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On G −Banach Bessel Sequences in Banach Spaces

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Abstract

Abdollahpour et.al [1] generalized the concepts of frames for Banach Spaces and defined g −Banach frames in Banach spaces. In the present paper, we define g − Banach Bessel sequences in Banach spaces and study their relationship with g −Banach frames. A sufficient condition for a sequence of bounded operators to be a g −Banach frame, in terms of g −Banach Bessel sequence has been given. Further, we study bounded linear operators on an associated Banach space X_d. Also, a necessary and sufficient condition for a kind of g −Banach Bessel sequence to be a g −Banach frame has been given. Furthermore, we give stability condition for the stability of g −Banach Bessel sequences.

1. INTRODUCTION

Frames are main tools for use in signal processing, image processing, data compression and sampling theory etc. Today even more uses are being found for the theory such as optics, filter banks, signal detection as well as study of Besov spaces, Banach space theory etc. Frames for Hilbert spaces were introduced by Duffin and Schaeffer [12]. Later, in 1986, Daubechies, Grossmann and Meyer [11] reintroduced frames and found a new application to wavelet and Gabor transforms. For a nice introduction to frames, one may refer [4].

Coifman and Weiss [10] introduced the notion of atomic decomposition for function spaces. Feichtinger and Gröchening [14] extended the notion of atomic decomposition to Banach spaces. Gröchening [16] introduced a more general concept for Banach spaces called Banach frame. He gave the following definition of a Banach frame:

Definition 1.1. ([16]) Let X be a Banach space and X_{d_1} an associated Banach space of scalar-valued sequences indexed by $\mathbb{N}$. Let $\{f_n\} \subset X^*$ and $S: X_{d_1} \to X$ be given. Then the pair $(\{f_n\}, S)$ is called a *Banach frame* for X with respect to X_{d_1}, if

1. $\{f_n(x)\} \in X_{d_1}$, for each $x \in X$.
2. there exist positive constants A and B with $0 < A \leq B < \infty$ such that
$$A\|x\|_X \leq \|\{f_n(x)\}\|_{X_{d_1}} \leq B\|x\|_X,\ x \in X. \tag{1.1}$$
3. S is a bounded linear operator such that $S(\{f_n(x)\}) = x,\ x \in X$.

The positive constants A and B, respectively, are called lower and upper frame bounds of the Banach frame $(\{f_n\}, S)$. The operator $S: X_{d_1} \to X$ is called the reconstruction operator (or the pre-frame operator). The inequality (1.1) is called the frame inequalty.

Banach frames and Banach Bessel sequence in Banach spaces were further studied in [3, 17, 18, 19, 20].

Over the last decade, various other generalizations of frames for Hilbert spaces have been introduced and studied. Some of them are bounded quasi-projectors by Fornasier [15]; Pseudo frames by Li and Ogawa [21]; Oblique frames by Eldar [13]; Christensen and Eldar [5]; (,; Y)-Operators frames by Cao et.al [2]. W. Sun [24] introduced and defined $g-$frames in Hilbert spaces and observed that bounded quasi-projectors, frames of subspaces, Pseudo frames and Oblique frames are particular cases of $g-$frames in Hilbert spaces. $G-$frames are further studied in [6, 7, 8, 9, 22, 25].

Abdollahpour et.al [1] generalized the concepts of frames for Banach Spaces and defined $g-$Banach frames in Banach spaces. In the present paper, we define $g-$Banach Bessel sequences in Banach spaces and study their relationship with $g-$Banach frames. A sufficient condition for a sequence of bounded operators to be a $g-$Banach frame, in terms of $g-$Banach Bessel sequence has been given. Further, we study bounded linear operators on an associated Banach space X_d. Also, a necessary and sufficient condition for a kind of $g-$Banach Bessel sequence to be a $g-$Banach frame has been given. Furthermore, we give stability condition for the stability of $g-$Banach Bessel sequences.

2. PRELIMINARIES

Throughout this paper, X will denote a Banach space over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), X_d is an associated Banach space of vector valued sequences indexed by $\mathbb{N}$. H is a separable Hilbert space and $\mathfrak{B}(X, H)$ is the collection of all bounded linear operators from X into H.

A Banach space of vector valued sequences (or BV-space) is a linear space of sequences with a norm which makes it a Banach space. Let X be a Banach space and $1 < p < \infty$, then

$$Y = \left\{ \{x_n\}: x_n \in X;\ (\textstyle\sum_{n\in\mathbb{N}} \|x_n\|^p)^{\frac{1}{p}} < \infty \right\}$$

and

$$\ell_\infty = \left\{ \{x_n\}: x_n \in X;\ \sup_{n\in\mathbb{N}} \|x_n\| < \infty \right\}$$

are BV space of X.

Definition 2.1. ([1]) Let X be a Banach space and H be a separable Hilbert space. Let X_d be an associated Banach space of vector-valued sequences indexed by $\mathbb{N}$. Let $\{\Lambda_n\}_{n\in\mathbb{N}} \subset \mathfrak{B}(X, H)$ and $S: X_d \to X$ be given. Then the pair $(\{\Lambda_n\}, S)$ is called a $g-$Banach frame for X with respect to H and X_d, if

1. $\{\Lambda_n(x)\} \in X_d$, for each $x \in X$.
2. there exist positive constants A and B with $0 < A \le B < \infty$ such that
$$A\|x\|_X \le \|\{\Lambda_n(x)\}\|_{X_d} \le B\|x\|_X,\ x \in X. \tag{2.1}$$
3. S is a bounded linear operator such that $S(\{\Lambda_n(x)\}) = x,\ x \in X$.

The positive constants A and B, respectively, are called the lower and upper frame bounds of the $g-$Banach frame $(\{\Lambda_n\}, S)$. The operator $S : X_d \to X$ is called the reconstruction operator and the inequality (2.1) is called the frame inequality for $g-$Banach frame $(\{\Lambda_n\}, S)$.

The following lemma, proved in [23], is used in the sequel:

Lemma 2.2. Let X be a Banach space and H be a separable Hilbert space. Let $\{\Lambda_n\} \subset \mathfrak{B}(X, H)$ be a sequence of non-zero operators. If $\{\Lambda_n\}$ is total over X, i.e.,

$\{x \in X: \Lambda_n(x) = 0,\ n \in \mathbb{N}\} = \{0\}$, then $\mathcal{A} = \{\{\Lambda_n(x)\}: x \in X\}$ is a Banach space with the norm given by $\|\{\Lambda_n(x)\}\|_{\mathcal{A}} = \|x\|_X,\ x \in X$.

3. G −BANACH BESSEL SEQUENCES

We begin this section with the following definition of g −Banach Bessel Sequences in Banach spaces.

Definition 3.1. Let X be a Banach space and H be a separable Hilbert space. Let X_d be an associated Banach space of vector-valued sequences indexed by $\mathbb{N}$. A sequence of operators $\{\Lambda_n\} \subset \mathfrak{B}(X, H)$ is said to be a g −Banach Bessel sequence for X with respect to X_d, if

1. $\{\Lambda_n(x)\} \in X_d$, for each $x \in X$.
2. there exist a positive constant M such that

$$\|\{\Lambda_n(x)\}\|_{X_d} \leq M\|x\|_X,\ x \in X. \tag{3.1}$$

The positive constant M is called a g −Banach Besssel bound (or simply, a bound) for the g −Banach Bessel sequence $\{\Lambda_n\}$.

In the view of Definition 3.1, the following observations arise naturally:

(I). If $(\{\Lambda_n\}, S)$ $(\{\Lambda_n\} \subset \mathfrak{B}(X, H),\ S : X_d \to X)$ is a g − Banach frame for X with respect to X_d and with bounds A, B, then $\{\Lambda_n\}$ is a g −Banach Bessel sequence for X with respect to X_d and with bound B.

(II). A sequence $\{\Lambda_n\} \subset \mathfrak{B}(X, H)$ is a g −Banach Bessel sequence for X with respect X_d if and only if the coefficient mapping $T : X \to X_d$ given by $T(x) = \{\Lambda_n(x)\},\ x \in X$ is bounded.

(III). For each $i \in \{1,2, \dots., k\}$, if $\{\Lambda_{i,n}\} \subset \mathfrak{B}(X, H)$ is a g − Banach Bessel sequence for X with respect to X_d and with bound M_i, then $\{\sum_{i=1}^{k} \alpha_i \Lambda_{i,n}\}$ is also a g − Banach Bessel sequence for X with respect to X_d with bound $\sum_{i=1}^{k} |\alpha_i| M_i$, where $\alpha_i \in \mathbb{R},\ i = 1,2, \dots, k$.

(IV). Let $(\{\Lambda_n\}, S)$ $(\{\Lambda_n\} \subset \mathfrak{B}(X, H),\ S : X_d \to X)$ be a g −Banach frame for X with respect to X_d and with bounds A, B. Let $\{\Theta_n\} \subset \mathfrak{B}(X, H)$ be a sequence of operators such that $\{\Lambda_n + \Theta_n\}$ is a g −Banach Bessel sequence for X with respect to X_d and with bound K. Then $\{\Theta_n\}$ is also a g −Banach Bessel sequence for X with respect to X_d with bound $K + B$.

Remark 3.2. The converse of the observation (I) need not be true. In this regard, we give the following example:

Example 3.3. Let H be a separable Hilbert space and let $E = l^\infty(H) = \{\{x_n\} : x_n \in H;\ \sup_{1\le n<\infty}\|x_n\|_H < \infty\}$ be a Banach space with the norm given by

$$\|\{x_n\}\|_E = \sup_{1\le n<\infty} \|x_n\|_H ,\ \{x_n\} \in E$$

and $E_d \subset l^\infty(H)$.

Now, for each $n \in \mathbb{N}$, define $\Lambda_n : E \to H$ by

$$\left.\begin{array}{r}\Lambda_1(x) = \delta_2^{x_2} \\ \Lambda_n(x) = \delta_{n+1}^{x_{n+1}},\ n > 1\end{array}\right\},\ x = \{x_n\} \in E,$$

where, $\delta_n^x = (0,0, \dots, \underbrace{x}_{\substack{\downarrow \\ n^{th} place}} ,0,0, \dots)$, for all $n \in \mathbb{N}$ and $x \in H$.

Then $\{\Lambda_n(x)\} \in E_d$, for all $x \in E$ and $\|\{\Lambda_n(x)\}\|_{E_d} \le \|x\|_E,\ x \in E$. Therefore, $\{\Lambda_n\}$ is a g −Banach Bessel sequence for E with respect to E_d and with bound 1. But, there exists no reconstruction operator $S : E_d \to E$ such that $(\{\Lambda_n\}, S)$ is a g −Banach frame for E with respect to E_d.

Indeed, if possible, let $(\{\Lambda_n\}, S)$ be a g −Banach frame for E with respect to E_d. Let A and B be choice of bounds for $(\{\Lambda_n\}, S)$. Then

$$A\|x\|_E \le \|\{\Lambda_n(x)\}\|_{E_d} \le B\|x\|_E,\ x \in E. \tag{3.2}$$

Let $x = (1,0,0, \ldots,0, \ldots)$ be a non-zero element in E such that $\Lambda_n(x) = 0$, for all $n \in \mathbb{N}$. Then, by frame inequality (3.2), we get $x = 0$. Which is a contradiction.

Remark 3.4. In view of observation (IV), there exists, in general, no reconstruction operator $S_0 : X_{d_0} \to X$ such that $(\{\Theta_n\},\ S_0)$ is a g −Banach frame for X with respect to X_{d_0}, where X_{d_0} is some associated Banach space of vector-valued sequences indexed by $\mathbb{N}$. Regarding this, we give the following example:

Example 3.5. Let H be a separable Hilbert space and let $E = l^\infty(H) = \{\{x_n\} :\ x_n \in H;\ \sup_{1\le n<\infty}\|x_n\|_H < \infty\}$ be a Banach space with the norm given by

$$\|\{x_n\}\|_E = \sup_{1\le n<\infty} \|x_n\|_H ,\ \{x_n\} \in E.$$

Now, for each $n \in \mathbb{N}$, define $\Lambda_n: E \to H$ by

$$\Lambda_n(x) = \delta_n^{x_n},\ x = \{x_n\} \in E,$$

where, $\delta_n^x = (0,0,\dots,\underbrace{x}_{\substack{\downarrow \\ n^{th} place}},0,0,\dots)$, for all $n \in \mathbb{N}$ and $x \in H$.

Thus $\{\Lambda_n\}$ is total over E. Therefore, by Lemma 2.2, there exists an associated Banach space $\mathcal{A} = \{\{\Lambda_n(x)\}: x \in E\}$ with the norm given by $\|\{\Lambda_n(x)\}\|_{\mathcal{A}} = \|x\|_E$, $x \in E$.

Define $S : \mathcal{A} \to E$ by $S(\{\Lambda_n(x)\}) = x,\ x \in E$. Then $(\{\Lambda_n\}, S)$ is a $g-$Banach frame for E with respect to $\mathcal{A}$ with bounds $A = B = 1$.

Let $\{\Theta_n\} \subset \mathfrak{B}(E,H)$ be a sequence of operators defined as

$$\left.\begin{array}{c}\Theta_1(x) = \Lambda_2(x),\\ \Theta_n(x) = \Lambda_n(x),\ n \geq 2,\ n \in \mathbb{N}\end{array}\right\},\ x \in E.$$

Then, for each $x \in E$, $\{\Theta_n(x)\} \in \mathcal{A}$ and

$$\begin{aligned}\|\{(\Lambda_n + \Theta_n)(x)\}\|_{\mathcal{A}} &\leq \|\{\Lambda_n(x)\}\|_{\mathcal{A}} + \|\{\Theta_n(x)\}\|_{\mathcal{A}}\\ &\leq \|\{\Lambda_n(x)\}\|_{\mathcal{A}} + \|\{\Lambda_n(x)\}\|_{\mathcal{A}}\\ &= 2\|\{\Lambda_n(x)\}\|_{\mathcal{A}}\\ &= 2\|x\|_{E.}\end{aligned}$$

Therefore, $\{\Lambda_n + \Theta_n\}$ is a $g-$Banach Bessel sequence for E with respect $\mathcal{A}$ and with bound 2. Thus, by observation (IV), $\{\Theta_n\}$ is a $g-$Banach Bessel sequence for E with respect to $\mathcal{A}$ and with bound 3. But, there exists no reconstruction operator $S_0: \mathcal{A}_0 \to E$ such that $(\{\Theta_n\}, S_0)$ is a $g-$Banach frame for E with respect to $\mathcal{A}_0$, where $\mathcal{A}_0$ is some associated Banach space of vector-valued sequences indexed by $\mathbb{N}$.

Note. If the Bessel bound of $g-$Banach Bessel sequence $\{\Lambda_n + \Theta_n\}$, $K < \|S\|^{-1}$, then there exists an associated Banach space $\mathcal{A}_0$ and a reconstruction operator $S_0: \mathcal{A}_0 \to X$ such that $(\{\Theta_n\}, S_0)$ is a normalized tight $g-$Banach frame for X with respect to $\mathcal{A}_0$.

In this regard, we give the following result:

Theorem 3.6. Let $(\{\Lambda_n\}, S)$ $(\{\Lambda_n\} \subset \mathfrak{B}(X,H),\ S: X_d \to X)$ be a $g-$Banach frame for X with respect to X_d. Let $\{\Theta_n\} \subset \mathfrak{B}(X,H)$ be a sequence of operators such that

$\{\Lambda_n + \Theta_n\}$ be a $g-$Banach Bessel sequence for X with respect to X_d and with bound $K < \|S\|^{-1}$. Then, there exists a reconstruction operator $S_0: X_{d_0} \to X$ such that $(\{\Theta_n\}, S_0)$ is a normalized tight $g-$Banach frame for X with respect to $X_{d_0} = \big\{\{\Theta_n(x)\}: x \in X\big\}$.

Proof: Let $(\{\Lambda_n\}, S)$ be a $g-$Banach frame for X with respect to X_d and with bounds A, B. Then

$$A\|x\|_X \leq \|\{\Lambda_n(x)\}\|_{X_d} \leq B\|x\|_X, \; x \in X.$$

Since $\{\Lambda_n + \Theta_n\}$ is a $g-$Banach Bessel sequence for X with respect to X_d and with bound K, then

$$(\|S\|^{-1} - K)\|x\|_X \leq \|\{\Lambda_n(x)\}\|_{X_d} - \|\{(\Lambda_n + \Theta_n)(x)\}\|_{X_d}$$
$$\leq \|\{\Theta_n(x)\}\|_{X_d}, \text{ for all } x \in X.$$

Thus $\{\Theta_n\}$ is total over X. Therefore, by Lemma 2.2, there exists an associated Banach space $X_{d_0} = \big\{\{\Theta_n(x)\}: x \in X\big\}$ with the norm given by $\|\{\Theta_n(x)\}\|_{X_{d_0}} = \|x\|_X$, $x \in X$. Define $S_0: X_{d_0} \to X$ by $S_0(\{\Theta_n(x)\}) = x$, $x \in X$. Then S is a bounded linear operator such that $(\{\Theta_n\}, S_0)$ is a normalized tight $g-$Banach frame for X with respect to X_{d_0}.

Towards, the converse of the Theroem 3.6, we give the following result:

Theorem 3.7. Let $(\{\Lambda_n\}, S)$ and $(\{\Theta_n\}, T)$ be two $g-$Banach frame for X with respect to X_d. Let $L : X_d \to X_d$ be a linear homeomorphism such that $L(\{\Lambda_n(x)\}) = \{\Theta_n(x)\}$, $x \in X$. Then $\{\Lambda_n + \Theta_n\}$ is a $g-$Banach Bessel sequence for X with respect to X_d and with bound

$$K = \min\{\|U\|\|I + L\|, \|V\|\|I + L^{-1}\|\}$$

where $U, V: X_d \to X$ are the coefficient mappings given by $U(x) = \{\Lambda_n(x)\}$ and $V(x) = \{\Theta_n(x)\}$, $x \in X$ and I is an identity mapping on X_d.

Proof. For each $x \in X$, we have

$$\|\{(\Lambda_n + \Theta_n)(x)\}\|_{X_d} = \|\{\Lambda_n(x)\} + \{\Theta_n(x)\}\|_{X_d}$$
$$= \|I(\{\Lambda_n(x)\}) + L(\{\Lambda_n(x)\})\|_{X_d}$$
$$\leq \|I + L\|\|\{\Lambda_n(x)\}\|_{X_d}$$
$$= \|I + L\|\|U(x)\|_{X_d}$$
$$\leq (\|I + L\|\|U\|)\|x\|_X.$$

Therefore

$$\|\{(\Lambda_n+\Theta_n)(x)\}\|_{X_d} \leq (\|U\|\|I+L\|)\|x\|_X,\ x\in X. \tag{3.3}$$

Similarly, we have

$$\|\{(\Lambda_n+\Theta_n)(x)\}\|_{X_d} \leq (\|V\|\|I+L^{-1}\|)\|x\|_X,\ x\in X. \tag{3.4}$$

Using (3.3) and (3.4), it follows that $\{\Lambda_n+\Theta_n\}$ is a g −Banach Bessel sequence for X with respect to X_d and with bound

$$K=\min\{\|U\|\|I+L\|,\|V\|\|I+L^{-1}\|\}.$$

Remark 3.8. One may observe that, if $(\{\Lambda_n\},S)$ and $(\{\Theta_n\},T)$ are two g −Banach frames for X with respect to X_d, then $\{\Lambda_n+\Theta_n\}$ is a g −Banach Bessel sequence for X with respect to X_d with bound $\|U\|+\|V\|$, where $U,V\colon X\to X_d$ are the coefficient mappings given by $U(x)=\{\Lambda_n(x)\}$ and $V(x)=\{\Theta_n(x)\},\ x\in X$.

4. BOUNDED LINEAR OPERATORS ON X_d

We begin this section with following result which shows that the action of a bounded linear operator on X_d to construct a g −Banach Bessel sequence, using a given g −Banach Bessel sequence.

Theorem 4.1. Let $\{\Lambda_n\}\subset\mathfrak{B}(X,H)$ be a g −Banach Bessel sequence for X with respect to X_d. Let $\{\Theta_n\}\subset\mathfrak{B}(X,H)$ be such that $\{\Theta_n(x)\}\in X_d,\ x\in X$ and let $W\colon X_d\to X_d$ be a bounded linear operator such that $W(\{\Lambda_n(x)\})=\{\Theta_n(x)\},\ x\in X$. Then $\{\Theta_n\}$ is a g −Banach Bessel sequence for X with respect to X_d.

Proof. Let $\{\Lambda_n\}$ be a g −Banach Bessel sequence for X with Bessel bound M. Then, for each $x\in X$, we have

$$\begin{aligned}\|\{\Theta_n(x)\}\|_{X_d} &= \|W(\{\Lambda_n(x)\})\|_{X_d}\\ &\leq \|W\|\|\{\Lambda_n(x)\}\|_{X_d}\\ &\leq (\|W\|M)\|x\|_X.\end{aligned}$$

Remark 4.2. In the view of Theorem 4.1, we observe that if $(\{\Lambda_n\},S)$ $(\{\Lambda_n\}\subset\mathfrak{B}(X,H),\ S\colon X_d\to X)$ is a g −Banach frame for X with respect to X_d and $W\colon X_d\to X_d$ is a bounded linear operator such that $W(\{\Lambda_n(x)\})=\{\Theta_n(x)\},\ x\in X$. Then, there exists, in general no reconstruction operator $T\colon X_d\to X_d$ such that $(\{\Theta_n\},T)$ is a g −Banach frame for X with respect to X_d.

In this regarding, we give the following example:

Example 4.3. Let H be a separable Hilbert space and let $E = l^\infty(H) = \{\{x_n\} : x_n \in H;\ \sup_{1\le n<\infty}\|x_n\|_H < \infty\}$ be a Banach space with the norm given by

$$\|\{x_n\}\|_E = \sup_{1\le n<\infty} \|x_n\|_H,\ \{x_n\} \in E\,.$$

Now, for each $n \in \mathbb{N}$, define $\Lambda_n: E \to H$ by

$$\Lambda_n(x) = \delta_n^{x_n},\ x = \{x_n\} \in E,$$

where, $\delta_n^x = (0,0,\dots, \underset{n^{th} place}{\underbrace{x}}, 0,0,\dots)$, for all $n \in \mathbb{N}$ and $x \in H$.

Thus $\{\Lambda_n\}$ is total over E. Therefore, by Lemma 2.2, there exists an associated Banach space $E_d = \{\{\Lambda_n(x)\}: x \in E\}$ with the norm given by $\|\{\Lambda_n(x)\}\|_{E_d} = \|x\|_E$, $x \in E$.

Define $S : E_d \to E$ by $S(\{\Lambda_n(x)\}) = x,\ x \in E$. Then $(\{\Lambda_n\}, S)$ is a $g-$Banach frame for E with respect to E_d with bounds $A = B = 1$.

Now, define $W: E_d \to E_d$ as $W(\{\Lambda_n(x)\}) = \{\Lambda_n(x)\}_{n\neq 1}$. Then W is a bounded linear operator on E_d. But, there exists no reconstruction operator $T: E_d \to E$ such that $(\{\Lambda_n\}_{n\neq 1}, T)$ is a $g-$Banach frame for E with respect to E_d.

Indeed, if possible, let $(\{\Lambda_n\}_{n\neq 1}, T)$ is a $g-$Banach frame for E with respect to E_d. Let A and B be choice of bounds for $g-$Banach frame $(\{\Lambda_n\}_{n\neq 1}, T)$. Then

$$A\|x\|_E \le \|\{\Lambda_n(x)\}_{n\neq 1}\|_{E_d} \le B\|x\|_E,\ x \in E. \tag{4.1}$$

Let $x = (1,0,0,\dots,0,\dots)$ be a non-zero element in E such that $\Lambda_n(x) = 0, n > 1, n \in \mathbb{N}$. Then, by frame inequality (4.1), we get $x = 0$. Which is a contradiction.

In the view of Theorem 4.1 and Example 4.3, we give a necessary and sufficient condition for a $g-$Banach Bessel sequence to be a $g-$Banach frame.

Theorem 4.4. Let $(\{\Lambda_n\}, S)$ $(\{\Lambda_n\} \subset \mathfrak{B}(X,H),\ S: X_d \to X)$ be a $g-$Banach frame for X with respect to X_d. Let $\{\Theta_n\} \subset \mathfrak{B}(X,H)$ be such that $\{\Theta_n(x)\} \in X_d,\ x \in X$. Let $W: X_d \to X_d$ be a bounded linear operator such that $W(\{\Lambda_n(x)\}) = \{\Theta_n(x)\}$, $x \in X$. Then, there exists a reconstruction operator $U: X_d \to X$ such that $(\{\Theta_n\}, U)$ is a $g-$Banach frame for X with respect to X_d if and only if

$$\|\{\Theta_n(x)\}\|_{X_d} \ge M\|L(\{\Theta_n(x)\})\|_{X_d},\ x \in X$$

where, M is a positive constant and $L: X_d \to X_d$ is a bounded linear operator such that $L(\{\Theta_n(x)\}) = \{\Lambda_n(x)\},\ x \in X$.

Proof. Let A_Λ and B_Λ be the frame bounds for the $g-$Banach frame $(\{\Lambda_n\}, S)$. Then, for all $x \in X$, we have

$$\begin{aligned} MA_\Lambda\|x\|_X &\leq M\|\{\Lambda_n(x)\}\|_{X_d} \\ &= M\|L(\{\Theta_n(x)\})\|_{X_d} \\ &\leq \|\{\Theta_n(x)\}\|_{X_d} \\ &= \|W(\{\Lambda_n(x)\})\|_{X_d} \\ &\leq \|W\|\|\{\Lambda_n(x)\}\|_{X_d} \\ &\leq \|W\|B_\Lambda\|x\|_X. \end{aligned}$$

Therefore

$$(MA_\Lambda)\|x\|_X \leq \|\{\Theta_n(x)\}\|_{X_d} \leq (\|W\|B_\Lambda)\|x\|_X, \ x \in X.$$

Put $U = SL$. Then $U: X_d \to X$ is a bounded linear operator such that $U(\{\Theta_n(x)\}) = x, \ x \in X$. Hence $(\{\Theta_n\}, U)$ is a $g-$Banach frame for X with respect to X_d.

Conversely, let $(\{\Theta_n\}, U)$ is a $g-$Banach frame for X with respect to X_d and with frame bounds A_Θ, B_Θ. Let $T: X \to X_d$ be the coefficient mapping given by $T(x) = \{\Lambda_n(x)\}, \ x \in X$. Then $L = TU: X_d \to X_d$ is a bounded linear operator such that $L(\{\Theta_n(x)\}) = \{\Lambda_n(x)\}, \ x \in X$.

Thus, on writing $M = \left(\frac{A_\Theta}{B_\Lambda}\right)$, where B_Λ is an upper bound for $g-$Banach frame $(\{\Lambda_n\}, S)$, we have

$$\begin{aligned} \|\{\Theta_n(x)\}\|_{X_d} &\geq A_\Theta\|x\|_X \\ &= MB_\Lambda\|x\|_X \\ &\geq M\|\{\Lambda_n(x)\}\|_{X_d} \\ &= M\|L(\{\Theta_n(x)\})\|_{X_d}. \end{aligned}$$

In the following result, we provide a necessary and sufficient condition for a $g-$Banach frame, in terms of $g-$Banach Bessel sequence.

Theorem 4.5. Let $(\{\Lambda_n\}, S)$ $(\{\Lambda_n\} \subset \mathfrak{B}(X, H), \ S: X_d \to X)$ be a $g-$Banach frame for X with respect to X_d. Let $\{\Theta_n\} \subset \mathfrak{B}(X, H)$ be a sequence of operators such that $\{\Lambda_n + \Theta_n\}$ is a $g-$Banach Bessel sequence for X with respect to X_d and with bound $K < \|S\|^{-1}$. Then, there exists a reconstruction operator $U: X_d \to X$ such that $(\{\Theta_n\}, U)$ is a $g-$Banach frame for X with respect to X_d if and only if there exists a bounded linear operator $L: X_d \to X_d$ such that $L(\{\Theta_n(x)\}) = \{\Lambda_n(x)\}, \ x \in X$.

Proof. Let $(\{\Theta_n\}, S)$ be a g −Banach frame for X wih respect to X_d. Let $T: X \to X_d$ be the coefficient mapping given by $T(x) = \{\Lambda_n(x)\},\ x \in X$. Then $L = TU: X_d \to X_d$ is a bounded linear operator such that $L(\{\Theta_n(x)\}) = \{\Lambda_n(x)\},\ x \in X$.

Conversely, by hypothesis $\{\Theta_n(x)\} \in X_d,\ x \in X$. Also, for each $x \in X$, we have

$$\begin{aligned}(\|S\|^{-1} - K)\|x\|_X &= \|S\|^{-1}\|x\|_X - K\|x\|_X \\ &\le \|\{\Lambda_n(x)\}\|_{X_d} - \|\{(\Lambda_n + \Theta_n)(x)\}\|_{X_d} \\ &\le \|\{\Theta_n(x)\}\|_{X_d} \\ &\le \|\{(\Lambda_n + \Theta_n)(x)\}\|_{X_d} + \|\{\Lambda_n(x)\}\|_{X_d} \\ &\le \|T\|\|x\|_X + K\|x\|_X \\ &= (\|T\| + K)\|x\|_X,\end{aligned}$$

where T is a coefficient mapping.

Now, put $U = SL$. Then $U: X_d \to X$ is a bounded linear operator such that $U(\{\Theta_n(x)\}) = x,\ x \in X$. Hence $(\{\Theta_n\}, U)$ is a g −Banach frame for X with respect to X_d and frame bounds $(\|S\|^{-1} - K)$ and $(\|T\| + K)$.

Next, we give a necessary and sufficient condition for a g −Banach Bessel sequence to be a g −Banach frame.

Theorem 4.6 Let $(\{\Lambda_n\}, S)$ $(\{\Lambda_n\} \subset \mathfrak{B}(X, H),\ S: X_d \to X)$ be a g −Banach frame for X with respect to X_d. Let $\{\Theta_n\} \subset \mathfrak{B}(X, H)$ be a g −Banach Bessel sequence for X with respect to X_d. Let $V: X_d \to X_d$ be a bounded linear operator such that $V(\{\Theta_n(x)\}) = \{\Lambda_n(x)\},\ x \in X$. Then, there exists a reconstruction operator $U: X_d \to X$ such that $(\{\Theta_n\}, U)$ is a g −Banach frame for X with respect to X_d if and only if there exists a constant $M > 0$ such that

$$\|\{(\Lambda_n - \Theta_n)(x)\}\|_{X_d} \le M\|\{\Theta_n(x)\}\|_{X_d},\ x \in X.$$

Proof. Let $K > 0$ be a constant such that

$$\|\{\Theta_n(x)\}\|_{X_d} \le K\|x\|_X,\ x \in X.$$

Let A_Λ be a lower bound for the g −Banach frame for $(\{\Lambda_n\}, S)$. Then

$$\begin{aligned}A_\Lambda\|x\|_X &\le \|\{\Lambda_n(x)\}\|_{X_d} \\ &\le \|\{\Theta_n(x)\}\|_{X_d} + \|\{(\Lambda_n - \Theta_n)(x)\}\|_{X_d} \\ &\le (1 + M)\|\{\Theta_n(x)\}\|_{X_d}.\end{aligned}$$

Put $U = SV$, then $U: X_d \to X$ is a bounded linear operator such that $U(\{\Theta_n(x)\}) =$

$x,\ x \in X$. Hence $(\{\Theta_n\}, U)$ is a g −Banach frame for X with respect to X_d.

Conversely, let A_Λ, B_Λ; A_Θ, B_Θ be bounds for the $g-$ Banach frames $(\{\Lambda_n\}, S)$ and $(\{\Theta_n\}, U)$, respectively. Then

$$A_\Lambda \|x\|_X \leq \|\{\Lambda_n(x)\}\|_{X_d} \leq B_\Lambda \|x\|_X,\ x \in X$$

and

$$A_\Theta \|x\|_X \leq \|\{\Theta_n(x)\}\|_{X_d} \leq B_\Theta \|x\|_X,\ x \in X.$$

Therefore

$$\begin{aligned}\|\{(\Lambda_n - \Theta_n)(x)\}\|_{X_d} &\leq \|\{\Lambda_n(x)\}\|_{X_d} + \|\{\Theta_n(x)\}\|_{X_d} \\ &\leq (B_\Lambda + B_\Theta)\|x\|_X \\ &\leq M\|\{\Theta_n(x)\}\|_{X_d},\ x \in X,\end{aligned}$$

where $M = \frac{1}{A_\Theta}(B_\Lambda + B_\Theta)$.

5. STABILITY OF G −BANACH BESSEL SEQUENCES

In the following result, we show that g −Banach Bessel sequences are stable under small perturbation of given g −Banach Bessel sequence.

Theorem 5.1. Let $\{\Lambda_n\} \subset \mathfrak{B}(X, H)$ be a g −Banach Bessel sequence for X with respect to X_d and let $\{\Theta_n\} \subset \mathfrak{B}(X, H)$ be such that $\{\Theta_n(x)\} \in X_d,\ x \in X$. Then $\{\Theta_n\}$ is a g −Banach Bessel sequence for X with respect to X_d, if there exist non-negative constants $\lambda, \mu\ (\mu < 1)$ and δ such that

$$\|\{(\Lambda_n - \Theta_n)(x)\}\|_{X_d} \leq \lambda\|\{\Lambda_n(x)\}\|_{X_d} + \mu\|\{\Theta_n(x)\}\|_{X_d} + \delta\|x\|_X, \qquad x \in X.$$

Proof. Let M be the Bessel bound for the g −Banach Bessel sequence $\{\Lambda_n\}$. Then

$$\begin{aligned}\|\{\Theta_n(x)\}\|_{X_d} &\leq \|\{\Lambda_n(x)\}\|_{X_d} + \|\{(\Lambda_n - \Theta_n)(x)\}\|_{X_d} \\ &= (1+\lambda)M\|x\|_X + \mu\|\{\Theta_n(x)\}\|_{X_d} + \delta\|x\|_X, x \in X \\ &\leq \left(\frac{(1+\lambda)M + \delta}{1-\mu}\right)\|x\|_X, x \in X.\end{aligned}$$

Hence $\{\Theta_n\} \subset \mathfrak{B}(X, H)$ is a g −Banach Bessel sequence for X with respect to X_d.

REFERENCES

[1] M.R. Abdollahpour, M.H. Faroughi and A. Rahimi: $PG-$ Frames in Banach Spaces, Methods of Functional Analysis and Topology, **13**(3)(2007), 201-210.

[2] Huai-Xin Cao, Lan Li, Qing-Jiang Chen and Guo-Xing Li: (p, Y)-operator frames for a Banach space, J. Math. Anal. Appl., **347**(2)(2008), 583-591.

[3] P. G. Casazza, D. Han and D. R. Larson: Frames for Banach spaces, Contem. Math., **247** (1999), 149-182.

[4] O. Christensen: An introduction to Frames and Riesz Bases, Birkhauser, 2003.

[5] O. Christensen and Y. C. Eldar: Oblique dual frames with shift-invariant spaces, Appl. Compt. Harm. Anal., **17**(1)(2004), 48-68.

[6] Renu Chugh, S.K. Sharma and Shashank Goel: A note on $g-$ frame sequences, Int. J. Pure and Appl. Mathematics, **70**(6)(2011), 797-805.

[7] Renu Chugh, S.K. Sharma and Shashank Goel: Block Sequences and G-Frames, Int. J. Wavelets, Multiresolution and Information Processing, **13**(2)(2015), 1550009-1550018.

[8] Renu Chugh and Shashank Goel: On sum of G-frames in Hilbert spaces, Jordanian Journal of Mathematics and Statistics(JJMS), **5**(2)(2012), 115-124.

[9] Renu Chugh and Shashank Goel: On finite sum of G-frames and near exact G-frames, Electronic J. Mathematical Anal. and Appl., **2**(2)(2014), 73-80.

[10] R.R. Coifman and G. Weiss: Extensions of Hardy Spaces and their use in Analysis, Bull. Amer. Math. Soc., **83**(4)(1977), 569-645.

[11] I.Daubechies, A. Grossman and Y. Meyer: Painless non-orthogonal expansions, J. Math. Phys., **27**(1986), 1271–1283.

[12] R.J. Duffin and A.C. Schaeffer: A class of non-harmonic Fourier series, Trans. Amer. Math. Soc., **72**(1952), 341–366.

[13] Y. C. Eldar: Sampling with arbitrary sampling and reconstruction spaces and oblique dual frame vectors, J. Fourier Anal. Appl., **9**(1)(2003), 77-96.

[14] H. G. Feichtinger and K. H. Gröchenig: A unified approach to atomic decompositions via integrable group representations, Function spaces and Applications, Lecture Notes in Mathematics, Vol. 1302 (1988), 52-73.

[15] M. Fornasier: Quasi-orthogonal decompositions of structured frames, J. Math. Anal. Appl., **289**(2004), 180–199.

[16] K.H. Gröchenig, Describing functions: Atomic decompositions versus frames, Monatsh. fur Mathematik, **112**(1991), 1-41.

[17] P.K. Jain, S. K. Kaushik and L. K. Vashisht: On Banach frames, Indian J. Pure Appl. Math., **37**(5)(2006), 265-272.

[18] P.K. Jain, S. K. Kaushik and L. K. Vashisht: On perturbation of Banach frames, Int. J. Wavelets, Multiresolution and Information Processing, **4**(3)(2006), 559-565.

[19] P.K. Jain, S. K. Kaushik and L. K. Vashisht: Bessel sequences and Banach frames in Banach spaces, Bulletin of the Calcutta Mathematical Society, **98**(1)(2006), 87-92.

[20] P.K. Jain, S. K. Kaushik and L. K. Vashisht: On stability of Banach frames, Bull. Korean Math. Soc., **44**(1) (2007), 73-81.

[21] S. Li and H. Ogawa: Pseudo-duals of frames with applications, Appl. Comp. Harm. Anal., **11**(2001), 289-304.

[22] A.Najati, M.H. Faroughi and A. Rahimi: G-frames and stability of G-frames in Hilbert spaces, Methods of Functional Analysis and Topology, **14**(3)(2008), 271–286.

[23] G.S. Rathore and Tripti Mittal: On G-Banach frames, Poincare Journal of Analysis and Applications(submitted).

[24] W. Sun: G-frame and g-Riesz base, J. Math. Anal. Appl., **322**(1)(2006), 437–452.

[25] W. Sun: Stability of $g-$frames, J. Math. Anal. Appl., **326**(2)(2007), 858–868.