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On Fuzzy Automata Homotopy

[1]K.Saranya and [2]B.Amudhambigai

[1,2]*Department of Mathematics, Sri Sarada College for Women, Salem, Tamilnadu, India.*

Abstract

In this paper, the concepts of fuzzy automata structure space, fuzzy automata homotopy, fuzzy automata contractible spaces, fuzzy automata path homotopy, fuzzy automata retraction, fuzzy automata deformation retraction, fuzzy au- tomata path connected space and fuzzy automata homotopy equivalent space are introduced and several interesting properties are established.

Keywords: Fuzzy automata structure space, fuzzy automata homotopy, fuzzy automata path connected space, fuzzy automata homotopy equivalent space.

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1 INTRODUCTION

The concept of fuzzy automata was introduced by Santos[12] and Wee[15] in 1960's. The concept of fuzzy set was innovated by Zadeh[16]. The concept of fuzzy topological space was introduced by Chang[1]. Malik, Mordeson and Sen[6] introduced the concept of fuzzy finite state machine(which is almost identical to a fuzzy automaton). Shukla and Srivastava[13] showed that it is possible to associate a topology with an automaton and demonstrated that many properties of an automaton can be characterized in terms of that topology. In application point of view, fuzzy automata have been useful in numerous engineering applications such as pattern recognition, clinical monitoring and also used to model fuzzy discrete event systems(cf. [5, 9, 10, 11]). Classical homotopy theory and fundamental group were introduced and developed by Massey[8]. Cuvalcioglu and Citil[2] introduced the concept of fuzzy homotopy theory.

In this paper, the concepts of fuzzy automata structure space, fuzzy automata homotopy, fuzzy automata contractible spaces, fuzzy automata path homotopy, fuzzy automata retraction, fuzzy automata deformation retraction, fuzzy automata path connected space and fuzzy automata homotopy equivalent space are introduced and several interesting properties are established.

2 PRELIMINARIES

Definition 2.1. [14] A fuzzy automaton is a triple $M = (Q, X, \delta)$ where Q is a non-empty set (of states of M), X is a monoid(the input monoid of M), whose identity shall be denoted as e and δ is a fuzzy subset of $Q \times X \times Q$, i.e a map $\delta : Q \times X \times Q \rightarrow [0, 1]$ such that for all $q, p \in Q$ and $x, y \in X$,

(1) $\delta(q, e, p) = \begin{cases} 1 \ if \ q = p \\ 0 \ \ if \ q \neq p \end{cases}$

(2) $\delta(q, xy, p) = \bigvee_{r \in Q} \{\delta(q, x, r) \wedge \delta(r, y, p)\}$

Notation 2.1. For any non-empty set of states Q, I^Q denotes the collection of all functions from Q into I, where I is the unit interval [0, 1].

Definition 2.2. [7] $\lambda \in I^Q$ is called a fuzzy subsystem of (Q, X, δ) if

$\lambda(q) \geq \lambda(p) \wedge \delta(p, x, q), \forall p, q \in Q, x \in X.$

Proposition 2.1. [3] The function $c : I^Q \rightarrow I^Q$ defined as

$c(\lambda)(q) = \bigvee \{\bigvee \{\lambda(p) \wedge \delta(p, x, q) : x \in X\} : p \in Q,$ for all $\lambda \in I^Q$ and $q \in Q$

is a Kuratowski saturated fuzzy closure operator on Q.

Proposition 2.2. [3] $\lambda \in I^Q$ is a fuzzy subsystem of (Q, X, δ) if and only if $c(\lambda) = \lambda$ (*i.e.*, if and only if λ is closed with respect to the fuzzy topology induced by c on Q).

Definition 2.3. [1] Let (X, T) and (Y, S) be any two fuzzy topological spaces. Let f be a function from the fuzzy topological space (X, T) to the fuzzy topological space (Y, S). Let λ be a fuzzy set in (Y, S). The inverse image of λ under f written as $f^{-1}(\lambda)$ is the fuzzy set in (X, T) defined by $f^{-1}(\lambda)(x) = \lambda(f(x))$, for all $x \in X$.

Also the image of λ in (X, T) under f written as $f(\lambda)$ is the fuzzy set in (Y, S) defined by,

$$f(\lambda)(y) = \begin{cases} sup_{x\in f^{-1}(y)}\lambda(x), & \text{if } f^{-1}(y) \text{ is non} - \text{empty, for each } y \in Y \\ 0, & \text{otherwise} \end{cases}$$

Definition 2.4. [4] Euclidean space R is the set of all real numbers together with the topology determined by the Euclidean metric, $d(x, y) = |x - y|$ for all $x, y \in$ R.

3 FUZZY AUTOMATA STRUCTURE SPACE AND FUZZY AUTOMATA HOMOTOPY

In this section, the concepts of fuzzy automata structure space, fuzzy automata homo- topy, fuzzy automata nullhomotopy, fuzzy automata homotopy equivalent spaces and fuzzy automata contractible spaces are introduced and some interesting properties are discussed.

Notation 3.1. Throughout this paper, 0_Q denotes $\mu_{0_Q}(q) = 0$ for all $q \in Q$ and 1_Q denotes $\mu_{1_Q}(q) = 1$ for all $q \in Q$.

Definition 3.1. Let $M = (Q, X, \delta)$ be a fuzzy automaton. For all $\lambda \in I^Q$ and $q \in Q$,

$$c(\lambda)(p) = \bigvee_{q\in Q}\left\{\bigvee_{x\in X}\{\lambda(q)\wedge\delta(q,x,p)\}\right\}$$

Is a fuzzy closure operator on Q. Let $\tau = \{\lambda : c(\lambda) = \lambda\}$ be the collection of fuzzy subsystems which satisfies the following conditions:

(i) $0_Q, 1_Q \in \tau$,

(ii) If $\lambda, \gamma \in \tau$, then $\lambda \wedge \gamma \in \tau$,

(iii) If $\lambda_i \in \tau$ for each $i \in J$, then $\vee\lambda_i \in \tau$.

Then τ is said to be the fuzzy automata structure generated by the fuzzy topology associated with a fuzzy automaton and the ordered pair (Q,τ) is said to be a fuzzy automata structure space (in short, *FASS*). Moreover, members of τ are said to be fuzzy automata open subsystems and their complements are said to be fuzzy automata closed subsystems.

Notation 3.2. Throughout this paper, fuzzy automata open subsystem, fuzzy automata closed subsystem and fuzzy automata subsystem are denoted by *FAOS, FACS* and *FAS* respectively.

Definition 3.2. Let $M = (Q, X, \delta)$ be a fuzzy automaton and let (Q, τ) be a fuzzy automata structure space. Let $p, q \in Q$ and $\mu : Q \rightarrow [0, 1]$. Then the fuzzy automata

Subsystem,

$$q_t(p) = \begin{cases} t, & for\ q = p \\ 0, & otherwise \end{cases}$$

Definition 3.3. Let (Q, τ) and (R, σ) be any two fuzzy automata structure spaces. A function $f : (Q, \tau) \to (R, \sigma)$ is said to be

(i) fuzzy automata continuous if $f^{-1}(\lambda)$ is a fuzzy automata open(resp. fuzzy automata closed) subsystem in (Q, τ) for every fuzzy automata open(resp. fuzzy automata closed) subsystem λ in (R, σ).

(ii) fuzzy automata homeomorphism if f is bijective and f, f^{-1} are fuzzy automata continuous functions.

Definition 3.4. Let (Q, τ) be a fuzzy automata structure space. Let $P \subset Q$ and let $\chi_P : Q \to \{0,1\}$ be a fuzzy automata function. Then the fuzzy automata characteristic function χ_P of P is defined as,

$$\chi_P(P) = \begin{cases} 1, & if \quad q \in P \\ 0, & otherwise \end{cases}$$

Definition 3.5. Let (Q, τ) be a fuzzy automata structure space. If $P \subset Q$ and χ_P is the fuzzy automata characteristic function of P, then the collection $\tau_{/P} = \{\mu_{/P} = \mu \wedge \chi_P : \mu \in \tau\}$ is a fuzzy automata structure on P, called the fuzzy automata subspace structure and the pair $(P, \tau_{/P})$ is called a fuzzy automata structure subspace of (Q, τ).

Definition 3.6. Let (Q, T) be a structure space where Q is the non-empty set of states of a fuzzy automaton M. The collection,

$$\omega(T) = \{\chi_U : U \in T\}$$

such that $c(\chi_U) = \chi_U$ is a fuzzy automata structure on Q introduced by T. Then $(Q, \omega(T))$ is called the fuzzy automata structure space introduced by (Q, T).

Note 3.1. Let ξ be an Euclidean topology on I, where $I = [0, 1]$ and $(I, \omega(\xi))$ be a fuzzy automata structure space introduced by the Euclidean space (I, ξ).

Definition 3.7. Let (Q, τ) and (R, σ) be any two fuzzy automata structure spaces and $(I, \omega(\xi))$ be a fuzzy automata structure introduced by the Euclidean space (I, ξ). Let $f, g : (Q, \tau) \to (R, \sigma)$ be any two fuzzy automata continuous functions. If there

exists a fuzzy automata continuous function $H : (Q, \tau) \times (I, \omega(\xi)) \to (R, \sigma)$ such that
$H(q_\delta, 0) = f(q_\delta)$ and $H(q_\delta, 1) = g(q_\delta)$, for each fuzzy automata point q_δ of Q, then f is said to be fuzzy automata homotopic to g. Moreover, the function H is said to be a fuzzy automata homotopy between f and g, denoted as $f \simeq g$.

Proposition 3.1. Let (Q, τ) and (R, σ) be any two fuzzy automata structure spaces. Let U and V be subsets of Q. Let $\lambda = \lambda_{/U} \vee \lambda_{/V}$, where $\lambda_{/U}$ and $\lambda_{/V}$ are the fuzzy automata open subsystems in (Q, τ). Let $f : (U, \tau_{/U}) \to (R, \sigma)$ and $h : (V, \tau_{/V}) \to (R, \sigma)$ be any two fuzzy automata continuous functions. If $f_{/(U\cap V)} = h_{/(U\cap V)}$, then $g : (Q, \tau) \to (R, \sigma)$ which is defined by,

$$g(q) = \begin{cases} f(q), & for \quad q \in U \\ h(q), & for \quad q \in V \end{cases}$$

is a fuzzy automata continuous function.

Proof Let μ be a fuzzy automata open subsystem in (R, σ). Let $\lambda_{/U}$ and $\lambda_{/V}$ be the fuzzy automata open subsystems in (Q, τ). Then, $\lambda_{/U} \vee \lambda_{/V}$ is a fuzzy automata open subsystem in (Q, τ). Now,

$$\begin{aligned} g^{-1}(\mu) &= g^{-1}(\mu) \wedge \lambda \\ &= g^{-1}(\mu) \wedge (\lambda_{/U} \vee \lambda_{/V}) \\ &= (g^{-1}(\mu) \wedge \lambda_{/U}) \vee (g^{-1}(\mu) \wedge \lambda_{/V}) \\ &= f^{-1}(\mu) \vee h^{-1}(\mu) \end{aligned}$$

Since f and h are fuzzy automata continuous functions, $f^{-1}(\mu)$ and $h^{-1}(\mu)$ are the fuzzy automata open subsystems in $(U, \tau_{/U})$ and $(V, \tau_{/V})$ respectively. Thus $g^{-1}(\mu)$ is a fuzzy automata open subsystem in (Q, τ). Hence g is a fuzzy automata continuous function.

Proposition 3.2. The homotopy relation $\simeq$ is an equivalence relation.

Proof Let (Q, τ) and (R, σ) be any two fuzzy automata structure spaces and $(I, \omega(\xi))$
be a fuzzy automata structure space introduced by the Euclidean space (I, ξ). Let
$f, g, h : (Q, \tau) \to (R, \sigma)$ be any three fuzzy automata functions.

(i) Let $G : (Q, \tau) \times (I, \omega(\xi)) \to (R, \sigma)$ be defined as $G(q_\delta, t) = f(q_\delta)$, where $t \in [0, 1]$, for each fuzzy automata point q_δ of Q. Then by ***Proposition* 3.1,** G is a fuzzy automata continuous function and $G(q_\delta, 0) = G(q_\delta, 1) = f(q_\delta)$. Thus $f \simeq f$.

(ii) Let $f \simeq g$, then it is to be proved that $g \simeq f$. Since $f \simeq g$ there exists a fuzzy automata continuous function $G : (Q, \tau) \times (I, \omega(\xi)) \to (R, \sigma)$ such that $G(q_\delta, 0) = f(q_\delta)$ and $G(q_\delta, 1) = g(q_\delta)$. Let $H : (Q, \tau) \times (I, \omega(\xi)) \to (R, \sigma)$ be defined by $H((q_\delta, t) =$
$G(q_\delta, 1 - t)$ for all $t \in I$. By ***Proposition*3.1,** H is a fuzzy automata continuous function. For each fuzzy automata point q_δ of Q, $H(q_\delta, 0) = G(q_\delta, 1) = g(q_\delta)$ and $H(q_\delta, 1) = G(q_\delta, 0) = f(q_\delta)$. Therefore $g \simeq f$.

(iii) Let $f \simeq g$ and $g \simeq h$, it is to be proved that $f \simeq h$. Since $f \simeq g$ and $g \simeq h$ there exist fuzzy automata continuous functions, $G : (Q, \tau) \times (I, \omega(\xi)) \to (R, \sigma)$ and $H : (Q, \tau) \times (I, \omega(\xi)) \to (R, \sigma)$ such that $G(q_\delta, 0) = f(q_\delta)$, $G(q_\delta, 1) = g(q_\delta)$, $H(q_\delta, 0) = g(q_\delta)$ and $H(q_\delta, 1) = h(q_\delta)$, for each fuzzy automata point q_δ of Q. Then $Z : (Q, \tau) \times (I, \omega(\xi)) \to (R, \sigma)$ is defined by,

$$Z(q_\delta, t) = \begin{cases} G(q_\delta, 2t), & 0 \le t \le 1/2 \\ H(q_\delta, 2t - 1), & 1/2 \le t \le 1 \end{cases}$$

for all $t \in I$. By ***Proposition*3.1,** Z is a fuzzy automata continuous function. Hence $f \simeq h$. Therefore $\simeq$ is an equivalence relation.

Notation 3.3. The equivalence class of the fuzzy automata function 'f' under the equivalence relation '$\simeq$' is denoted by $[f]$.

Definition 3.8. Let (Q, τ) and (R, σ) be any two fuzzy automata structure spaces and let $f, g : (Q, \tau) \to (R, \sigma)$ be any two fuzzy automata continuous functions and let $f \simeq g$. If g is a constant function then f is called a fuzzy automata null-homotopic function.

Definition 3.9. Let (Q, τ) and (R, σ) be any two fuzzy automata structure spaces. Let $f : (Q, \tau) \to (R, \sigma)$ and $g : (R, \sigma) \to (Q, \tau)$ be fuzzy automata continuous functions such that
$f \circ g \simeq \boldsymbol{id}_R$ and $g \circ f \simeq \boldsymbol{id}_Q$, where $\boldsymbol{id}_Q$ and $\boldsymbol{id}_R$ are the identity functions of (Q, τ) and (R, σ) respectively. Then (Q, τ) and (R, σ) are said to be fuzzy automata homotopic equivalent spaces and f is called a fuzzy automata homotopy equivalence.

Definition 3.10. Let (Q, τ) and (R, σ) be any two fuzzy automata structure spaces. A fuzzy automata function $f : (Q, \tau) \to (R, \sigma)$ is said to be a constant function if for each fuzzy automata point q_δ in Q, $f(q_\delta) = r_\lambda$ where r_λ is a fixed fuzzy automata point in R.

Definition 3.11. A fuzzy automata structure space (Q, τ) is said to be fuzzy automata contractible space if the identity function $\boldsymbol{id}_Q : (Q, \tau) \rightarrow (Q, \tau)$ is fuzzy automata homotopic to a constant function $h : (Q, \tau) \rightarrow (Q, \tau)$.

Definition 3.12. Let (Q, τ) and (R, σ) be any two fuzzy automata structure spaces. Let $f, g : (Q, \tau) \rightarrow (R, \sigma)$ be any two fuzzy automata continuous functions and $f \simeq g$.

If g is a constant function then f is said to be a fuzzy automata null-homotopic function.

Proposition 3.3. A fuzzy automata structure space (Q, τ) is fuzzy automata contractible iff for any arbitrary fuzzy automata structure space (R, σ) every fuzzy automata function $f : (Q, \tau) \rightarrow (R, \sigma)$ is fuzzy automata null-homotopic.

Proof Assume that for any arbitrary fuzzy automata structure space (R, σ) the function $f : (Q, \tau) \rightarrow (R, \sigma)$ is fuzzy automata null-homotopic. Then the identity function $\boldsymbol{id}_Q : (Q, \tau) \rightarrow (Q, \tau)$ is fuzzy automata null-homotopic and hence (Q, τ) is fuzzy automata contractible. Now, assume that (Q, τ) is a fuzzy automata contractible space. Then there exists a constant function $h : (Q, \tau) \rightarrow (Q, \tau)$ defined by $h(q_\delta) = q'_\delta$ and a fuzzy automata homotopy $F : (Q, \tau) \times (I, \omega(\xi)) \rightarrow (Q, \tau)$ such that

$$F(q_\delta, 0) = \boldsymbol{id}_Q(q_\delta);$$
$$F(q_\delta, 1) = h(q_\delta) = q'_\delta$$

If there is a fuzzy automata function $f : (Q, \tau) \rightarrow (R, \sigma)$, then $H : (Q, \tau) \times (I, \omega(\xi)) \rightarrow (R, \sigma)$ defined by $H(q_\delta, s) = f(F(q_\delta, s))$ where $s \in I$ has the following properties:

$$H(q_\delta, 0) = f(F(q_\delta, 0)) = f(\boldsymbol{id}_Q(q_\delta)) = f(q_\delta) = (f(q))_\delta;$$

$$H(q_\delta, 1) = f(F(q_\delta, 1)) = f(h(q_\delta)) = f(q'_\delta)$$

Hence H is a fuzzy automata homotopy from f to a constant map with value $f(q'\ )$. Thus, f is fuzzy automata null-homotopic.

Proposition 3.4. Let (Q, τ), (R, σ) and (S, ρ) be any three fuzzy automata structure spaces. Let $f, g : (Q, \tau) \rightarrow (R, \sigma)$ be fuzzy automata continuous functions such that

$f \simeq g$. If
$h : (R, \sigma) \to (S, \rho)$ is a fuzzy automata continuous function, then $h \circ f$, $h \circ g : (Q, \tau) \to (S, \rho)$ are fuzzy automata continuous and $h \circ f \simeq h \circ g$.

Proof Let $(I, \omega(\xi))$ be a fuzzy automata structure space introduced by the Euclidean space (I, ξ). Since h, f and g are fuzzy automata continuous functions, $h \circ f$ and $h \circ g$ are fuzzy automata continuous. Since $f \simeq g$ by definition of fuzzy automata homotopy there is a fuzzy automata continuous function $G : (Q, \tau) \times (I, \omega(\xi)) \to (R, \sigma)$ such that $G(q_\delta, 0) = f(q_\delta)$, $G(q_\delta, 1) = g(q_\delta)$, for each fuzzy automata point q_δ of Q.

Now, $H : (Q, \tau) \times (I, \omega(\xi)) \to (S, \rho)$ is given by $H(q_\delta, t) = h(G(q_\delta, t))$ where $t \in I$. Since h and g are fuzzy automata continuous functions, $H = h \circ g$ is a fuzzy automata
continuous function. Moreover, H satisfies the following conditions.

$$H(q_\delta, 0) = h(G(q_\delta, 0)) = h(f(q_\delta)) = (h \circ f)(q_\delta);$$

$$H(q_\delta, 1) = h(G(q_\delta, 1)) = h(g(q_\delta)) = (h \circ g)(q_\delta).$$

Hence $h \circ f \simeq h \circ g$.

Proposition 3.5. Let (Q, τ), (R, σ) and (S, ρ) be any three fuzzy automata structure spaces. Suppose that $f_1, f_2 : (Q, \tau) \to (R, \sigma)$ are fuzzy automata homotopic functions and that $g_1, g_2 : (R, \sigma) \to (Q, \tau)$ are fuzzy automata homotopic functions, then $g_1 \circ f_1 \simeq g_2 \circ f_2$.

Proof Let $H : (Q, \tau) \times (I, \omega(\xi)) \to (R, \sigma)$ be a fuzzy automata homotopy from f_1 to f_2 and let $G : (R, \sigma) \times (I, \omega(\xi)) \to (S, \rho)$ be a fuzzy automata homotopy from g_1 to g_2. Now let us define a function $F : (Q, \tau) \times (I, \omega(\xi)) \to (S, \rho)$ by $F(q_\delta, s) = G(H(q_\delta, s), s)$.

Since F is a composition of two fuzzy automata continuous functions G and H, F is also a fuzzy automata continuous function. It is seen that

$$F(q_\delta, 0) = G(H(q_\delta, 0), 0) = g_2(f_2(q_\delta))$$

$$F(q_\delta, 1) = G(H(q_\delta, 1), 1) = g_1(f_1(q_\delta))$$

Hence F is a fuzzy automata homotopy from $g_1 \circ f_1$ to $g_2 \circ f_2$. Therefore $g_1 \circ f_1 \simeq g_2 \circ f_2$.

Definition 3.13. Let (Q, τ) and (R, σ) be any two fuzzy automata structure spaces. Let $f_1 : (Q, \tau) \to (R, \sigma)$ be fuzzy a automata continuous function. If there exists a fuzzy automata continuous function $f_2 : (R, \sigma) \rightarrow (Q, \tau)$ which satisfies the following conditions

(i)$f_1 \circ f_2 \simeq \boldsymbol{id}_R$ and (ii) $f_2 \circ f_1 \simeq \boldsymbol{id}_Q$, where $\boldsymbol{id}_Q$ and $\boldsymbol{id}_R$ are the identity functions of Q and R respectively, then the function f_1 is called a fuzzy automata homotopy equivalence and the fuzzy automata structure spaces (Q, τ) and (R, σ) are called fuzzy automata homotopic equivalent spaces.

Proposition 3.6. Let (Q, τ) and (R, σ) be any two fuzzy automata structure spaces. Every fuzzy automata function that is fuzzy automata homotopic to a fuzzy automata homotopy equivalence is a fuzzy automata homotopy equivalence.

Proof let $f_1 : (Q, \tau) \rightarrow (R, \sigma)$ be a fuzzy automata homotopy equivalence and let $g : (Q, \tau) \rightarrow (R, \sigma)$ be fuzzy automata homotopic to f_1. Then by definition of fuzzy automata homotopy equivalence there exists a fuzzy automata continuous function f_2 :
$(R, \sigma) \rightarrow (Q, \tau)$ such that $f_2 \circ f_1 \simeq \boldsymbol{id}_Q$ and $f_1 \circ f_2 \simeq \boldsymbol{id}_R$ respectively. Since f_1, g :
$(Q, \tau) \rightarrow (R, \sigma)$ are fuzzy automata homotopic and by ***Proposition*3.5** it is seen that $f_2 \circ f_1 \simeq f_2 \circ g \simeq \boldsymbol{id}_Q$

$f_1 \circ f_2 \simeq g \circ f_2 \simeq \boldsymbol{id}_R$

Therefore g is a fuzzy automata homotopy equivalence.

Definition 3.14. Let (Q, τ) and (R, σ) be any two fuzzy automata structure spaces. If the bijective function $f : (Q, \tau) \rightarrow (R, \sigma)$ and its inverse function are fuzzy automata
continuous functions, then the function f is said to be a fuzzy automata homeomorphism. Moreover, (R, σ) are said to be fuzzy automata homeomorphic spaces.

Proposition 3.7. Every fuzzy automata homeomorphic spaces are fuzzy automata homotopy equivalent spaces.

Proof Let (Q, τ) and (R, σ) be any two fuzzy automata structure spaces. Since f_1 :
$(Q, \tau) \rightarrow (R, \sigma)$ and $f_2 : (R, \sigma) \rightarrow (Q, \tau)$ are fuzzy automata homeomorphisms, $f_1 \circ f_2 =$
$\boldsymbol{id}_R$ and $f_2 \circ f_1 = \boldsymbol{id}_Q$, where $\boldsymbol{id}_Q$ and $\boldsymbol{id}_R$ are the identity functions of Q and R respectively. Hence by ***Proposition*3.2** (Q, τ) and (R, σ) are fuzzy automata homotopy equivalent spaces.

4 FUZZY AUTOMATA PATH CONNECTED SPACES

In this section, the concepts of fuzzy automata path connected spaces, fuzzy automata retractions, fuzzy automata deformation retractions, fuzzy automata path homotopy and fuzzy automata fundamental groups are introduced and several interesting properties are discussed.

Definition 4.1. Let $(I, \omega(\xi))$ be a fuzzy automata structure space introduced by the Euclidean space (I, ξ) and (Q, τ) be any fuzzy automata structure space. Let q_δ and p_t be any two fuzzy automata points of Q. A fuzzy automata path ϑ in (Q, τ) from q_δ to p_t is a fuzzy automata continuous function $\vartheta : (I, \omega(\xi)) \to (Q, \tau)$ such that $\vartheta(0) = q_\delta$ and $\vartheta(1) = p_t$. Then the fuzzy automata points $\vartheta(0)$ and $\vartheta(1)$ are called the origin and endpoints of ϑ.

Definition 4.2. Let (Q, τ) be a fuzzy automata structure space. Let q_δ and p_t be any fuzzy automata points of Q. A fuzzy automata structure space (Q, τ) is said to be a fuzzy automata path connected space if there exists a fuzzy automata path in (Q, τ) with origin q_δ and endpoint p_t.

Definition 4.3. Let (Q, τ) be a fuzzy automata structure space and let $\lambda \in I^Q$. Let $P \subseteq Q$. A fuzzy automata subsystem $\lambda_{/P}$ is called a fuzzy automata retract of Q if there exists a fuzzy automata continuous function $\tilde{r} : (Q, \tau) \to (P, \tau_{/P})$ such that $\tilde{r}(q_\delta) = q_\delta$ for each fuzzy automata point q_δ of P. Then the fuzzy automata function $\tilde{r}$ is said to be a fuzzy automata retraction.

Definition 4.4. Let $(I, \omega(\xi))$ be a fuzzy automata structure space introduced by the Euclidean space (I, ξ). Let (Q, τ) be a fuzzy automata structure space and $P \subseteq Q$. A fuzzy automata subsystem $\lambda_{/P}$ is called a fuzzy automata deformation retract of Q if there exists a fuzzy automata retraction $\tilde{r} : (Q, \tau) \to (P, \tau_{/P})$ and a fuzzy automata homotopy $H : (Q, \tau) \times (I, \omega(\xi)) \to (Q, \tau)$ such that $H(q_\delta, 0) = q_\delta$, $H(q_\delta, 1) = \tilde{r}(q_\delta)$, for each fuzzy automata point q_δ of Q and $H(p_\delta, s) = p_\delta$, for each fuzzy automata point p_δ of P, $s \in I$. Then the fuzzy automata function H is said to be a fuzzy automata deformation retraction.

Definition 4.5. Let (Q, τ) be any fuzzy automata structure space. Let ϑ and θ be fuzzy automata paths in (Q, τ) from q_δ to q'_μ and from q'_μ to q''_γ. The product of ϑ and θ is the fuzzy automata path $\vartheta\theta$ in (Q, τ) from q_δ to q''_γ defined by,

$$\vartheta\theta = \begin{cases} \vartheta((2s)_\alpha), & 0 \le s \le 1/2 \\ \theta((2s-1)_\alpha), & 1/2 \le s \le 1 \end{cases}$$

for all s $\in I$.

Definition 4.6. Let ϑ be the fuzzy automata path in (Q, τ) from q_δ to q'_μ. The inverse of ϑ is the fuzzy automata path in (Q, τ) from q'_μ to q_δ defined by $\vartheta^{-1}(s) = \vartheta(1-s)$ for all $s \in I$.

Definition 4.7. Let $(I, \omega(\xi))$ and $(I, \omega(\zeta))$ be any two fuzzy automata structure spaces introduced by the Euclidean spaces (I, ξ) and (I, ζ) respectively. Let (Q, τ) be a fuzzy automata structure space. Two fuzzy automata paths ϑ and θ in (Q, τ) from q_δ to q'_μ are said to be a fuzzy automata path homotopy(in short, $\vartheta \simeq \theta$) if there exists a fuzzy automata continuous function $H : (I, \omega(\zeta)) \times (I, \omega(\xi)) \to (Q, \tau)$ such that, $H(0, s_\alpha) = q_\delta$ and $H(1, s_\alpha) = q'_\mu$ for all $s_\alpha \in (I, \omega(\xi))$ and $H(s_\beta, 0) = \vartheta(s_\beta)$ and $H(s_\beta, 1) = \theta(s_\beta)$ for all $s_\beta \in (I, \omega(\zeta))$.

Proposition 4.1. Let (Q, τ) be a fuzzy automata structure space. Let $\vartheta_0, \vartheta_1, \theta_0, \theta_1$ be the fuzzy automata paths in (Q, τ) respectively. If $\vartheta_0 \simeq \vartheta_1$, $\theta_0 \simeq \theta_1$ and $\vartheta_0\theta_0$ is defined, then $\vartheta_1\theta_1$ is defined and $\vartheta_1\theta_1 \simeq \vartheta_0\theta_0$.

Proof Let $(I, \omega(\xi))$ and $(I, \omega(\zeta))$ be any two fuzzy automata structure spaces introduced by the Euclidean spaces (I, ξ) and (I, ζ) respectively. It is clear that $\vartheta_1\theta_1$ is well-defined. Let $H, J : (I, \omega(\zeta)) \times (I, \omega(\xi)) \to (Q, \tau)$ be the fuzzy automata path homotopies from ϑ_0 to ϑ_1 and from θ_0 to θ_1 respectively. Then there exists a fuzzy automata path homotopy, $E : (I, \omega(\zeta)) \times (I, \omega(\xi)) \to (Q, \tau)$ from $\vartheta_0\theta_0$ to $\vartheta_1\theta_1$ is given by,

$$E(s, s'_\delta) = \begin{cases} H(2s, s'_\delta), & 0 \le s \le 1/2 \\ J(2s-1, s'_\delta), & 1/2 \le s \le 1 \end{cases}$$

for each $s'_\delta \in (I, \omega(\xi)), s \in I$. By ***Proposition* 3.1**, E is a fuzzy automata continuous function. Hence, $\vartheta_1\theta_1 \simeq \vartheta_0\theta_0$.

Proposition 4.2. Let (Q, τ) be a fuzzy automata structure space. Let ϑ_0, ϑ_1 be any two fuzzy automata paths in (Q, τ). If $\vartheta_0 \;\tilde{=}\; \vartheta_1$, then $\vartheta_0^{-1} \cong \vartheta_1^{-1}$.

Proof Let $(I, \omega(\xi))$ and $(I, \omega(\zeta))$ be any two fuzzy automata structure spaces introduced by the Euclidean spaces (I, ξ) and (I, ζ) respectively. Let $H : (I, \omega(\zeta)) \times (I, \omega(\xi)) \to (Q, \tau)$ be a fuzzy automata path homotopy from ϑ_0 to ϑ_1. A fuzzy automata path homotopy

$J : (I, \omega(\zeta)) \times (I, \omega(\xi)) \to (Q, \tau)$ from ϑ_0^{-1} to ϑ_1^{-1} is given by, $J(s, s'_\delta) = H(1 - s, s'_\delta)$ for each $s'_\delta \in \big(I, \omega(\xi)\big)$. Hence $\vartheta_0^{-1} \cong \vartheta_1^{-1}$.

Notation 4.1. $\langle\langle \vartheta \rangle\rangle$ denotes the set of all fuzzy automata paths homotopic to ϑ, that is, the equivalence class of ϑ.

Definition 4.8. The product and inverse of the equivalence classes of the fuzzy automata

paths ϑ and θ are defined by $\langle\langle \vartheta \rangle\rangle \circ \langle\langle \theta \rangle\rangle = \langle\langle \vartheta\theta \rangle\rangle$ and $\langle\langle \vartheta \rangle\rangle^{-1} = \langle\langle \vartheta^{-1} \rangle\rangle$

Definition 4.9. Let e : $(I, \omega(\zeta)) \to (Q, \tau)$ be a fuzzy automata path in (Q, τ), defined by $e(s_\gamma) = q_\delta$ for each fuzzy automata points s_γ and q_δ of I and Q respectively. The set of all fuzzy automata homotopic equivalence classes of paths with origin and end point q_δ is called a fuzzy automata fundamental group of Q at q_δ, if the following conditions are satisfied,

1. If $\langle\langle \vartheta \rangle\rangle$ has origin and end q_δ, then $\langle\langle e \rangle\rangle \circ \langle\langle \vartheta \rangle\rangle = \langle\langle \vartheta \rangle\rangle \circ \langle\langle e \rangle\rangle = \langle\langle \vartheta \rangle\rangle$.
2. If $\langle\langle \vartheta \rangle\rangle$ has origin and end q_δ, then $\langle\langle \vartheta \rangle\rangle \circ \langle\langle \vartheta^{-1} \rangle\rangle = \langle\langle \vartheta^{-1} \rangle\rangle \circ \langle\langle \vartheta \rangle\rangle = \langle\langle e \rangle\rangle$.
3. If $(\vartheta\theta)\kappa$ is defined, then $(\langle\langle \vartheta \rangle\rangle \circ \langle\langle \theta \rangle\rangle) \circ \langle\langle \kappa \rangle\rangle = \langle\langle \vartheta \rangle\rangle \circ (\langle\langle \theta \rangle\rangle) \circ \langle\langle \kappa \rangle\rangle)$.

Notation 4.2. The fuzzy automata fundamental group of Q at q_δ is denoted by $\pi_1(Q, q_\delta)$.

Definition 4.10. Let $\pi_1(Q, q_\delta)$ and $\pi_1(R, p_\mu)$ be any two fuzzy automata fundamental groups. A fuzzy automata function $g : \pi_1(Q, q_\delta) \to \pi_1(R, p_\mu)$ is said to be a fuzzy automata homomorphism if $g(\langle\langle \theta \rangle\rangle \circ \langle\langle \vartheta \rangle\rangle) = g(\langle\langle \theta \rangle\rangle) \circ g(\langle\langle \vartheta \rangle\rangle)$ for all $\langle\langle \theta \rangle\rangle, \langle\langle \vartheta \rangle\rangle \in \pi_1(Q, q_\delta)$. The fuzzy automata homomorphism is said to be a fuzzy automata isomorphism if it is bijective.

Proposition 4.3. Let (Q, τ) be a fuzzy automata path connected space. Let q_δ, q'_μ be any two fuzzy automata points of Q. Then there exists a fuzzy automata group isomorphism of $\pi_1(Q, q_\delta)$ onto $\pi_1(Q, q'_\mu)$.

Proof Let ϑ be a fuzzy automata path from q'_μ to q_δ in (Q, τ). Let $\vartheta_\#: \pi_1(Q, q_\delta) \to \pi_1(Q, q'_\mu)$ be defined by

$\vartheta_\#(\langle\langle \theta \rangle\rangle) = \langle\langle \vartheta \rangle\rangle^{-1} \circ \langle\langle \theta \rangle\rangle \circ \langle\langle \vartheta \rangle\rangle = \langle\langle \vartheta^{-1}\theta\vartheta \rangle\rangle$ for each $\langle\langle \theta \rangle\rangle \in \pi_1(Q, q_\delta)$. Now for all $\langle\langle\theta\rangle\rangle$, $\langle\langle\kappa\rangle\rangle \in \pi_1(Q, q_\delta)$,

$$\vartheta_\#(\langle\langle\theta\rangle\rangle)\vartheta_\#(\langle\langle\kappa\rangle\rangle) = \langle\langle\vartheta\rangle\rangle^{-1} \circ \langle\langle\theta\rangle\rangle \circ \langle\langle\vartheta\rangle\rangle \circ \langle\langle\vartheta\rangle\rangle^{-1} \circ \langle\langle\kappa\rangle\rangle \circ \langle\langle\vartheta\rangle\rangle$$

$= \langle\langle\vartheta\rangle\rangle^{-1} \circ \langle\langle\theta\vartheta\, \vartheta^{-1}\kappa\rangle\rangle \circ \langle\langle\vartheta\rangle\rangle$ (By **definition 4.8**)

$$= \langle\langle\vartheta\rangle\rangle^{-1} \circ \langle\langle\theta e\kappa\rangle\rangle \circ \langle\langle\vartheta\rangle\rangle$$

$$= \langle\langle\vartheta\rangle\rangle^{-1} \circ \langle\langle\theta\rangle\rangle \circ \langle\langle\kappa\rangle\rangle \circ \langle\langle\vartheta\rangle\rangle$$

$$= \vartheta_\#(\langle\langle\theta\rangle\rangle \circ \langle\langle\kappa\rangle\rangle)$$

Therefore $\vartheta_\#$ is a fuzzy automata homomorphism. As $\vartheta_\#$ has an inverse namely $(\vartheta_\#)^{-1} = (\vartheta^{-1})_\#$, it is seen that $\vartheta_\#$ is a fuzzy automata isomorphism.

Note 4.1. If $g : (Q, \tau) \to (R, \sigma)$ is a fuzzy automata continuous function and if ϑ, θ are the fuzzy automata paths in (Q, τ) with $\vartheta(1) = \theta(0)$, then $g(\vartheta\theta) = (g\vartheta).(g\theta)$.

Definition 4.11. Let (Q, τ) and (R, σ) be any two fuzzy automata path connected spaces and $g : (Q, \tau) \to (R, \sigma)$ be a fuzzy automata continuous function. Then the function,

$g_* : \pi_1(Q, q'_\mu) \to \pi_1(R, g(q'_\mu))$ is called the fuzzy automata homomorphism induced by g if

$g_*(\langle\langle\vartheta\rangle\rangle) = \langle\langle g\vartheta\rangle\rangle$ for all $\langle\langle\vartheta\rangle\rangle \in \pi_1(Q, q'_\mu)$.

Proposition 4.4. Let (Q, τ), (R, σ) and (S, ρ) be any three fuzzy automata path connected spaces. Let $g : (Q, \tau) \to (R, \sigma)$ be a fuzzy automata continuous function and q'_μ be a

a fuzzy automata point of Q. Then g induces a automata homomorphism, $g_* : \pi_1(Q, q'_\mu) \to \pi_1(R, g(q'_\mu))$.

Proof For each $\langle\langle\theta\rangle\rangle$, $\langle\langle\vartheta\rangle\rangle \in \pi_1(Q, q'_\mu)$,

$$g_*(\langle\langle\theta\rangle\rangle \circ \langle\langle\vartheta\rangle\rangle) = g_*(\langle\langle\theta\vartheta\rangle\rangle)$$

$$= \langle\langle g(\theta\vartheta)\rangle\rangle$$

$$= \langle\langle(g\theta).(g\vartheta)\rangle\rangle$$

$$= \langle\langle(g\theta)\rangle\rangle \circ \langle\langle(g\vartheta)\rangle\rangle$$

$$= g_*(\langle\langle\theta\rangle\rangle) \circ g_*(\langle\langle\vartheta\rangle\rangle)$$

Thus g_* is a fuzzy automata homomorphism.

Proposition 4.5. Let (Q, τ), (R, σ) and (S, ρ) be any three fuzzy automata path

connected spaces. Let q'_δ be a fuzzy automata point of Q. If $f : (Q, \tau) \rightarrow (R, \sigma)$ and
$g : (R, \sigma) \rightarrow (S, \rho)$ are fuzzy automata continuous functions, then $(g \circ f)_* = g_* \circ f_*$.

Proof For each $\langle\langle\vartheta\rangle\rangle \in \pi_1(Q, q'_\delta)$,

$$\begin{aligned}(g \circ f)_*(\langle\langle\vartheta\rangle\rangle) &= \langle\langle(g \circ f)\vartheta\rangle\rangle \\ &= \langle\langle g(f(\vartheta))\rangle\rangle \\ &= g_*(\langle\langle f(\vartheta)\rangle\rangle) \\ &= g_*(f_*(\langle\langle\vartheta\rangle\rangle)) \\ &= (g_* \circ f_*)(\langle\langle\vartheta\rangle\rangle)\end{aligned}$$

Hence $(g \circ f)_* = g_* \circ f_*$.

Proposition 4.6. Let j be a fuzzy automata homeomorphism between the fuzzy automata path connected spaces (Q, τ) and (R, σ). Then $j_* : \pi_1(Q, q'_\mu) \rightarrow \pi_1(R, j(q'_\mu))$ is a fuzzy automata isomorphism.

Proof Let (Q, τ) and (R, σ) be any two fuzzy automata structure spaces. Let $\boldsymbol{id}_Q : (Q, \tau) \rightarrow (Q, \tau)$ and $\boldsymbol{id}_{\pi_1(Q,q'_\mu)} : \pi_1(Q, q'_\mu) \rightarrow \pi_1(Q, q'_\mu)$ be any two identity functions. For each $\langle\langle\vartheta\rangle\rangle \in \pi_1(Q, q'_\mu)$, $(\boldsymbol{id}_Q)_*(\langle\langle\vartheta\rangle\rangle) = \langle\langle \boldsymbol{id}_Q\vartheta\rangle\rangle = \langle\langle\vartheta\rangle\rangle = \boldsymbol{id}_{\pi_1(Q,q'_\mu)}(\langle\langle\vartheta\rangle\rangle)$.

Thus, $(\boldsymbol{id}_Q)_* = \boldsymbol{id}_{\pi_1(Q,q'_\mu)}$. Since there is a fuzzy automata homeomorphism j from (Q, τ) to (R, σ), j is a bijective function. Now, $(j^{-1})_* \circ j_* = (j^{-1} \circ j)_* = (\boldsymbol{id}_Q)_* = \boldsymbol{id}_{\pi_1(Q,q'_\mu)}$ and similarly, $j_* \circ (j^{-1})_* = \boldsymbol{id}_{\pi_1(\mathrm{R},j(q'_\mu))}$. Since $(j_*)^{-1} = (j^{-1})_*$, it is seen that j_* is bijective. Therefore by ***Proposition 3.4*** , j induces a fuzzy automata homomorphism j_* with an inverse $(j_*)^{-1}$. Hence j_* is a fuzzy automata isomorphism from $\pi_1(Q, q'_\mu)$ to $\pi_1(R, j(q'_\mu))$.

Proposition 4.7. Let $P \subseteq Q$ and q_δ be a fuzzy automata point of P . If $\lambda_{/P}$ is the fuzzy automata deformation retract of Q, then $\pi_1(P, q_\delta)$ is fuzzy automata isomorphic to $\pi_1(Q, q_\delta)$.

Proof Let (Q, τ), $(P, \tau_{/P})$ be any fuzzy automata structure spaces. Let $(I, \omega(\xi))$ be a fuzzy automata structure space introduced by the Euclidean space (I, ξ). Since $\lambda_{/P}$ is the fuzzy automata deformation retract of Q, there exists a fuzzy automata retraction,
$\tilde{r} : (Q, \tau) \rightarrow (P, \tau_{/P})$ and a fuzzy automata homotopy $G : (Q, \tau) \times (I, \omega(\xi)) \rightarrow (Q, \tau)$

such that $G(q_\delta, 0) = q_\delta$, $G(q_\delta, 1) = \tilde{r}(q_\delta)$, for each fuzzy automata point q_δ of Q and $G(p_\delta, s) = p_\delta$, for each fuzzy automata point p_δ of P, $s \in I$. If ϑ is a fuzzy automata path in (Q, τ) with origin and end q'_μ, then $G(\vartheta(s), 1)$ is a fuzzy automata path in $(P, \tau_{/P})$ with origin and end q'_μ, for all $s \in I$. Let us define $f : \pi_1(Q, q_\delta) \to \pi_1(P, q_\delta)$

by $f(\langle\langle\vartheta\rangle\rangle) = \langle\langle G(\vartheta(s), 1)\rangle\rangle$, for all $s \in I$. Now for $\langle\langle\vartheta\rangle\rangle, \langle\langle\theta\rangle\rangle \in \pi_1(Q, q_\delta)$,

$$\begin{aligned} f(\langle\langle\vartheta\rangle\rangle \circ \langle\langle\theta\rangle\rangle) &= f(\langle\langle\vartheta\theta\rangle\rangle) \\ &= \langle\langle G(\vartheta\theta(s), 1)\rangle\rangle \\ &= \langle\langle \big(G(\vartheta(s), 1)\big).\,(G(\theta(s), 1))\rangle\rangle. \\ &= f(\langle\langle\vartheta\rangle\rangle) \circ f(\langle\langle\theta\rangle\rangle) \end{aligned}$$

Therefore, f is a fuzzy automata homomorphism. Since the fuzzy automata path $G(\vartheta(s), 1)$ is equivalent to $G(\vartheta(s), 0) = \vartheta$, we have f is a one-to-one function . If $\langle\langle\kappa\rangle\rangle$ is in $\pi_1(P, q_\delta)$, then $\langle\langle\kappa\rangle\rangle$ is also in $\pi_1(Q, q_\delta)$. Thus it is clear that f is an onto function.
Hence $\pi_1(P, q_\delta)$ is fuzzy automata isomorphic to $\pi_1(Q, q_\delta)$.

Proposition 4.8. Let (Q, τ) be a fuzzy automata contractible space, then (Q, τ) is a fuzzy automata path connected space.

Proof Let (Q, τ) be a fuzzy automata contractible space. Then by hypothesis, the identity function $\boldsymbol{id}_Q : (Q, \tau) \to (Q, \tau)$ is fuzzy automata homotopic to a constant function h :

$(Q, \tau) \to (Q, \tau)$ with value q_δ. Let $H : (Q, \tau) \times (I, \omega(\xi)) \to (Q, \tau)$ be a fuzzy automata homotopy from the identity function $\boldsymbol{id}_Q$ to the constant function h with value q_δ, where $(I, \omega(\xi))$ is the fuzzy automata structure space introduced by the Euclidean space (I, ξ). Let p_t be a fuzzy automata point of Q. It suffices to prove that there exists a fuzzy automata path with origin p_t and end q_δ. It is seen that the fuzzy automata homotopy function H maps the fuzzy automata structure space (Q, τ) to (Q, τ) itself but the fuzzy automata structure space $(I, \omega(\xi))$ introduced by (I, ξ) to the fuzzy automata point q_δ.
Now consider a fuzzy automata function f: $(I, \omega(\xi)) \to (Q, \tau) \times (I, \omega(\xi))$ by $f(s) = (p_t, s)$ and it is seen that f is a fuzzy automata continuous function. Then $\eta = H \circ f$ is a fuzzy automata path in (Q, τ) such that,

$$\eta(0) = H(f(0)) = H(p_t, 0) = p_t$$

$\eta(1) = H(f(1)) = H(p_t, 1) = q_\delta$.

Hence η is a fuzzy automata path with origin p_t and end q_δ , proving that (Q, τ) is a fuzzy automata path connected space.

Proposition 4.9. Let (Q, τ) be a fuzzy automata structure space and let h : $(Q, \tau) \to (Q, \tau)$ be a constant function. Then the following are equivalent:

(i) (Q, τ) is a fuzzy automata contractible space.

(ii) An arbitrary fuzzy automata function $f : (R, \sigma) \to (Q, \tau)$ where (R, σ) is a fuzzy automata structure space is fuzzy automata homotopic to a constant function h.

Proof $(i) \Rightarrow (ii)$. Let (Q, τ) be a fuzzy automata contractible space. Then by hypothesis, the identity function $\boldsymbol{id}_Q : (Q, \tau) \to (Q, \tau)$ is fuzzy automata homotopic to a constant function say $h : (Q, \tau) \to (Q, \tau)$ with value q_δ . Now let f : $(R, \sigma) \to (Q, \tau)$ be any fuzzy automata function. Then by **Proposition 3.5,** it is seen that $\boldsymbol{id}_Q \circ f$ is fuzzy automata homotopic to

$h \circ f$. But $\boldsymbol{id}_Q \circ f = f$ and $h \circ f : (R, \sigma) \to (Q, \tau)$ is the constant function. Hence $(i) \Rightarrow (ii)$ holds.

$(ii) \Rightarrow (i)$. To prove the converse, let $(R, \sigma) = (Q, \tau)$ and let the fuzzy automata function $f : (R, \sigma) \to (Q, \tau)$ be the identity function. Then by the hypothesis given, it is seen that $\boldsymbol{id}_Q : (Q, \tau) \to (Q, \tau)$ is fuzzy automata homotopic to a constant function h. Thus (Q, τ) is a fuzzy automata contractible space. Hence $(ii) \Rightarrow (i)$ holds.

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