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Relation-theoretic fuzzy Banach contraction principle and fuzzy Eldestein contraction theorem

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Abstract

In this paper we extend and generalize the classical Banach's contraction principle and Edelstein's contraction theorem on a fuzzy metric spaces endowed with a binary relation. We extend and generalize some earlier results in the literature.

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1. Introduction

The best known result from the Fixed Point Theory is Banach's Contraction Principle, which is one of the most important results of analysis and considered as the begining and main source of metric fixed point theory. It guarantees the existence and uniqueness of fixed points of certain self-maps of matric spaces, and provides a constructive method to find those fixed points. Fixed point theorems have applications not only in the different branches of mathematics, but also in physics, chemistry, engineering, biology, computer science and others. Due to its importance, Banach contraction principle has been extended in many different directions. Infact there is vast amount of literature dealing with generalizations and extentions of this theorem (cf. [5]–[7], [13], [17], [18]).

The concept of fuzzy set was initiated by Zadeh [27]. Thereafter, it was developed extensively by many authors which also include interesting applications of this theory in diverse areas. To use this concept, several researchers have defined fuzzy metric space in several ways (e.g. [2], [8], [10], [15], [20]). George and Veeramani [11] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [15]. The origin of fixed point theory in fuzzy metric spaces is often traced back to the paper of Grabiec [12] wherein he extended classical fixed point theorems of Banach and Edelstein to complete and compact fuzzy metric spaces respectively. Very recently Alam and Imdad [1] presented new varient of classical Banach contraction principle on a complete metric space endowed with a binary relation which, under universal relation reduces to Banach contraction principle. Many authors proved fixed point theorems in fuzzy metric space including [3], [4], [9], [21]–[25].

The aim of this paper is to extend and generalize the, Banach contraction principle to a complete fuzzy metric space and Edelstein contraction theorem to a compact fuzzy metric space endowed with a binary relation to the contractive conditions which are relatively weaker than usual contraction as it is required to hold only on those elements which are related under the underlying relation rather than the whole space. We extend and generalize the results of Alam and Imdad [1] and Grabiec [12].

2. Preliminaries

To set up our main results in the sequel, we recall some necessary definitions and preliminary concepts in this section. Let N, N_0, Q, and R denote the sets of positive integers, non-negative integers, rational numbers and real numbers respectively.

Definition 2.1. [20] A binary operation $* \to [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous t-norm if $([0, 1], *)$ is an Abelian topological monoid with the unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b.c.d \in [0, 1]$.

Definition 2.2. [11] The 3-tuple is called a fuzzy metric space if X is an arbitrary nonempty set, $*$ a continuous t-norm and M a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions, for all $x, y, z \in X$ and $t, s > 0$:

(FM-1) $M(x, y, t) > 0$,
(FM-2) $M(x, y, t = 1)$ for all $t > 0$ if and only if $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$
(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$, and
(FM-5) $M(x, y, t) : (0, \infty) \to [0, 1]$ is continuous.

Example 2.3. [11] Let (x, d) be a metric space. Define $(a * b) = ab$ (or $a * b =$

$\min\{a, b\}$) and for all $x, y \in X$ and $t > 0$,

$$M(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then $(X, M, *)$ is a fuzzy metric space. We call this fuzzy metric M induced by the metric d the standard fuzzy metric.

Definition 2.4. [11] Let $(X, M, *)$ be a fuzzy metric space:

(a) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$, if

$$\lim_{n\to\infty} M(x_n, x, t) = 1, \text{ for all } t > 0.$$

(b) A sequence $\{x_n\}$ in X is called Cauchy sequence if

$$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1,$$

for all $t > 0$ and $p > 0$.

(c) A fuzzy metric space $(X, M, *)$ in which every Cauchy sequence is convergent, is said to be complete.

(d) A fuzzy metric space $(X, M, *)$ is said to be compact if every sequence contains a convergent subsequence.

Lemma 2.5. [12] For all $x, y \in X$, $M(x, y, \cdot)$ is non decreasing.

Definition 2.6. [14] Let $(X, M, *)$ be a fuzzy metric space. M is said to be continuous on $X^2 \times (0, \infty)$ if

$$\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t),$$

whenever $\{(x_n y_n, t_n)\}$ is a sequence in $X^2 \times (0, \infty)$ which converges to a point $(x, y, t) \in X^2 \times (0, \infty)$; i.e.,

$$\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(y_n, y, t) = 1 \text{ and } \lim_{n\to\infty} M(x, y, t_n) = M(x, y, t).$$

Definition 2.7. [12] Let $\lim_n x_n = x$ and $\lim_n y_n = y$. Then

(7.1) $\lim_{n\to\infty} M(x_n, y_n, t) \geq M(x, y, t)$ for all $t > 0$;

(7.2) If $M(x, y, \cdot)$ is continuous, then $\lim_{n\to\infty} M(x_n, y_n, t) = M(x, y, t)$ for all $t > 0$.

Definition 2.8. [16] Let X be a nonempty set. A subset R of X^2 is called a binary relation on X.

Notice that for each pair $x, y \in X$, one of the following conditions hold:

(i) $(x, y) \in R$; which amounts to saying that "x is R-related to y" or "x relates to y under R". Sometimes, we write xRy instead of $(x, y) \in R$;

(ii) $(x, y) \notin R$; which means that "x is not R-related to y" or "x does not relate to y under R".

Trivally, X^2 and ϕ being subsets of X^2 are binary relations on X, which are respectively called the universal relation (or full relation) and empty relation.

Throughtout this paper, R stands for a nonempty binary relation, but for the sake of simplicity, we write only "binary relation" instead of "nonempty binary relation."

Definition 2.9. [1] Let R be a binary relation defined on a nonempty set X and $x, y \in X$. We say that x and y are R-comperative if either $(x, y) \in R$ or $(y, x) \in R$. We denote it by $[x, y] \in R$.

Definition 2.10. [1] Let X be a nonempty set and R a binary relation on X.

(i) The inverse transpose or duel relation of R, denoted by R^{-1}, is defined by $R^{-1} = \{(x, y) \in X^2 : (y, x) \in R\}$.

(ii) The symmetric closure of R, denoted by R^s, is defined to be the set $R \cup R^{-1}$(i.e., $R^s := R \cup R^{-1}$). Indeed, R^s is the smallest symmetric relation on X containing R.

Proposition 2.11. [1] For a binary relation R defined on a nonempty set X,

$$(x, y) \in R^s \iff [x, y] \in R.$$

Proof. The observation is straightforward as

$$\begin{aligned} (x, y) \quad \in \quad & R^s \iff (x, y) \in R \cup R^{-1} \\ \iff \quad & (x, y) \in R \text{ or } (x, y) \in R^{-1} \\ \iff \quad & (x, y) \in R \text{ or } (y, x) \in R \\ \iff \quad & [x, y] \in R \cup R^{-1}. \end{aligned}$$

■

Definition 2.12. [1] Let X be a nonempty set and R a binary relation on X. A sequence $\{x_n\} \subset X$ is called R-preserving if

$$(x_n, x_{n+1}) \in R \ \forall\, n \in N_0.$$

Definition 2.13. [1] Let $(X, M, *)$ be a fuzzy metric space. A binary relation R defined on X is called d-self-closed if whenever $\{x_n\}$ is an R-preserving sequence and

$$x_n \to x$$

then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, x] \in R$ for all $k \in N_0$.

Definition 2.14. Let X be a nonempty set and T a self mapping on X. Abinary relation R defined on X is called T-closed if for any $x, y \in X$,

$$(x.y) \in R \Longrightarrow (Tx, Ty) \in R.$$

Proposition 2.15. [1] Let X, T and R be the same as in definition 14. If R is T-closed, then R^s is also T-closed.

Definition 2.16. [1] Let X be a nonempty set and R a binary relation on X. A subset E of X is called R-directed if for each $x, y \in E$, there exists $z \in X$ such that $(x, z) \in R$ and $(y, z) \in R$.

Definition 2.17. [1] Let X be a nonempty set and R a binary relation on X. For $x, y \in X$, a path of length k (where k is a natural number) in R from x to y is a finite sequence $\{z_0, z_1, z_2, \ldots, z_k\} \subset X$ satisfying the following conditions:

(i) $z_0 = x$ and $z_k = y$,

(ii) $(z_i, z_{i+1}) \in R$ for each $i (0 \leq i \leq k-1)$.

Notice that a path of length k involves $k + 1$ elements of X, although they are not distinct.

In this paper, we use the following notations:

(i) $F(T) =$ the set of all fixed points of T,

(ii) $X(T; R) := \{x \in X : (x, Tx) \in R\}$,

(iii) $\Upsilon(x, y, R) :=$ the class of all paths in R from x to y.

3. Main Results

In order to establish our main result, we first prove the following proposition:

Proposition 3.1. If $(X, M, *)$ is a fuzzy metric space, R is a binary relation on X, T is a self-mapping on X, $0 < k < 1$ and $t > 0$, then the following contractivity conditions are equivalent:

(i) $M(Tx, Ty, kt) \geq M(x, y, t)\ \forall\, x, y \in X$ with $(x, y) \in R$,

(ii) $M(Tx, Ty, kt) \geq M(x, y, t)\ \forall\, x, y \in X$ with $[x, y] \in R$.

Proof. The implication (ii) $\Longrightarrow$ (i) is trivial. Conversely, suppose that (i) holds. Take $x, y \in X$ with $[x, y] \in R$, then (ii) directly follows from (i). Otherwise, if $(y, x) \in R$. then using the symmetry of M and (i), we obtain

$$M(Tx, Ty, kt) = M(Ty, Tx, kt) \geq M(y, x, t) = M(x, y, t).$$

This shows that (i)$\Longrightarrow$(ii). ■

Fuzzy Banach contraction principle:

Theorem 3.2. Let $(X, M, *)$ be a complete fuzzy metric space with $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$, R a binary relation on X and T a self mapping on X. Suppose that the following conditions hold:

(a) $X(T; R)$ is non empty,

(b) R is T-closed,

(c) either T is continuous or R is d-self closed.

(d)
$$M(Tx, Ty, kt) \geq M(x, y, t) \ \forall\, x, y \in X \ \ with (x, y) \in R$$
where $0 < k < 1$ and $t > 0$.

Then T has a fixed point. Moreover, if

(e) $\Upsilon(x, y, R^s)$ is non empty, for each $x, y \in X$, then T has a unique fixed point.

Proof. Let x_0 be an ordinary point of $X(T, R)$. Define an iterative sequence $\{x_n\}$ by $x_n = T^n(x_0)$ or $x_{n+1} = Tx_n \ \forall\, n \in N_0$. As $(x_0, Tx_0) \in R$, using assumption (b), we obtain
$$(Tx_0, T^2x_0),\ (T^2x_0, T^3x_0), \ldots (T^nx_0, T^{n+1}x_0) \in R$$
so that
$$(x_n, x_{n+1}) \in R \ \forall\, n \in N_0 \tag{1}$$
Thus the sequence $\{x_n)$ is R-preserving. Applying the contractive condition (d) to (1), we deduce for $n \in N_0$, that
$$\begin{aligned} M(Tx_n, Tx_{n+1}, kt) = M(x_{n+1}, x_{n+2}, kt) &\geq M\left(Tx_0, Tx_1, \frac{t}{k^{n-1}}\right) \\ &= M\left(x_1, x_2, \frac{t}{k^{n-1}}\right) \end{aligned}$$
for all $n \in N_0$ and $t > 0$.

Thus for any positive integer p, we have
$$\begin{aligned} M(Tx_n, Tx_{n+p}, t) &= M(x_{n+1}, x_{n+p+1}, t) \\ &\geq M\left(x_{n+1}, x_{n+2}, \frac{t}{p}\right) * \overset{(p)}{\cdots} * M\left(x_{n+p}, x_{n+p+1}, \frac{t}{p}\right) \\ &\geq M\left(x_1, x_2, \frac{t}{pk^n}\right) * \cdots * M\left(x_1, x_2, \frac{t}{pk^n}\right) \end{aligned}$$

since $M(x, y, t) \longrightarrow 1$ as $t \longrightarrow \infty$ we get

$$\lim_n M(Tx_n, Tx_{n+p}, t) \geq 1 * ... * 1 = 1$$

which implies that the sequence $\{x_n\}$ is Cauchy, hence convergent. so there exists $x \in X$ such that

$$\lim_{n \longrightarrow \infty} x_n = x.$$

Now assume that T is continuous, we have

$$\lim_{n \longrightarrow \infty} x_{n+1} = \lim_{n \longrightarrow \infty} Tx_n = Tx$$

Owing to the uniqueness of limit, we obtain $Tx = x$, i.e., x is a fixed point of T.
Now let us assume that R is d-self-closed. As $\{x_n\}$ is R-preserving sequence and

$$\lim_{n \longrightarrow \infty} x_n = x$$

there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with

$$[x_{n_k}, x] \in R \ \forall \ k \in N_0.$$

Using (d) and proposition(18), $[x_{n_k}, x] \in R$ and $\lim_{n \longrightarrow \infty} x_{n_k} = x$, we obtain

$$M(x_{n_{k+1}}, Tx, kt) = \ M(Tx_{n_k}, Tx, kt) \geq M(x_{n_k}, x, t) \longrightarrow 1 \ as \ k \longrightarrow \infty,$$

i.e.,

$$\lim_{n \longrightarrow \infty} M(x_{n_k}, x, t) = 1$$

so that

$$\lim_{n \longrightarrow \infty} x_{n_{k+1}} = Tx,$$

Again owing to the uniqueness of limit, we obtain

$$Tx = x$$

so that x is a fixed point of T.

To prove uniqueness, let us take $x, y \in F(t)$, i.e.,

$$T(x) = x \quad \& \quad T(y) = y \tag{2}$$

By assumption (e), there exists a path (say$\{z_0, z_1, z_2, ...z_r\}$) of some finite length r in R^s from x to y so that

$$z_0 = x, \ z_r = y, \ [z_i, z_{i+1}] \in R \ \text{ for each } i (0 \leq i \leq r-1) \tag{3}$$

As R is T-closed, by using proposition (15), we have

$$[T^n z_i, T^n z_{i+1}] \in R \ \text{ for each } i (0 \leq i \leq r-1) \text{ and for each } n \in N_0 \tag{4}$$

Now making use of (2), (3), (4), Fm-4, assumption (d) and proposition 18, we obtain

$$\begin{aligned}
M(x, y, t) &= M(Tx, Ty, t) \\
&= M(Tz_0, Tz_r, t) \\
&= M(T^n z_0, T^n z_r, t) \\
&\geq M\left(T^n z_0, T^n z_1, \frac{t}{p}\right) * M\left(T^n z_1, T^n z_2, \frac{t}{p}\right) * \cdots * M\left(T^n z_{r-1}, T^n z_r, \frac{t}{p}\right) \\
&\geq M\left(z_0, z_1, \frac{t}{pr^n}\right) * M\left(z_0, z_1, \frac{t}{pr^n}\right) * \overset{(p)}{\cdots} * M\left(z_0, z_1, \frac{t}{pr^n}\right) \\
&\geq 1 * 1 * \ldots * 1 \\
&= 1
\end{aligned}$$

so that $x = y$. Hence T has a unique fixed point. ∎

If R is complete or X is R^s-directed, then the following consequence is worth recording:

Corollary 3.3. Theorem (19) remains true if we replace condition (e) by one of the following conditions (besides retaining the rest of the hypothesis):

(e$'$) R is complete

(e$''$) X is R^s-directed.

Proof. If R is complete,then for each $x, y \in X$, $[x, y] \in R$, which amounts to saying that $\{x, y\}$ is a path of length 1 in R^s from x to y so that $\Upsilon(x, y, R^s)$ is nonempty. Hence Theorem 19 gives rise to the conclusion.

Otherwise, if X is R^s-directed, then for each $x, y \in X$, there exists $z \in X$ such that $[x, z] \in R$ and $[y, z] \in R$ so that $\{x, z, y\}$ is a path of length 2 in R^s from x to y. Hence $\Upsilon(x, y, R^s)$ is nonempty, for each $x, y \in X$ and again by Theorem 19 the conclusion is immediate. ∎

Fuzzy Edelstein contraction theorem:

Theorem 3.4. Let $(X, M, *)$ be a compact fuzzy metric space with $M(x, y, \cdot)$ continuous for all $x, y \in X$, R a binary relation on X and T a self mapping on X. Suppose that the following conditions hold:

(a) $X(T; R)$ is non empty,

(b) R is T-closed,

(c) either T is continuous or R is d-self closed.

(d$'$) $M(Tx, Ty, t) > M(x, y, t)\ \forall x, y \in X\ with\ (x, y) \in R\ \ \&\ \ x \neq y\ \ \&\ \ t > 0.$

Then T has a fixed point. Moreover, if

(e) $\Upsilon(x, y, R^s)$ is non empty, for each $x, y \in X$, then T has a unique fixed point.

Proof. Let x_0 be an ordinary point of $X(T, R)$. Define an iterative sequence $\{x_n\}$ by $x_n = T^n(x_0)$ or $x_{n+1} = Tx_n \; \forall \; n \in N_0$. As $(x_0, Tx_0) \in R$, using assumption (b), we obtain

$$(Tx_0, T^2x_0),\; (T^2x_0, T^3x_0), \ldots (T^nx_0, T^{n+1}x_0) \in R$$

so that

$$(x_n, x_{n+1}) \in R \; \forall \; n \in N_0$$

Thus the sequence $\{x_n)$ is R-preserving.

Now since X is compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$.

Let $\lim\limits_i x_{n_i} = x$. Now assume that T is continuous, we have

$$\lim_i Tx_{n_i} = Tx \tag{5}$$

$$\lim_i T^2x_{n_i} = T^2x \tag{6}$$

As $(x_{n_1}, Tx_{n_1}) \in R$.

Now observe that

$$\begin{aligned} M(x_{n_1}, Tx_{n_1}, t) &< M(Tx_{n_1}, T^2x_{n_1}, t) < \cdots < M(x_{n_i}, Tx_{n_i}, t) \\ &< M(Tx_{n_i}, T^2x_{n_i}, t) < \cdots < M(x_{n_{i+1}}, Tx_{n_{i+1}}, t) \\ &< M(Tx_{n_{i+1}}, T^2x_{n_{i+1}}, t) < \cdots < 1 \; \forall \; t > 0. \end{aligned}$$

Thus $\{M(x_{n_i}, Tx_{n_i}, t)\}$ and $\{M(Tx_{n_i}, T^2x_{n_i}, t)\}$ $(t > 0)$ are convergent to a common limit [19]. So by (5),(6) and (7.2) we get

$$\begin{aligned} M(x, Tx, t) &= M(\lim x_{n_i}, T \lim x_{n_i}, t) \\ &= \lim M(x_{n_i}, Tx_{ni}, t) \\ &= \lim M(Tx_{n_i}, T^2x_{n_i}, t) \\ &= M(\lim Tx_{n_i}, \lim T^2x_{n_i}, t) \\ &= M(Tx, T^2x, t) \end{aligned}$$

for all $t > 0$. Now suppose $x \neq Tx$. Then by (d$'$), we have

$$M(x, Tx, t) < M(Tx.T^2x, t), t > 0$$

which is a contradiction. Hence $x = Tx$, i.e., x is a fixed point of T.

Now assume that R is d-self-closed. As $\{x_n\}$ is an R-preserving sequence and $\lim\limits_{n\to\infty} x_n = x$, there exists a sequence $\{x_{n_i}\}$ of $\{x_n\}$ with $[x_{n_i}, x] \in R \;\; \forall \; i \in N$ using (d$'$), proposition 18, $[x_{n_i}, x] \in R$ and $\lim\limits_{i\to\infty} x_{n_i} = x$, we obtain

$$M(x_{n_i+1}, Tx, t) = M(Tx_{n_i}, Tx, t) > M(x_{n_i}, x, t) \to 1 \text{ as } i \to \infty$$

Thus

$$x_{n_i+1} \to Tx$$

Owing to the uniqueness of limit, we obtain $Tx = x$. so that x is a fixed point of T. Uniqueness follows from (d$'$).

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