with $A \neq 0$, then gr-A isomorphic gr-B if and only if A and B are isomorphic.

This simplifies and extends Zhang's results. A research on graded annihilators of modules over the frobenius skew polynomial ring and tight clousre has been explored in [105].

Let R be a commutative ring and let G be an abelian group. A graded ring R is called gr-Noetherian if it satisfies the ascending chain condition on graded ideals of R. Equivalently, R is gr-Noetherian if and only if every graded ideal of R is finitely generated (see[62]). A commutative ring R is called a Q-ring if every ideal in R is a finite product of primary ideals in R. Khashan [62] gave a generalizations of Q-rings to graded case and defined the QGR-ring as graded rings in which every graded ideal is a finite product of gr-primary ideals.

Definition 3.9. [62] Let R be a graded ring. Then R is said to be a QGR-ring if every graded ideal of R is a finite product of gr-primary ideals of R.

He [62] also proved some basic properties of QGR-ring and then he gave a characterization of gr-Noetherian QGR-ring.

In the book of Nastasescu and Van oystaeyen [75] on group graded rings, two equivalent description of graded Jacobson radical for rings with unity are given. Several investigations of graded Jacobson radical have appeared (see [1]- [29]) all for rings with unity. A comprehensive account of special radicals of graded rings without unity was presented in [18]. Unfortunately the descriptions given in the section for the Jacobson radical came from [75] on group graded rings with unity. After an extensive literature search, authors [39] seems that no actual definition of the graded Jacobson radical for rings without unity has appeared.

Definition 3.10. (Jacobson radical for ring without unity) [39] The graded Jacobson radical for group graded rings without unity is as the intersection of annihilators of simple modules.

The definition state above is the most natural one-the intersection of the annihilators of all simple graded module and it is meaningful more generally for semigroup graded rings, though for semigroups in general it may not be a graded ideal. As an example of consequence of this investigation, they have show that 1984 result of Nastasescu [78] that $nJ(R) \subseteq J_{gr}(R)$ (for a finite group G of order $n \in \mathbb{Z}^+$ where R is a G-graded ring with unity and J_{gr} is the G-graded Jacobson radical) can be the extended to group graded rings without unity.

The purpose of work in [71] to explore multi-graded analogues of some results in the algebra of modules, and particularly local cohomology modules, over a commutative Noetherian ring that is graded by the additive semigroup N_0 of non-negative integers. In 1995, T. Marley [72] had established connections between finitely graded local cohomology modules of M and local behaviour of M across Proj(R). For a finitely generated graded module$M$ over a positively-graded commutative Noetherian ring R, Sharp [104] established in 1999 some restrictions, which can be formulated in terms of the Castelnuovo regularity of M or the so-called a^*- invariant of M, on the supporting degrees of a graded-indecomposable graded injective direct summand, with associated prime ideal containing the irrelevant ideal of R, of any term in the minimal graded injective resolution of M. The purpose of [71] is to present some multi-graded analogues of the above-mentioned work.

In [108], author considered The first Weyl algebra, $A = L[X, Y]/(XY - YX - 1)$, where K is algebracally closed field of characteristic zero, in the Euler gradation, and completely classify graded rings B that are graded equivalent to A: that is, the categories gr-A and gr-B are equivalent. This included some surprising examples: in particular, A is graded equivalent to an idealizer in a localization of A. They obtained this classification as an application of a general Moritatype characterization of equivalences of graded module categories and proved:

Theorem 3.11. [108] Let S be a $\mathbb{Z}$-graded ring. Then S is graded equivalent to A if and only if S is graded Morita equivalent to some $S(J, n)$.

Another ring occurring in Theorem 2.19 is the Veronese ring $A^{(2)} = \oplus_{n \in Z} A_{2n} \cong S(\phi, 2)$. By Theorem 2.19. A and $A^{(2)}$ are graded equivalent. Of course one expects that Proj A (in the appropriate sense) and Proj $A^{(2)}$ will be equivalent, but, this is the first nontrivial example of an equivalence between the graded module categories of a ring and its Veronese.

Theorem 2.19 is an application of general results on equivalences of graded module categories. Given a $\mathbb{Z}$-graded ring R, an autoequivalence F of gr-R, and a finitely generated graded right R-module P, Sierra [108] proved that there is simpler way to construct a twisted endomorphism ring $End_R^F(P)$ and proved:

Theorem 3.12. [108] Let R and S be $\mathbb{Z}$-graded rings. Then R and S are graded equivalent if and only if there are a finitely generated graded projective right R-module P and an autoequivalence F of gr-R such that $\{F^n P\}_{n \in \mathbb{Z}}$ generates gr-R and $S \cong End_R^F(P)$.

In particular, Sierra[108] characterized graded Morita equivalences and Zhang twists in term of the picard group and analyzed the graded module category of the Weyl algebra and its Picard group.

They described Sierra [108] the graded K-theory of A, and in particular show that, in contrast to the ungraded case, if $P \oplus Q \cong P \oplus Q'$ where P, Q, and Q' are finitely generated graded projective modules, then $Q \cong Q'$.

Numerical invariants of a minimal free resolution of a module M over a regular local ring (R, m) can be studied in [99] by taking advantage of the rich literature on the graded case. The key is to fix suitable m-stable filtrations M of M and to compare the Betti numbers of M with those of the associated graded module $gr_M(M)$. This approach has the advantage that the same module M can be detected by using different filtrations on it. It provided interesting upper bounds for the Betti numbers and they [99] studied the modules for which the extremal values are attained. Among others, the Koszul modules have this behavior.

Oinert [80] gave a review of the basics of graded ring theory and also described the background to the problems that they have considered. They [80] laid out the general theory of (group) graded rings and describe some special cases; skew group rings, twisted group rings, crossed products, strongly graded rings, pre-crystalline graded rings, crystalline graded rings and crossed product-like rings.

In [35] the Betti table of a graded module M over a graded ring R is numerical data consisting of the minimal number of generators in each degree required for each syzygy module of M. In their remarkable paper [111], Mats Boij and Jonas Soderberg conjectured that the Betti table of a Cohen-Macaulay module over a polynomial ring is a positive linear combination of Betti tables of modules with pure resolutions. They [35] proved a strengthened form of their conjectures in [35].

A variety X admits homogeneous coordinates if there exists an affine variety Z together with an action of a diagonalizable algebraic group H and an open subset W such that X is a good quotient of W by H. Then the coordinate ring S of $\mathbb{Z}$ acquires a grading by the character group of H and S serves as a homogeneous coordinate ring for X with respect to this grading. This setting comes with a natural sheafification functor $F \to \tilde{F}$, which maps a graded S-modules to a quasi-coherent sheaf on X. This generalized the usual homogeneous coordinate rings for projective spaces and toric varities in [88].

For primary decomposition of sheaves on X it would seem to be sufficient to look at primary decompositions of graded S-modules. However, it is not clear in [88] that a graded primary decomposition of some S-module F yields a proper primary decomposition of sheaves of $\tilde{F}$. One can get the proof in [88] that this at least holds if X is a geometric quotient of W by H. G is an algebraic group which contains H as a normal group. Graded G-equivariant primary decomposition over S with $\frac{G}{H\text{âŁ¢}}$- equivariant primary decomposition over X. It has been compared in [88] as an explicit application, equivariant primary decomposition for sheaves of Zariski diferential over toric varieties has been constructed in [88].

The local cohomology of finitely generated bigraded modules over a standard bigraded polynomial ring which have only one non vanishing local cohomology with respect to one of the irrelevant bigraded ideals have been studied in [91].

Let G be a group with identity e, R be a G-graded commutative ring, and M be a graded R-module. Graded primary submodules of graded multiplication modules characterized in [42] . Second submodules of modules over commutative rings were introduced in [121] as the dual notion of prime submodules. This submodule class has been studied in detail by some authors ([8], [10]). Second modules over arbitrary rings were defined in [4] and used as a tool for the study of attached prime ideals over noncommutative rings. In [26], second modules have been studied in detail in the noncommutative ring. The authors [9] have introduced and studied graded second modules over commutative graded rings. Most of their results are related to [121] which have been proved for second submodules.

The concept of coprimary module which is generalization of second modules has been intoduced [68]. They have characterizations and properties of this module class and study coprimary decomposition of modules.

Secondary modules are generalization of second modules over commutative rings. In [103] secondary module were considered over commutative graded rings. Sharp [103] defined graded secondary modules and used them as a tool for the study of asymptotic behavior of attached prime ideals.

Authors [26] introduced the concept of graded second and graded coprimary modules which are different from second and coprimary modules over arbitrary graded rings and also study graded prime submodules of modules with gr-coprimary decomposition. They [26] have deal with graded secondary representations for graded injective modules over commutative graded rings. By using the concept of σ-suspension $(\sigma)M$ of a graded module M, they [26] proved that a graded injective module over a commutative graded Noetherian ring has a graded secondary representation.

Definition 3.13. (Graded second Modules) [26] Let R be a G-graded ring. A graded R-module M is said to be a graded second (or gr-second) R-module $M \neq 0$ and $ann_R(M) = ann_R(M/N)$ for every proper submodule N of M.

The work presented in [20] has two objectives. First, discussed the application of the theory developed in [86] for G-associated ideals, that is, the behavior of G-associated ideal (Ass^G) with short exact sequences. Second, they introduced strong Krull G-associated ideal (Ass^{SG}) with flat base change of rings, over rings graded by finitely generated abelian groups, and established a relationship between strong Krull G-associated ideals and G-associated ideals with the corresponding associated ideals in polynomial rings by using technique developed in [54]. The reason to discuss the strong Krull G-associated prime ideals is that for non-Noetherian rings the G-graded primary decomposition may not exist. They studied the properties of non-Noetherian rings more closely by using strong Krull G-associated prime ideal. These results have application in algebraic geometry, for instance for the study of toric varieties which are not neccessarily Noetherian and which arise in the study of representation of Kac-Moody groups. Using the theory developed in [20] proved this theorem as an application on polynomial rings.

Theorem 3.14. [20] Let M be a G-graded R-module and T an indeterminate. Then

- $Ass^G(M \otimes_A A[T]) \subseteq Ass^{SG}(M \otimes_A A[T])$.
- $\{PA[T] : P \in Ass^{SG}(M)\} \subseteq Ass^{SG}(M \otimes_A A[T])$.
- $Ass^G(M \otimes_A A[T]) \subseteq \{PA[T] : P \in Ass^{SG}(M)\}$.

In commutative Noetherian ring, every ideal has primary decomposition and this decomposition can be created as a generlization of the factorization of an integer $n \in \mathbb{Z}$ into the product of prime powers. For polynomial ring it was proved by Lasker Notherian. But this is not true in non commutative ring for example the ring of 2×2 upper triangular matrices with entries from the field of rational number does not have primary decomposition (see [65]). If there is primary decomposition but it is not unique, we can see in [69], $\mathbb{Z}$ and ring of polynomial $K[X_1, \ldots, X_n]$ where K is a field, both are unique factorization domain. But this is not true for arbitrary commutative rings, even if they are integral domains for example ring $Z[\sqrt{-5}]$, 6 has two essential distinct factorizations, 2.3 and $(1 + \sqrt{-5})(1 - \sqrt{-5})$. Some types of graded ring appear as homogenous cordinated ring for toric varieties. Perling [84] has shown that for any toric variety X there exist a homogenous cordinate ring $R = \oplus R_g$ such that X can be identified with set of homogenous prime ideals of R minus certain exceptional subset although a lot of work has been done in the area of graded ring for the last decades (see [86]).

The uniqueness of a graded primary decomposition of graded over finitely generated abelian groups has been established in [65]. They also gave a new proof of the main result in [86] on existence of G- primary decomposition as a by product and also introduced the concept of G-graded primary submodules, G-graded P-primary subodules and their properties for rings graded over a finitely generated abelian group G. This work is motivated by articles [25], [63], [79] and [95].

Definition 3.15. [65] A G-graded submodule N of M is called G-graded primary or G-primary if $N \neq M$ and for each $a \in h(R)$, the homothety $\Lambda_a : M/N \to M/N$ defined by $\lambda_a(x+N) = ax + N$ is either injective or nilpotent. An ideal I of R is called G-graded primary ideal if it is a G-graded primary submodule of R.

Definition 3.16. [65] If N is a G-gaded primary submodule of M and $P = Gr_M(N)$, then N is called a G-graded P-primary.

Theorem 3.17. (First Uniqueness Theorem) [65] Let M be a finitely generated G-graded module over a G-graded Noetherian ring R. If $N = \bigcap_{i \in I} N_i$ is a reduced G-graded primary decomposition of N, N_i being graded G-graded P_i -primary, then P_i are uniquely determined by N.

Theorem 3.18. (Second Uniqueness Theorem) [65] Let $N = \bigcap_{i \in I} N_i$ be a reduced primary decomposition of N, N_i being G-graded P_i -primary. If P_i is minimal, then N_i is uniquely determined by N.

Let Γ be a cancelation monoid with the neutral element e. Consider a Γ-graded ring $R = \oplus_{\gamma \in \Gamma} R_\gamma$, which is not necessarily commutative. Huishi [51], proved that R_e, the degree-e part of R, is a local ring in the classical sense if and only if the graded two-sided ideal M of R generated by all non-invertible homogeneous elements is a proper ideal. He [51] defined a Γ-graded local ring R in terms of this equivalence, it is proved that any two minimal homogeneous generating sets of a finitely generated Γ-graded R-module have the same number of generators.

In literature review many algebraist studied the graded primary submodules of a G-graded R-module. A number of results concerning this class of submodules are given in [16].

The graded primary decomposition of graded module has been studied in [50]. They have discussed some preliminary results which are extensively used their work. They [50] established that if M is a graded free R-module and I, a proper graded ideal of R with graded primary decomposition then a graded submodule IM has a graded primary decomposition. If N is a graded submodule of M with gr-primary decomposition then there exists a graded ideal $(N :_R M)$ of R with gr-primary decomposition. They [50] have proved that if R is a graded ring with gr-dim$(R) = 1$ and M is gr-Noetherian R-module then for any gr-submodule N of M, the graded ideal $(N :_R M)$ can be expressed as product of graded primary ideals $(N_i : M)$ of $R(i = 1, 2, \ldots, k)$, where N_i is gr-primary submodule of M.

Let G be a group, R a G-graded ring and M a G-graded R- module. Then the relation between the category of gr -R - modules and their identity components for the weak multiplication property studied [2]. Some results concerning graded prime submodules introduced [2].

In [7], Ansari et al. introduced the notion of graded comultiplication modules and obtained some related results. Authors [6] introduced the dual notion of multiplication modules and investigated some properties of this class of modules. Secondary modules, completely irreducible submodule and p-interior has been given in [11].

Definition 3.19. (Secondary Module) [11] A non zero R-module M is said to be secondary if for each $a \in R$ the endomorphism of M given by multiplication by a is either surjective or nilpotent.

Authors [11] have got some results concerning second modules by using the notion of the P-interior of N relative to M. Moreover, they [11] had given some characterization for secondary module.

A commutative ring is graded by an abelian group if the ring has a direct sum decomposition by additive subgroups of the ring indexed over the group, with the additional condition that multiplication in the ring is compatible with the group operation. Johnson [56] developed a theory of graded rings by defining analogues of familiar properties such as chain conditions, dimension, and Cohen-Macaulayness. The preservation of these properties when passing to gradings induced by quotients of the grading group has been studied in [56].

G-graded twisted algebras were introduced in [36], and independently [119], as distinguished mathematical structures which arise naturally in theoretical physics (see[117], [118]). From this algebra one can understand the following : In [59] let G denote a group. An R-algebra W (not necessarily commutative, neither associative) will be called a G-graded twisted algebra if there exists a G-grading, i.e., $W = \oplus_{g \in G} W_g$, with $W_a W_b \subset W_{ab}$, in which each summand W_g is an R -module of free rank one. They assumed that W has an identity element $1 \in W_e$, where W_e denotes the graded component corresponding to the identity element e of G and required that W has no monomial zero divisors, i.e., for each pair of nonzero elements $w_a \in W_a$, and $w_b \in W_b$, their product must be non zero, $w_a w_b \neq 0$. Besides its interest for physicists, these algebras are natural objects of study for mathematicians, since they are related to generalizations of Lie algebras. In [59], methods of group cohomology are used to study the general problem of classification under graded isomorphisms. A full description of these algebras in the associative cases, for complex and real algebras. In the nonassociative case, an analogous result is obtained under a symmetry condition of the corresponding associative function of the algebra, and when the group providing the grading is finite cyclic.

The positively $\mathbb{Z}$-graded polynomial ring $R = K[X, Y]$ over an arbitrary field K

and Hilbert series of finitely generated graded R-modules has been considered in [57]. The central result is an arithmetic criterion for such a series to be the Hilbert series of some R-module of positive depth.In the generic case, that is, $deg(X)$ and $deg(Y)$ being coprime, this criterion can be formulated in terms of the numerical semi group generated by those degrees.

Let G be an arbitrary group with identity e and let R be a G-graded ring. Graded semiprime ideals of a commutative G-graded ring with nonzero identity defined in [38] and they [38] gave a number of results concerning such ideals. Also, they [38] extended some results of graded semiprime ideals to graded weakly semiprime ideals.

For any graded commutative Noetherian ring, where the grading group is abelian and where commutativity is allowed to hold in a quite general sense, they [32] established an inclusion-preserving bijection between, on the one hand, the twist-closed localizing subcategories of the derived category, and, on the other hand, subsets of the homogeneous spectrum of prime ideals of the ring.

Let R be a Noetherian local ring. They [89] defined the minimal j - multiplicity and almost minimal j - multiplicity of an arbitrary R - ideal on any finite R - module. For any ideal I with minimal j - multiplicity or almost minimal j - multiplicity on a CohenâŁ“Macaulay module M, they [89] proved that under some residual conditions, the associated graded module $gr_1(M)$ is Cohen - Macaulay or almost Cohen - Macaulay, respectively. Their work generalized the results for minimal multiplicity.

Let G be a multiplicative group, R a G-graded commutative ring and M a G-graded R-module. Then various properties of multiplicative ideals in a graded ring are discussed in [61] and authors extended this to graded modules over graded rings. The set of P -primary ideals and modules of R when P is a graded multiplication prime ideals and modules are studied in [61].

Cox [31] introduced the homogeneous coordinate ring S of a toric variety X and compute its graded pieces in terms of global sections of certain coherent sheaves on X. The ring S is a polynomial ring with one variable for each one-dimensional cone in the fan $\triangle$ determining X, and S has a natural grading determined by the monoid of effective divisor classes in the Chow group $A_{n-1}(X)$ of X (where $n = dimX$). Using this graded ring, X behaves like projective space in many ways has been shown in [31].

It was shown by Bergman that the Jacobson radical of a Z-graded ring is homogeneous. In [110] the analogous result holds for nil radicals namely, that the nil radical of a Z-graded ring is homogeneous. It is obvious that a subring of a nil ring is nil, but generally a subring of a Jacobson radical ring need not be a Jacobson radical ring. In [110], it is shown that every subring which is generated by homogeneous elements in a graded Jacobson radical ring is always a Jacobson radical ring. It is also observed that a ring whose all subrings are Jacobson radical rings is nil. Some new results on graded-nil rings are also obtained in [110].

A new set of invariants associated to the linear strands of a minimal free resolution of a $\mathbb{Z}$-graded ideal $I \subseteq R = K[x_1, \ldots, x_n]$ introduced in [74]. They also proved that these invariants satisfy some properties analogous to those of Lyubeznik numbers of local rings. For the case of squarefree monomial ideals they achieved more insight on the relation between Lyubeznik numbers and the linear strands of their associated Alexander dual ideals. Finally, in [74] proved that Lyubeznik numbers of Stanley-Reisner rings are not only an algebraic invariant but also a topological invariant, meaning that depend on the homeomorphic class of the geometric realization of the associated simplicial complex and the characteristic of the base field.

In [53] the aim is twofold. First, studied generalizations of graded injective modules. Second, provided a characterization of graded quasi-Frobenius rings in terms of graded mini-injective rings.

Let R be a polynomial ring over a field. In [52] authors proved an upper bound for the multiplicity of R/I when I is a homogeneous ideal of the form $I = J+(F)$, where J is a Cohen-Macaulay ideal and $F \notin J$. The bound is given in terms of two invariants of R/J and the degree of F. They have shown that ideals achieving this upper bound have high depth, and provided a purely numerical criterion for the Cohen-Macaulay property. Applications to quasi-Gorenstein rings and almost complete intersections are given in [52].

Emil [40] investigated the graded Brown–McCoy and the classical Brown–McCoy radical of a graded ring, which is the direct sum of a family of its additive subgroups indexed by a nonempty set, under the assumption that the product of homogeneous elements is again homogeneous. There are two kinds of the graded BrownâŁ"McCoy radical, the graded BrownâŁ"McCoy and the large graded BrownâŁ"McCoy radical of a graded ring. In [40] proved that the large graded BrownâŁ"McCoy radical of a graded ring is the largest homogeneous ideal contained in the classical Brown–McCoy radical of that ring.

In [41] Let I be a homogenous ideal of a polynomial ring $S = K[X_1, \dots, X_d]$ over a field K with usual grading. Bertram, Ein and Lazarsfeld [23] have initiated the study of the Castelnuovo-Mumford regularity of I^n as a function of n by proving that if I is the defining ideal of a smooth complex projective variety, then reg(I^n) is bounded by a linear function of n. Let $R = R_0[X_1, \dots, X_d]$ be a Noetherian standard N-graded algebra over Artinian local ring (R_0, m). In particular, R can be a coordinate ring of any projective variety over any field with usual grading. Let $I_1, \dots, I_t$ be homogenous ideals of R and M a finitely generated N-graded R-module. Ghosh [41] proved that there exist two integers k, k' such that

$$reg(I_1^{n_1} \dots I_t^{n_t M}) \le (n_1 + \dots + n_t)k + k'$$

$$\text{for all } n_1, \dots, n_t \in N.$$

Using E-algebraic branching systems, various graded irreducible representations of a Leavitt path K-algebra L of a directed graph E are constructed. The concept of a Laurent vertex is introduced and it is shown that the minimal graded left ideals of Lare generated by the Laurent vertices or the line points leading to a detailed description of the graded socle of L. Following this, a complete characterization was obtained of the Leavitt path algebras over which every graded irreducible representation is finitely presented. A useful result is that the irreducible representation $V_{[p]}$ induced by infinite paths tail-equivalent to an infinite path p (say this a Chen simple module) is graded if and only if p is an irrational path [48]. They also have showen that every one-sided ideal of L is graded if and only if the graph E contains no cycles. Since by [46] every Leavitt path algebra is graded von Neumann regular, it is natural to consider the subclass of Leavitt path algebras which are graded self injective. They [48] have shown that L is graded.

Consider a generalization $K_0^{gr}(R)$ of the standard Grothendieck group $K_0(R)$ of a graded ring R with involution. If Γ is an abelian group, proved K_0^{gr} completely classifies graded ultramatricial $*$-algebras over a Γ-graded $*$-field A such that (1) each nontrivial graded component of A has a unitary element in which case we say that A has enough unitaries, and (2) the zero-component A_0 is 2-proper (for any $a, b \in A_0$, $aa^* + bb^* = 0$ implies $a = b = 0$) and $*$-pythagorean (for any $a, b \in A_0$, $aa^* + bb^* = cc^*$ for some $c \in A_0$) (see [49]). If the involutive structure is not considered, their result implies that K_0^{gr} completely classifies graded ultramatricial algebras over any graded field A. If the grading is trivial and the involutive structure is not considered, some well known results obtained as corollaries.

As an application of their results, proved that the graded version of the Isomorphism Conjecture holds for a class of Leavitt path algebras: if E and F are countable, row-finite, no-exit graphs in which every path ends in a sink or a cycle and K is a 2-proper and $*$-pythagorean field, then the Leavitt path algebras $L_K(E)$ and $L_K(F)$ are isomorphic as graded rings if any only if they are isomorphic as graded $*$-algebras. Also presented examples which illustrated that K_0^{gr} produces a finer invariant than K_0.

Let K be a field, and let $R = K[X_1, \ldots, X_n]$ be the positively $\mathbb{Z}$-graded polynomial ring with $degX_i = d_i \geq 1$ where $i = 1, \ldots, n$. Consider a finitely dimensional R-module $M = \oplus_k M_k$ over R. The graded components M_k of M are finitely dimensional K-vector spaces and since R is positively graded, $M_k = 0$ for $k << 0$. The formal Laurent series

$$H_M(t) := \sum_{k \in z} (dim_k M_k) t^k \in \mathbb{Z}[[t]][t^{-1}]$$

is called the Hilbert Series of M. Obviously every coefficient of this series is non negative. Moreover, it is well-known that $H_M(t)$ can be written as a rational function with denominator $(1 - t^{d_1}), \ldots, (1 - t^{d_n})$ (see [67]).

In [67] authors discussed a result on formal Laurent series and some of its implications for Hilbert series of finitely generated graded modules over standard-graded polynomial rings: For any integer Laurent function of polynomial type with non-negative values the associated formal Laurent series can be written as sum of rational function of the form of $\frac{Q_j^t}{(1-t)^j}$, where the numerator is Laurent polynomials with non negative integer coefficient. Hence any such series is the Hilbert series of some finitely generated graded module over a suitable polynomial ring $K[X_1, \ldots, X_n]$. They have given two further applications, namely an investigation of the maximal depth of module with a given Hilbert series and characterization of Laurent polynomials which may occur as numerator in the presentation of Hilbert series as a rational function with powers of $(1-t)$ as denominator. In fact, in the standard graded case i.e $(d_1 = \cdots = d_n = 1)$, these two properties characterize the Hilbert series of finitely generated R-module among the formal Laurent series $\mathbb{Z}[[t]][t^{-1}]$, (see [114] Cor.2.3).

In the non-standard graded case, the situation is more involved. A characterization of Hilbert series was obtained by Fernandez and Uliczka in [57]:

Theorem 3.20. (Mayano-Uliczka) [67] Let $P(t) \in Z[[t]][t^{-1}]$ be a formal Laurent series which is rational with denominator $(1-t^{d_1}), \ldots, (1-t^{d_n})$. Then P can be realized as Hilbert Series of some finitely generated R-module if and only if it can be written in the form

$$P(t) = \sum_{I \subseteq \{1,\ldots,n\}} \frac{Q_I(t)}{\prod_{i \in I}(1-t^{d_i})} \tag{2}$$

with non negative $Q_I \in \mathbb{Z}[t, t^{-1}]$. However, it remained an open question in [67] if the condition of the Theorem (2.15) is satisfied by every rational function with the given denominator and nonnegative coefficient. They [57] answered this question to negative and they provided example of rational function that do not admit a decomposition (1) and are thus not realizable as Hilbert series (see [67]).

Example 3.21. [67] Consider the rational function

$$\begin{aligned} P(t) &:= \frac{1}{(1-t^2)(1-t^5)} - \frac{t^4}{(1-t^3)(1-t^5)} \\ &= \frac{1}{2}(1+t^2+\frac{t^6}{1-t^2}+\frac{1+t^6}{1-t^5}+\frac{t^{12}}{(1-t^3)(1-t^5}) \end{aligned}$$

$P(t)$ can not be written as a non negative integral linear combination. On the other hand, the second line gives a rational number decomposition. This shows in particular that the coefficient of the series of P are non negative.

On the other hand, claimed that the answer is positive by showing the following theorem:

Theorem 3.22. [67] Assume that the degrees $d_1, \ldots, d_n$ are pairwise either co-prime or equal. Then the following holds:

- If $n = 2$, then every rational function $P(t) \in Z[[t]][t^{-1}]$ with the given denominator and nonnegative coefficients admits a decomposition as in 1 of Theorem (2.15).
- In general, the same still holds up to multiplication with a scalar.

Moreover, they have provided following example:

Example 3.23. [67] The condition that degree $\delta_1, \ldots, \delta_r$ are pairwise co-prime is essential, as the following example shows. Consider the rational function

$$P(t) := \frac{1+t-t^6-t^{10}-t^{11}-t^{15}+t^{20}+t^{21}}{(1-t^6)(1-t^{10})(1-t^{15})}$$

$$= \frac{1 + t + t^7 + t^{13} + t^{19} + t^{20}}{1 - t^{30}}.$$

One can read off from the second line that P(t) can not be written as a sum with positive coefficients and the required denominator: The coefficient of t^0.

The Weyl algebra over a field K of characteristic 0 is a simple ring of Gelfand-Kirillov dimension 2, which has a grading by the group of integers. All $\mathbb{Z}$-graded simple rings of GK-dimension 2 classified in [21] and show that they are graded Morita equivalent to generalized Weyl algebras as defined by Bavula (see [21]). More generally, Bell et al studied $\mathbb{Z}$-graded simple rings R of any dimension which have a graded quotient ring of the form $K[t, t^{-1}; \sigma]$ for a field K. By some further hypotheses, classified all such R in terms of a new construction of simple rings which is introduced in [21]. In the special case $GKdimR = tr.deg(K/k) + 1$, they [21] have shown that K and σ must be of a very special form. They [21] defined the new simple ring study from the perspective of non commutative geometry.

In [5] if R is a graded ring then defined a valuation on R induced by graded structure, and proved some properties and relations for R. Later they [5] have shown that if R is a graded ring and M a graded R-module then there exists a valuation on of M which was derived from graded structure and also proved some properties and relation R. They [5] gave a new method for finding the Kurll dimension of a valuation ring.

References

[1] Abrams G and Menini C, Embedding modules in graded modules over a semigroup-graded ring, *Comm. Algebra*, 2001, 29, 2611–2625.

[2] Abu-Dawwas R, Some Remarks on Graded Weak Multiplication Modules, *Int. J. Contemp. Math. Sciences*, Vol. 6, 2011, no. 14, 681–686.

[3] Altinok S, Hilbert series and application to graded rings, *IJMMS 2003:7*, 397–403.

[4] Annin S, Attached primes over noncommutative rings, *J. Pure Appl. Algebra*, 212 (2008), no. 3, 510–521.

[5] Anjom and Hosseini M.H, Valuation Derived from Graded Ring and Module and Krull Dimension Properties, *3s. v. 35 2 (2017): 93–103.c SPM – ISSN-2175-1188 on line ISSN-00378712 in press doi:10.5269/bspm.v35i2.29429.*

[6] Ansari-Toroghy H and Farshadifar F, The dual notion of multiplication modules, *Taiwanese Journal of Mathematics*, 2007; 11(4): 1,189–1,301.

[7] Ansari-Toroghy H and Farshadifar F, Graded Comultiplication Modules, *Chiang Mai J. Sci.* 2011; 38(3):311–320.

[8] Ansari-Toroghy H and Farshadifar F On the dual notion of prime submodules II, *Mediterr. J. Math.*, 9 (2012), no. 2, 327–336.

[9] Ansari-Toroghy H and Farshadifar F, On grade second modules, *Tamkang Journal of Mathematics*, 43 (2012), no. 4, 499–505.

[10] Ansari-Toroghy H and Farshadifar F, On the dual notion of prime submodules, *Algebra Colloq.*, 19 (2012), no. 1, 1109–1116.

[11] Ansari-Toroghy H,Farshadifar F,Pourmortazavi S.S and Khalipane F On Secondary Modules, *International journal of Algebra*, Vol. 6, 2012, no. 16, 769–774.

[12] Aoki H, Simple graded rings of siegel modular forms,differntial operators and borcherds products, *Int. J. Math.*, 16, 249 (2005). DOI: http://dx.doi.org/10.1142/S0129167X05002837.

[13] Apostolidi T and Vavatsoulas H, On strongly graded Gorestein orders, *Algebra and Discrete Mathematics*, Number 2. (2005). pp. 80–89.

[14] Artin and J.J. Zhang, Noncommutative projective schemes, *Adv. Math.*, 109(2), 1994, 228–287.

[15] Atani S.E., On Graded Prime Submodules, *Chiang Mai J. Sci.*, 2006; 33(1):3–7.

[16] Atani S.E., and Farzalipour F, On graded secondary modules, *Turk J. Math*, 31, (2007).

[17] Atani S.E. and Tekir U, On the Graded Primary Avoidance Theorem, *Chiang Mai J. Sci.* 2007; 34(2):161–164.

[18] Balaba I.N, Special radical of graded ring, *Bul. Acad. Stiinte Repub. Mold., Mat*, 2004, No. 1(44), 2004, 26–33.

[19] Beattie A.M, A generalization of the smash product of graded ring, *J. Pure Apple. Algebra*, 1988, 51, 219–226.

[20] Behara S. and Kumar S.D, Group graded associated ideals with flat base change of rings and short exact sequences, *Proceedings-Mathematical Sciences*, (2011).

[21] Bell J. and Rogalski D, Z-graded simple ring *American Mathematical Society*, Volume 368, Number 6, June 2016, Pages 4461–4496.

[22] Bergman G, Everbody knows what a Hopf algebra is in:, *Contemp. Math.*, vol. 43, 1985, pp. 25–48.

[23] Betram A, Ein L. and Lazarsfeld R, Vainshing theorem,a theorem of severi,and the equations defining projective varities, *J.Amer. Math. Soc.*, 4 (1991) 587–602.

[24] Borovik A, Mathematics under the microscope, *American Mathematical Society Publication*, Providence, 2010.

[25] Bourbaki N, commutative algebra, *Berlin-Heideberg-New York: Springar-Verlag*.

[26] Ceken S. and Alkan M, On graded second and coprimary Modules and Graded Secondary Representations, *Akdeniz University Department of Mathematics Antalya, Turkey* in prepartion, 2010.

[27] Ceken S, Alkan M. and Smith P.F, H. Ansari-Toroghy and F. Farshadifar, Second modules over noncommutative ring, *communication in Algebra*, 41 (2013), no. 1, 83–89.

[28] Chifen N, semiprime graded rings of finite, *Bull. Math. Soc. Sc. Math. Roumanie*, Tome 19(97) No. 1, 2006, 25–30.

[29] Cohen M, Rowen L.H., Group graded ring, *Commun Algebra*, 1983, 11, 1253–1270.

[30] Cohen M. and Montgomery S., Group-graded rings, smash products, and group actions, *Trans. Amer. Math. Soc.*, 282(1), (1984), 237–258.

[31] Cox D.A., The homogenous coordinate ring of toric variety, *J. Algebraic Geom.*, 4(1), (1995), 17–50.

[32] Dell–Ambrogio I. and Stevenson G., On the derived category of a graded commutative Noetherian ring, *Journal of Algebra*, 373 (2013), 356–376.

[33] Divinsky N.J, Rings and Radical, *University of Toronto press*, 1965.

[34] Eisenbud D, Mustata M. and Stillman M, Cohomology on toric varities and local cohomology with monomial supports, *J. Symb. Comp.*, 29(2000), 583–600.

[35] Eisenbud D. and Schreyer F, Betti numbers of graded modules and cohomology of vector bundles, *American Mathematical Society*, Volume 22, Number 3, July 2009, Pages 859–888.

[36] Edwards C. and Lewis J, Twisted Group Algebras I, *Commun. math. Phys*. 13, 119–130 (1969).

[37] Fang H. and Stewart P, Radical theory of graded rings, *J. Austr. Math. Soc.*, 52 (series A) 1992, 143–153.

[38] Farzalipour F. and Ghiasvand P, On graded semiprime and graded weakly semiprime ideals, *International Electronic Journal of Algebra*, Volume 13 (2013), 15–22.

[39] Gardner B.J. and Plant A, The graded Jacobson radical of associative rings, *ISSN 1024-7696* no. 1(59), (2009), 31–36.

[40] Georgijevic E, On graded Brown–McCoy radicals of graded rings, *J. Algebra Appl.* DOI: http://dx.doi.org/10.1142/S0219498816501437,2015.

[41] Ghosh D, Asymtotic linear bounds of Castelnuovo-Mumford regularity in multigraded modules, *Journal of Algebra*, 445(2016), 103–114.

[42] Ghiasvand P, On Graded Primary Submodules of Graded Multiplication Modules, *International Journal of Algebra*, Vol. 4, 2010, no. 9, 429–432.

[43] Haefner J, On when a graded ring is graded equivalent to a crossed product, *American Mathematical Society*, Volume 124, Number 4, April 1996.

[44] Haefner J. and Pappacena C.J, Strongly graded hereditary orders, *arXiv:math/0108192v1* [math.RA] 28 Aug 2001.

[45] Hanes D, On the Cohen-Macaulay modules of graded subrings, *Transactions of the American Mathematical Society*, Volume 357, Number 2, Pages 735–756, S 0002-9947(04)03562-7.

[46] Hazrat R, Leavitt path algebras are graded von Neumann regular rings, *J. Algebra*, 401 (2014), 220–233.

[47] Hazrat R, Graded Rings and Graded Grothendieck Groups, *University of Western Sydney*, Australia July 3, 2015.

[48] Hazrata R. and Kulumani M.R, On graded irreducible representations of Leavitt path algebras, *Journal of Algebra*, 450 (2016), 458–486.

[49] Hazrat R, and Liavas, K-Theory classification of graded ultramatricial algebras with involution, *arXiv:1604.07797v1* [math.RA] 26 Apr 2016.

[50] Helen K, Saikia and Das Gayatri, On Graded Primary Decomposition of a Graded Module, *Indian Journal of Mathematical Science*, Vol. 7, No. 2, (December 2011): 121–126.

[51] Huishi Li, On Monoid Graded Local Rings, *arXiv:1108.0258v2* [math.RA] 18 Aug 2011.

[52] Huneke C, Mantero P, Mccullough J, and Seceleanu A, A multiplicity bound for graded rings and a criterion for the cohen-macauley property, *Proceedings of the American Mathematical Society*, Volume 143, Number 6, June 2015, Pages 2365–2377, S 0002-9939(2015)12612-3.

[53] Hussien S, Essam El-Seidy and Tabarak M.E, Graded Injective Modules and Graded Quasi-Frobenius Rings, *Applied Mathematical Sciences*, Vol. 9, 2015, no. 23, 1113–1123.

[54] Iroz J. and Rush E.D, Associated prime ideals in non Noetherian rings, *Canad. J. Math.*, 36(2), (1984), 237–298.

[55] Jensen A. and Jondrup S, Smash products,group actions and group graded rings, *Math. Scad.*, 68 (1991), 161–170.

[56] Johnson and Brian P, Commutative Rings Graded by Abelian Groups, Dissertations, *http://digitalcommons.unl.edu/mathstudent/37*, (2012).

[57] Jose J, Fernandez M. and Uliczka J., Hilbert depth of graded modules over polynomial rings in two varaible, *Journal of Algebra*, 373 (2013), 130–152.

[58] J.P. Serre, Faisceaux algebrique coherents, *Ann. of Math.*, 61, 1955, 197–278.

[59] Juan D. V Elez, Luts A. Wills and Natalia Agudelo, On the classification of G-graded twisted algebra, *arXiv:1301.5654v1 [math.RA] 23 Jan 2013*.

[60] Karpilovsky G, The Jacobson radical of classical rings, *Wiley and Sons, New York*, 1991.

[61] Khaksari A. and Rasti Jahromi F, Multiplication Graded Modules, *International Journal of Algebra*, Vol. 7, 2013, no. 1, 17–24.

[62] Khashan H.A, Graded Rings in which Every Graded Ideal is a Product of Gr-Primary Ideals, *International Journal of Algebra*, Vol. 2, 2008, no. 16, 779–788.

[63] Khashan, H.A., Graded rings in which every graded ideal is a product of Gr-primary ideals, *Int. J. Algebra*, (2008) 2:779–788.

[64] Kumar S.D, A note on a graded ring analogue of Quillen's theorem, *Expo. Math.* 22 (2004), 297–298.

[65] Kumar S.D. and Behara S., Uniqueness of graded primary decomposition of graded modules graded over finitely generated abelian groups, *Communications in Algebra*, **39**, (2011).

[66] Lazard D, Autour de la platitude, *Bull.Soc.Math*, France 97 (1969), 81–128.

[67] Lukas Katthana, Jose J, Fernandez M and Uliczkac J, Hilbert series of modules over positively graded polynomial rings, *Journal of Algebra* 459 (2016), 437–445.

[68] Maani-Shirazi M. and Smith P.F, Uniqueness of coprimary decompositions, *Turk. J. Math.*, 31 (2007), no. 1, 53–64.

[69] Macdonald A, Commutative Algebra, *Addison Wesley Publishing Company*, 1969.

[70] Macdonald I. G, Secondary representation of modules over a commutative ring, *Sympos. Math. XI*, (1973), 23–43.

[71] Markus P. Brodmann and Sharp R.Y, Supporting degrees of multi-graded local cohomology modules, *Journal of Algebra*, 321 (2009), 450–482.

[72] Marley T, Finitely graded local cohomology modules and the depths of graded algebras, *Proc. Amer. Math. Soc.*, 123 (1995), 3601–3607.

[73] Matsumura H, Commutative ring theory, *Cambridge University Press* (2002).

[74] Montaner J.A. and Yanagawa K, Lyubeznik numbers of local rings and linear strands of graded ideals, *arXiv:1409.6486v1* [math.AC] 23 Sep 2014.

[75] Nasataescu C. and Ostaeyens F, Graded ring theory, *North-Holland, Amsterdam-Newyork*, 1982.

[76] Nastasescu C. and Van Oystaeyen F, On strongly graded rings and crossed products, *Communications in Algebra*, 10 (19), 2085–2106, (1982).

[77] Nastasescu C, Strongly graded rings of infinite groups, *Communications in Algebra*, 11(10), 1033–1071, (1983).

[78] Nastasescu C, Group rings of graded rings, *J. Pure Appl. Algebra*, 1984, 33, 313–335.

[79] Nastasescu C. and van Oystaeyen F, Methods of graded rings, *Springer LNM*, (1836), 2004.

[80] Oinert J, Ideals and maximal commutative subrings of graded rings, *Doctoral Theses in Mathematical Sciences* 2009:5 ISSN 1404-0034 ISBN 978-91-628-7832-0 LUTFMA-1038-2009.

[81] Perling M, Toric varieties as spectra of homogeneous prime ideals, *Department of Mathematics, University of Kaiserslautern*, Germany, Oct 2001.

[82] Perling M, Graded ring and equivariant sheave and toric varieties, *arXiv:math/0205311v1* [math.AG] 29 May 2002.

[83] Perling M, Graded rings and equivariant sheaves on toric varieties, *Math. Nachr.*, 263 (2004), 181–197.

[84] Perling, M. Toric varieties as spectra of homogenous prime ideals, *Geom.Dedicata*, (2007), 127:121–129.

[85] Perling M. and Trautmann G, Equivariant primary decomposition and toric sheaves, *in prepartion*.

[86] Perling M. and Kumar S.D, Primary decomposition over rings graded by finitely generated abelian group, *J. Algebra*, 318 (2007), 553–561.

[87] Perling M. and Trautmann G, Equivariant primary decomposition and toric sheaves, *arXiv*:0802.0257vl[math.AG].

[88] Perling M. and Trautmann G, Equivariant primary decomposition and toric sheaves, *arXiv:0802.0257v2* [math.AG] 21 Jan 2010.

[89] Polini C. and Xie Y, j-Multiplicity and depth of associated graded modules, *Journal of Algebra*, 379 (2013), 31–49.

[90] Quinn D, Group-graded rings and duality, *Journal: Trans. Amer. Math. Soc.*, 292 (1985), 155–167.

[91] Rahimi A. and Curtkosky S.D, Relative Cohen-Macaulayness of bigraded modules, *Journal of Algebra*, 323 (2010), 1745–1757.

[92] Refai M, Augmented graded rings, *Turkish Journal of Mathematics*, 21 (3), 333–341, (1997).

[93] Refai M, Augmented Noetherian graded modules, *Turk J Math*, 23 (1999), 355–360.

[94] Refai M. and Al-Zoubi K, On Graded Primary Ideals, *Truk. J. Math*, **28** (2004), 217–229.

[95] Refia M. and Al-Zoubi K, On graded primary ideals, *Turk J. Math.*, (2004), 28:217–229,

[96] Refai M, Augmented G-graded modules, *To appear Tishreen University Journal (Pure Sciences Series)*.

[97] Reid M, Graded rings and varieties in weighted projective space, appeared in 2012.

[98] R. Hazrat, Graded Rings and Graded Grothendieck Groups, *University of Western Sydney Australia*, 2015.

[99] Rossi M.E. and Sharifan L, Minimal free resolution of a finitely generated module over a regular local ring, *Journal of Algebra.*, 322 (2009), 3693–3712.

[100] Sands A.D, Some graded radical of graded rings, *Mathematica Pannonica*, 16/2, (2005), 211–220.

[101] Sauer T, Ideal bases for graded polynomial rings and applications to interpolation, *MonografÂ´Ä±as de la Academia de Ciencias de Zaragoza*. 20: 97–110, (2002).

[102] Serre J.P, Local Algebra, *Springer Verlag* (2000).

[103] Sharp R.Y, Asymptotic behavior of certain sets of attached prime ideals, *J. London Math. Soc.*, **34** (1986), 212–218.

[104] Sharp R.Y, Bass numbers in the graded case, a-invariant formulas, and an analogue of Faltings Annihilator Theorem, *J. Algebra*, 222 (1999), 246–270.

[105] Sharp R.Y, Graded annihilators of modules over the frobenius skew polynomial ring, and tight closure, *Transaction of the American Mathematical Society*, Volume 359, Number 9, September 2007, Pages 4237–4258, S 0002-9947(07)04247-X.

[106] Shiro Goto, Hilbert coefficient and buchsbaumness of associated graded ring, *Journal of Pure and Applied Algebra.*, 181 (2003), 61–74.

[107] Sierra S.J, G-algebra twisting and equivalences of graded categories, *arxiv: Math/0608791v3.* [Math.RA], 23 july 2007.

[108] Sierra S.J, Rings graded equivalent to the Weyl algeb, *Journal of Algebra*, 321 (2009) 495–531.

[109] Smith S.P, Category equivalences involving graded modules over path algebras of quivers, *Adv. in Math.*, 230, 2012, 1780–1810.

[110] Smoktunowicz A, A note on nil and jacobson radicals in graded ring, *J. Algebra Appl.*, 13, 1350121 (2014) [8 pages] DOI: http://dx.doi.org/10.1142/S0219498813501211.

[111] Soderberg J, Graded Betti numbers and h-vectors of level modules. *Preprint, arxiv:math.AC/0612047*, 2006.

[112] Sudhir R. Ghorpade, and Jugal K. Verma, Primary decomposition of Modules, 2000.

[113] Szasz F.A, Radicals of rings, *John Wiley and Sons*, 1981.

[114] Uliczka J, Remarks on Hilbert series of graded modules over polynomial rings, *Manuscripta Math.*, 132 (2010), 159–168.

[115] Wiegandt R, Radical and semisimple classes of rings, *Queen's University Papers, Kingston, Ontanio*, no. 37, 1974.

[116] Wiegandt R, Radical theory of rings, *Math Student*, 51, no. 1-4 (1983), 145–185.

[117] Wills-Toro L., Trefoil Symmetry I: Clover Extensions Beyond Coleman-Mandula The-orem, *Journal of Mathematical Physics.*, 42, 3915 (2001).

[118] Wills-Toro L., SÂ´anchez L., Leleu X.and Bleecker D., Trefoil Symmetry V: Class Rep-resentations for the Minimal Clover Extension, *International Journal of Theoretical Physics*, Volume 42, No. 1, 73–83 (2003).

[119] Wills-Toro L, Finite group graded Lie algebraic extensions and Trefoil symmetric relativity, Standard Model, Yang-Mills and gravity theories, *University of Hawaii Manoa.*, Ph.D. Thesis (2004).

[120] Yahya H, Graded radical graded semisimple classes, *Acta Math Hungar.*, 66 (1995), 163–175.

[121] Yassemi, The notion of prime submodules, *Archivum Mathematicum.*, 37 (2001), no. 4, 273–278.

[122] Y. Yoshino, The canonical modules of graded rings defined by generic matrices, *Nagoya Math. J.* , 81 (1981), 105–112.