Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 9 (2017), pp. 6963–6972

http://www.ripublication.com/gjpam.htm

A Luzin and Lipschitz condition for the Primitive of Henstock Integral[1]

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Abstract

In the paper [3], the concept of SLi was introduced and many class of functions are investigated, such as absolutely continuous functions, functions having negligible variations and functions satisfying the strong Luzin conditions. Here, we prove directly that all Henstock primitives are SLi functions. Moreover, we also investigate all ACG^* functions and functions that satisfy some Lipschitz condition.

AMS subject classification: 26A46, 26A99, 28A05, 28A12.
Keywords: Covering relation, inner variation zero, Lipschitz condition, AC^*, primitive of Henstock integrable function, SLi-function.

1. Introduction

A larger class of functions, called SLi-functions, was introduced in [3]. In the said paper, all absolutely continuous functions and functions that satisfy the strong Luzin condition belong to the class of SLi functions. This function was defined by utilizing the concept of covering relation and inner variation introduced by Thomson in [12].

In this paper, we will show that the class of SLi functions includes all AC^* functions and every Lipschitz functions. In particular, we will show that a primitive of a Henstock integrable function is an SLi function. We also exhibit a function which is *not* a SLi function.

[1]Research is partially funded by the Department of Science and Technology (DOST)-Accelerated Science and Technology Human Resource Development Program(ASTHRDP), Philippines.

2. Basic Concepts

Definition 2.1. [12] Let $E \subseteq \mathbb{R}$. A *covering relation* β on E is a subset of $\mathcal{I} \times E$ with the property that for each $x \in E$, there exists $I \in \mathcal{I}$ such that $x \in I$ and $(I, x) \in \beta$. A covering relation β is said to be *full at a point* x if there exists $\delta(x) > 0$ such that $([y, z], x) \in \beta$ whenever

$$x \in [y, z] \subseteq (x - \delta(x), x + \delta(x)).$$

Such a relation is said to be a *full covering relation on a set* E if it is full at each point of E.

Definition 2.2. [12] A covering relation β is said to be *fine at a point* x if for each $\delta(x) > 0$, there exists an interval-point pair $([y, z], x) \in \beta$ such that

$$x \in [y, z] \subseteq (x - \delta(x), x + \delta(x)).$$

β is a *fine covering relation on a set* E if it is fine at each point of E. Henstock called a fine covering relation on $E \subseteq \mathbb{R}$ as *inner covering* of E.

Recall that if $\delta : [a, b] \to \mathbb{R}^+$ be a gauge and $[u, v] \subseteq [a, b]$, then a pair $([u, v], \xi)$ is said to be ***δ**-fine* if

$$\xi \in [u, v] \subseteq (\xi - \delta(\xi), \xi + \delta(\xi)).$$

The following results are found in [3]:

Theorem 2.3. [3] Let $E \subseteq \mathbb{R}$, $\mathcal{I}$ is the collection of all compact intervals in $\mathbb{R}$ and δ be a gauge on E. Then the collection

$$\beta = \{(I, x) \in \mathcal{I} \times E : (I, x) \text{ is } \delta\text{-fine}\}$$

is an inner covering on E.

Theorem 2.4. [3] Let $E \subseteq [a, b]$ and $F : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. If β is an inner covering of E, then

$$\gamma = \Big\{(F(I), F(x)) : (I, x) \in \beta\Big\}$$

is an inner covering of $F(E)$.

Definition 2.5. [12] Let β be a covering relation on $E \subseteq \mathbb{R}$, and φ be an interval-point function defined on β. Define

$$Var(\varphi, \beta) = \sup\left\{ \sum_{(I,x)\in P} |\varphi(I, x)| : \ P \text{ is a partial division with } P \subseteq \beta \right\}$$

and $Var_*(\varphi, E) = \inf \Big\{ Var(\varphi, \beta) : \ \beta \text{ is a fine covering relation of } E \Big\}$. $Var(\varphi, \beta)$ refers to the *variation* of φ taken relative to β and Var_* is known as the *inner variation*.

Althroughout this paper, we use that following notations for inner variation:

$$IV(E) = \inf \Big\{ Var(\beta) : \ \beta \text{ is an inner covering of } E \Big\},$$

where

$$Var(\beta) = \sup \Bigg\{ \sum_{(I,x)\in P} \ell(I) : \ P \text{ is a partial division with } P \subseteq \beta \Bigg\}.$$

If $IV(E) = 0$, then we say that E is of *inner variation zero*.

Theorem 2.6. [4] A set $E \subset \mathbb{R}$ is of inner variation zero if and only if for each $\epsilon > 0$ there exists an inner covering β of E such that for each partial division $P = \{([u, v], \xi)\}$ with $P \subseteq \beta$, we have

$$(P) \sum |v - u| < \epsilon.$$

Definition 2.7. A function F is said be an ***SLi****-function* on a set $S \subseteq [a, b]$ if for every $E \subseteq S$ with $IV(E) = 0$, then $IV(F(E)) = 0$.

The Dirichlet function $F : [0, 1] \to \mathbb{R}$ defined by

$$F(x) = \begin{cases} 1 & , \ x \in \mathbb{Q} \cap [0, 1], \\ 0 & , \ x \in \mathbb{Q}^c \cap [0, 1]. \end{cases}$$

is an SLi-function on $\mathbb{Q} \cap [0, 1]$.

It was shown in [3] that a function $F : [a, b] \to \mathbb{R}$ is an SLi-function on $[a, b]$ if and only if F satisfies the condition (N) on $[a, b]$. The Cantor function defined in [7] fails to satisfy the condition (N) on $[0, 1]$; hence, it is not an SLi-function on $[0, 1]$.

3. Primitives

Definition 3.1. [8] A function $f : [a, b] \to \mathbb{R}$ is said to be *Henstock integrable* to $A \in \mathbb{R}$ on $[a, b]$ if for each $\epsilon > 0$, there exists a gauge $\delta(\xi) > 0$ on $[a, b]$ such that whenever $D = \{([u, v], \xi)\}$ is a Henstock δ-fine division of $[a, b]$, we have

$$\Big| (D) \sum f(\xi)(v - u) - A \Big| < \epsilon.$$

If $f : [a, b] \to \mathbb{R}$ is Henstock integrable to A, then we write

$$A = (\mathcal{H}) \int_a^b f(x)\, dx$$

and call A the *Henstock integral* of f on $[a, b]$. The function $F : [a, b] \to \mathbb{R}$ defined by

$$F(x) = (\mathcal{H}) \int_a^x f(t)\, dt$$

is called a *primitive* of f.

Lemma 3.2. [Henstock Lemma] If $f : [a, b] \to \mathbb{R}$ is Henstock integrable on $[a, b]$ with primitive F, then for every $\epsilon > 0$ there exists $\delta(\xi) > 0$ on $[a, b]$ such that for any δ-fine partial division $P = \{([u, v], \xi)\}$ we have

$$(P)\sum \left|F(v) - F(u) - f(\xi)(v - u)\right| < \epsilon.$$

Theorem 3.3. A primitive F of a Henstock integrable function $f : [a, b] \to \mathbb{R}$ is an SLi-function on $[a, b]$.

Proof. For each $n \in \mathbb{N}$, define

$$A_n = \left\{x \in [a, b] : n - 1 \leq |f(x)| < n\right\}.$$

Then $[a, b] = \bigcup_{n\in\mathbb{N}} A_n$.

Next, we will show that F is an SLi-function on each A_n. Fix $n \in \mathbb{N}$ and put $A = A_n$. Let $E \subseteq A$ with $IV(E) = 0$ and let $\epsilon > 0$ be given. Then there exists an inner cover β of E such that for any partial division $P_0 \subseteq \beta$,

$$(P_0)\sum |v - u| < \frac{\epsilon}{2}.$$

By Lemma 3.2, there exists $\delta(\xi) > 0$ on $[a, b]$ such that for any δ-fine partial division $P = \{([u, v], \xi)\}$, we have

$$(P)\sum \left|F(v) - F(u) - f(\xi)(v - u)\right| < \frac{\epsilon}{2}.$$

Define $\beta_0 = \Big\{([u, v], x) \in \beta : ([u, v], x) \text{ is } \delta\text{-fine}\Big\}$. By Theorem 2.3, β_0 is an inner covering of E. Let

$$\gamma = \Big\{(F(I), F(x)) : (I, x) \in \beta_0\Big\}.$$

Since F is continuous, by Theorem 2.4, γ is an inner covering of $F(E)$. Let $\pi = \{([s, t], y)\}$ be a partial division with $\pi \subseteq \gamma$. Then for each $([s, t], y) \in \gamma$, there exists $(I, x) \in \beta_0$ such that $F(I) = [s, t]$ and $F(x) = y$. By continuity of F, there exist $u, v, \in I$ such that such that $F(u) = s$ and $F(v) = t$. Without loss of generality, we may assume that $u < v$. Let

$$P = \Big\{([u, v], x) : F(u) = s \text{ and } F(v) = t\Big\}.$$

Here, $([u, v], x) \in P$ may not be δ-fine. Let

$$\begin{aligned}
P_1 =&\{([u, v], x) \in P : ([u, v], x) - \delta\text{-fine}\},\\
P_2 =&\{([u, v], x) \in P : x < u\}, \quad \text{and}\\
P_3 =&\{([u, v], x) \in P : v < x\}.
\end{aligned}$$

Then for each $i = 1, 2, 3$, we have

$$(P_i)\sum |f(x)(v-u)| \leq n\sum |v-u| < \frac{\epsilon}{2}.$$

Note that in P_2 and P_3, $x \notin [u, v]$. Let

$$\begin{aligned}
P_2' &= \{([x, u], x) : ([u, v], x) \in P_2\}\\
P_2'' &= \{([x, v], x) : ([u, v], x) \in P_2\}\\
P_3' &= \{([u, x], x) : ([u, v], x) \in P_3\}\\
P_3'' &= \{([v, x], x) : ([u, v], x) \in P_3\}.
\end{aligned}$$

Then P_i' and P_i'' $(i = 2, 3)$ are δ-fine partial division of $[a, b]$. Thus,

$$(P_2')\sum \left|F(u) - F(x) - f(x)(u-x)\right| < \frac{\epsilon}{2} \quad \text{and}$$
$$(P_2'')\sum \left|F(v) - F(x) - f(x)(v-x)\right| < \frac{\epsilon}{2}.$$

Similarly, we have

$$(P_3')\sum \left|F(x) - F(u) - f(x)(x-u)\right| < \frac{\epsilon}{2} \quad \text{and}$$
$$(P_3'')\sum \left|F(x) - F(v) - f(x)(x-v)\right| < \frac{\epsilon}{2}.$$

Thus,

$$\begin{aligned}
(P_2)\sum |F(v) - F(u)| &= (P_2)\sum |F(v) - F(u) - f(x)(v-u) + f(x)(v-u)|\\
&\leq (P_2'')\sum \left|F(v) - F(x) - f(x)(v-x)\right|\\
&\quad + (P_2')\sum \left|F(u) - F(x) - f(x)(u-x)\right|\\
&\quad + (P_2)\sum |f(x)(v-u)|\\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
\end{aligned}$$

and

$$\begin{aligned}
(P_3)\sum |F(v) - F(u)| &= (P_2)\sum |F(v) - F(u) - f(x)(v-u) + f(x)(v-u)|\\
&\leq (P_3'')\sum \left|F(x) - F(v) - f(x)(x-v)\right|s\\
&\quad + (P_3')\sum \left|F(x) - F(u) - f(x)(x-u)\right|\\
&\quad + (P_3)\sum |f(x)(v-u)|\\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

Consequently,

$$\begin{aligned}
(\pi)\sum|t-s| &= (P)\sum\Big|F(v)-F(u)\Big| \\
&= (P_1)\sum\Big|F(v)-F(u)\Big| \\
&\quad + (P_2)\sum\Big|F(v)-F(u)\Big| \\
&\qquad + (P_3)\sum\Big|F(v)-F(u)\Big| \\
&< \frac{\epsilon}{5}+\frac{2\epsilon}{5}+\frac{2\epsilon}{5} \\
&= \epsilon.
\end{aligned}$$

Therefore, by Theorem 2.6, $F(E)$ is of inner variation zero. ■

4. ACG Functions and Lipschitz Condition

In this section, we investigate a relationship between genaralized absolutely continuous functions (ACG) and SLi-functions.

Definition 4.1. [9] Let $X \subseteq [a,b]$. A function $F:[a,b]\to\mathbb{R}$ is said to be $\boldsymbol{AC^*(X)}$ if the following condition is satisfied:

> For every $\epsilon > 0$, there exists $\eta > 0$ such that for any partial division $P=\{[u,v]\}$, where u or v belong to X, we have
>
> $$(P)\sum|v-u|<\eta \quad\text{implies}\quad (P)\sum|F(v)-F(u)|<\epsilon.$$

A function F is said to be $\boldsymbol{ACG^*}$ on $[a,b]$ if there exists a collection $\{X_n : n\in\mathbb{N}\}$ such that $[a,b]=\bigcup_{n=1}^{\infty} X_n$ and F is $AC^*(X_n)$, for each n.

Definition 4.2. [2] Let $X \subseteq [a,b]$. A function $F:[a,b]\to\mathbb{R}$ is said to be $\boldsymbol{AC^*_H(X)}$ if the following condition is satisfied:

> For every $\epsilon > 0$, there exists $\eta > 0$ and $\delta(\xi) > 0$ on $[a,b]$ such that for any Henstock δ-fine partial division $P=\{([u,v],x)\}$ with $x\in X$, we have
>
> $$(P)\sum|v-u|<\eta \quad\text{implies}\quad (P)\sum|F(v)-F(u)|<\epsilon.$$

A function $F:[a,b]\to\mathbb{R}$ is $\boldsymbol{ACG^*_H}$ on $[a,b]$ if if there exists a collection $\{X_n : n\in\mathbb{N}\}$ such that $[a,b]=\bigcup_{n=1}^{\infty} X_n$ and F is $AC^*_H(X_n)$, for each n.

It was seen in [2] that if f is continuous, then ACG^* and ACG^*_H are equivalent. The following result says that the class of SLi functions is larger that the class of AC^* functions.

Theorem 4.3. Let $F : [a, b] \to \mathbb{R}$ be an continuous function and $X \subseteq [a, b]$. If F is $AC^*_H(X)$, then F is SLi-function on X.

Proof. Let $E \subseteq X$ with $IV(E) = 0$. Let $\epsilon > 0$. Then there exist $\eta > 0$ and $\delta(\xi) > 0$ on $[a, b]$ such that for any δ-fine partial division $P = \{([u, v], x)\}$ with $x \in X$, we have

$$(P)\sum |v-u| < \eta \quad \text{implies} \quad (P)\sum |F(v) - F(u)| < \frac{\epsilon}{5}. \tag{4.1}$$

By Theorem 2.6, there exists an inner covering β of E such that for any partial division $P = \{([u, v], t)\}$ with $P \subseteq \beta$, we have

$$(P)\sum |v-u| < \eta. \tag{4.2}$$

Consider the inner covering $\beta_0 = \Big\{([u, v], x) \in \beta : ([u, v], x) \text{ is } \delta\text{-fine}\Big\}$ and

$$\gamma = \Big\{(F(I), F(x)) : (I, x) \in \beta_0\Big\}.$$

By Theorem 2.4, γ is an inner covering of $F(E)$.

Now, let $\pi = \{([s, t], y)\}$ be a partial division such that $\pi \subseteq \gamma$. Since $\pi \subseteq \gamma$, for each $([s, t], y) \in \pi$ there exists $(I, x) \in \beta_0$ such that $F(I) = [s, t]$ and $F(x) = y$. Set

$$P_0 = \{(I, x) \in \beta_0 : F(I) = [s, t] \text{ and } F(x) = y\}.$$

Then $P_0 \subseteq \beta_0 \subseteq \beta$. Hence,

$$(P_0)\sum \ell(I) < \eta.$$

By continuity of F, there exist $u, v, \in I$ such that such that $F(u) = s$ and $F(v) = t$. WLOG, we may assume that $u < v$. Let

$$P = \Big\{([u, v], x) : F(u) = s \text{ and } F(v) = t\Big\}.$$

Here, $([u, v], x) \in P$ may not be δ-fine. Let

$$\begin{aligned} P_1 &= \{([u, v], x) \in P : ([u, v], x) \text{ is } \delta\text{-fine}\}, \\ P_2 &= \{([u, v], x) \in P : x < u\}, \qquad \text{and} \\ P_3 &= \{([u, v], x) \in P : v < x\}. \end{aligned}$$

For $i = 1, 2, 3$, we have

$$(P_i)\sum |v-u| < \eta.$$

Note that in P_2 and P_3, $x \notin [u, v]$. Let

$$P_2' = \{([x,u],x) : ([u,v],x) \in P_2\}, \quad P_2'' = \{([x,v],x) : ([u,v],x) \in P_2\},$$
$$P_3' = \{([u,x],x) : ([u,v],x) \in P_3\}, \text{ and } \quad P_3'' = \{([v,x],x) : ([u,v],x) \in P_3\}.$$

Then P_i' and P_i'' $(i = 2, 3)$ are δ-fine partial division of $[a, b]$ with $x \in X$. Thus,

$$(P_2')\sum |u - x| \leq (P_0)\sum \ell(I) < \eta \text{ and } (P_2'')\sum |u - x| \leq (P_0)\sum \ell(I) < \eta.$$

Similarly, we have

$$(P_3')\sum |u - x| < \eta \text{ and } (P_3'')\sum |u - x| < \eta.$$

Hence, by (4.1) we have

$$\begin{aligned}
&(P_1)\sum |F(v) - F(u)| < \frac{\epsilon}{5},\\
&(P_2')\sum |F(u) - F(x)| < \frac{\epsilon}{5},\ (P_2'')\sum |F(v) - F(x)| < \frac{\epsilon}{5},\\
&(P_3')\sum |F(x) - F(u)| < \frac{\epsilon}{5},\ (P_3'')\sum |F(x) - F(v)| < \frac{\epsilon}{5}.
\end{aligned}$$

Thus,

$$(P_2)\sum |F(v) - F(u)| \leq (P_2'')\sum |F(v) - F(x)| + (P_2')\sum |F(x) - F(u)| < \frac{2\epsilon}{5}$$

and

$$(P_3)\sum |F(v) - F(u)| \leq (P_3'')\sum |F(v) - F(x)| + (P_3')\sum |F(u) - F(x)| < \frac{2\epsilon}{5}.$$

Consequently,

$$\begin{aligned}
(\pi)\sum |t - s| &= (P)\sum \left|F(v) - F(u)\right|\\
&= (P_1)\sum \left|F(v) - F(u)\right| + (P_2)\sum \left|F(v) - F(u)\right| + (P_3)\sum \left|F(v) - F(u)\right|\\
&< \frac{\epsilon}{5} + \frac{2\epsilon}{5} + \frac{2\epsilon}{5} = \epsilon.
\end{aligned}$$

Therefore, by Theorem 2.6, $F(E)$ is of inner variation zero. ■

We remark that if $F : [a, b] \to \mathbb{R}$ is monotonic and continuous on $[a, b]$, then F is ACG^* on $[a, b]$. Hence, by Theorem 4.3, we have the following corollary:

Corollary 4.4. If F is monotonic and continuous on $[a, b]$, then F is SLi on $[a, b]$.

We recall that a function $F : D \to \mathbb{R}$ is said to satisfy a *Lipschitz condition* on $S \subseteq D$ if there exists $L > 0$ such that for all $x, y \in S$,

$$|F(y) - F(x)| < L \cdot |y - x|.$$

Note that if $F : [a, b] \to \mathbb{R}$ satisfies a Lipschitz condition on $[a, b]$, then F is continuous on $[a, b]$.

Theorem 4.5. If $F : [a, b] \to \mathbb{R}$ satisfies a Lipschitz condition on $[a, b]$, then F is an SLi-function on $[a, b]$.

Proof. Choose $L > 0$ such that for any $x, y \in [a, b]$, we have

$$|F(x) - F(y)| < L \cdot |y - x|.$$

Let $E \subseteq [a, b]$ with $IV(E) = 0$ and $\epsilon > 0$ be given. Then there exists an inner cover β of E such that for any partial division $P \subseteq \beta$,

$$(P) \sum |v - u| < \frac{\epsilon}{L}.$$

Define $\gamma = \{(F(I), F(x)) : (I, x) \in \beta\}$. Since $F : [a, b] \to \mathbb{R}$ satisfies a Lipschitz condition on $[a, b]$, F is continuous on $[a, b]$. So, by Theorem 2.4, γ is an inner covering of $F(E)$.

Now, let $\pi = \{([s_k, t_k], y_k) : k = 1, 2, \ldots, p\}$ be a partial division such that $\pi \subseteq \gamma$. Since $\pi \subseteq \gamma$, for each k, $([s_k, t_k], y_k) \in \pi$ there exists $(I_k, x_k) \in \beta$ such that $F(I_k) = [s_k, t_k]$ and $F(x_k) = y_k$. By continuity of F on each I_k, there exists $u_k, v_k \in I_k$ such that $F(s_k) = u_k$ and $F(t_k) = v_k$. Thus,

$$P = \{(I_k, x_k) : k = 1, 2, \ldots, p\}$$

is a partial division of $[a, b]$ with $P \subseteq \beta$. Hence,

$$\sum_{k=1}^{p} |v_k - u_k| \leq \sum_{k=1}^{p} \ell(I_k) < \frac{\epsilon}{L}.$$

Thus,

$$\sum_{k=1}^{p} |t_k - s_k| = \sum_{k=1}^{p} |F(v_k) - F(u_k)| \leq \sum_{k=1}^{p} L \cdot |v_k - u_k|$$
$$\leq L \cdot \sum_{k=1}^{p} |v_k - u_k| < L \cdot \frac{\epsilon}{L} = \epsilon.$$

It follows that $IV(F(E)) = 0$. Therefore, F is an SLi-function on $[a, b]$. ∎

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