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k-Regular and k-Duo Γ-Semirings

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Abstract

The concept of a k-duo Γ-semiring is introdued. Several characterizations of a k-duo Γ-semiring in a k-regular Γ-semiring are furnished. Further characterizations of a k-regular and k-duo Γ-semiring are studied by using different kinds of k-ideals in a Γ semiring.

Keywords: k-ideal, k-bi-ideal, k-quasi-ideal, k-regular Γ-semiring, k-duo Γ-semiring.

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1. INTRODUCTION

The notion of a Γ-semiring was introduced by Rao [9] as a generalization of a semiring and studied it. Dutta and Sardar[1] discussed semiprime ideals in a Γ-semiring. Author studied quasi-ideals and minimal quasi-ideals of a Γ-semiring in [3] and bi-ideals of a Γ-semiring in [6]. In general ring ideal does not coincides with semiring ideal. Hence Henriksen [2] defined more restricted class of ideals in a semiring known as k-ideals. Sen and Adhikari [10, 11] studied k-ideals of semirings. Properties of k-ideals in a Γ-semiring were discussed by Rao [9] and Dutta and Sardar [1]. Also Author studied k-ideals and full k-ideals in Γ-semirings in [5].

Neumann [9] gave the definition of a regular ring. Analogously the concept of a regular semiring was introduced by Zelznikov [13]. This concept of regularity was extended to a Γ-semiring by Rao [9]. Author furnished some characterizations of regular Γ-semirings in [4]. In [7] Author gave definitions of k-quasi-ideal, k-bi-ideal and k-regular Γ-semiring and then some characterizations of k-regular Γ-semirings are furnished. The concept of a duo semiring was considered by Shabir, Ali and Batool [12] and proved some characterizations of it. In [4] author introduced the concept of a duo Γ-semiring and gave some characterizations of it.

In this paper the notions of a left k-duo Γ-semiring, right k-duo Γ-semiring and k-duo Γ-semiring are defined. Various characterizations of a k-duo Γ-semiring in a k-regular Γ-semiring are proved. Further some characterizations of a k-regular and k-duo Γ-semiring are discussed by using k-ideals, k-bi-ideals, k-quasi-ideals in a Γ-semiring.

2. PRELIMINARIES:

For the basic concepts of Γ-semirings we follow Dutta and Sardar [1].

Definition 2.1: *Let S and Γ be two additive commutative semigroups. S is called a Γ-semiring if there exists a mapping* $S \times \Gamma \times S \longrightarrow S$ *whose image is denoted by* $a\alpha b$; *for all* $a, b \in S$ *and for all* $\alpha \in \Gamma$ *satisfying the following conditions:*

(i) $a\alpha(b+c) = (a\alpha b) + (a\alpha c)$

(ii) $(b+c)\alpha a = (b\alpha a) + (c\alpha a)$

(iii) $a(\alpha+\beta)c = (a\alpha c) + (a\beta c)$

(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$; for all $a, b, c \in S$ *and* $\alpha, \beta \in \Gamma$.

Definition 2.2 : *An element* $0 \in S$ *is said to be an absorbing zero if*

$0\alpha a = 0 = a\alpha 0, a + 0 = 0 + a = a$; *for all* $a \in S$ *and for all* $\alpha \in \Gamma$.

Definition 2.3: *A non-empty subset T of a Γ-semiring S is said to be a sub-Γ-semiring of S if (T,+) is a subsemigroup of (S,+) and* $a\alpha b \in T$; *for all* $a, b \in T$ *and for all* $\alpha \in \Gamma$.

Definition 2.4: *A non-empty subset T of a Γ-semiring S is called a left (respectively right) ideal of S if T is a subsemigroup of (S,+) and* $x\alpha a \in T$ *(respectively* $a\alpha x \in T$*) for all* $a \in T, x \in S$ *and for all* $\alpha \in \Gamma$.

Definition 2.5: *If a non-empty subset T of a Γ-semiring S is both left and right ideal of S, then T is known as an ideal of S.*

Definition 2.6 : *A right ideal I of a Γ-semiring S is said to be a right k-ideal if $a \in I$ and $x \in S$ such that $a + x \in I$, then $x \in I$.*

Similarly we define a left k-ideal of a Γ-semiring S.

If an ideal I is both right k-ideal and left k-ideal of a Γ-semiring S, then I is known as a k-ideal of S.

Examples: (1) Let N_0 denotes the set of all positive integers with zero. $S = N_0$ is a semiring and with $\Gamma = S$, S forms a Γ-semiring. A subset $I = 3N_0 \setminus \{3\}$ of S is an ideal of S but not a k-ideal. Since $6, 9 = 3 + 6 \in I$ but $3 \notin I$.

(2) If $S = N$ is the set of all positive integers, then (S, max., min.) is a semiring and with $\Gamma = S$, S forms a Γ-semiring. $I_n = \{1, 2,3,\cdots\cdots\cdots, n\}$ is a k-ideal for any $n \in I$.

Definition 2.7 : *For a subset I of a Γ-semiring S define*

$\bar{I} = \{a \in S \mid a + x \in \mathrm{I}, for\ some\ x \in I \}$. *$\bar{I}$ is called a k-closure of I.*

Definition 2.8 [7]: *A non-empty subset B of a Γ-semiring S is said to be a k-bi-ideal of S if B is a sub-Γ-semiring of S, $\overline{B\Gamma S\Gamma B} \subseteq B$* and *if $a \in B$ and $x \in S$ such that $a + x \in B$, then $x \in B$.*

Definition 2.9 [7]: *A subsemigroup Q of $(S, +)$ is a k-quasi-ideal of S if $\overline{(S\Gamma Q)} \cap \overline{(Q\Gamma S)} \subseteq Q$ and if $a \in Q$ and $x \in S$ such that $a + x \in Q$, then*

$x \in Q$.

Definition 2.10 [7]: *An element a of a Γ-semiring S is said to be k-regular if $a \in \overline{a\Gamma S\Gamma a}$.*

If all elements of a Γ-semiring S are k-regular, then S is known as k-regular Γ-semiring.

Now onwards S denotes a Γ-semiring with absorbing zero unless otherwise stated.

Some basic properties of k-closure are given in the following lemma .

Lemma 2.11 :- *For non-empty subsets A and B of S we have,*

1) *If $A \subseteq B$, then $\bar{A} \subseteq \bar{B}$.*

2) $\bar{A}$ *is the smallest (left k-ideal, right k-ideal, k-quasi-ideal, k-bi-ideal) k-ideal containing (left k-ideal, right k-ideal, k-quasi-ideal, k-bi-ideal) k-ideal* A *of* S.

3) $\bar{A} = A$ *if and only if* A *is (left k-ideal, right k-ideal, k-quasi-ideal, k-bi-ideal) k-ideal of* S.

4) $\bar{\bar{A}} = \bar{A}$, where A *is (left k-ideal, right k-ideal, k-quasi-ideal, k-bi-ideal) k-ideal of* S.

5) $\overline{\bar{A}\Gamma\bar{B}} = \overline{A\Gamma B}$, where A *and* B *are (left k-ideals, right k-ideals, k-quasi-ideals, k-bi-ideals) k-ideals of* S.

Theorem 2.12 [7]:- *In S following statements are equivalent.*

1) S *is* k*-regular.*

2) *For every left* k*-ideal L and right* k*-ideal* R *of* S, $\overline{R\Gamma L} = R \cap L$.

3) *For every left* k*-ideal L and right* k*-ideal* R *of* S,

(i) $\overline{R^2} = \overline{R\Gamma R} = R$ (ii) $\overline{L^2} = \overline{L\Gamma L} = L$

(iii) $\overline{R\Gamma L} = R \cap L$ *is a k-quasi-ideal of* S.

4) *Every k-quasi-ideal* Q *of* S *is of the form* $\overline{Q\Gamma S\Gamma Q} = Q$.

3. k-DUO Γ-SEMIRING :

Now we define a k-duo Γ-semiring as follows.

Definition 3.1 :- *A* Γ*-semiring* S *is said to be left (right) k-duo* Γ*-semiring if every left (right) k-ideal of S is a right (left) k-ideal.*

A Γ*-semiring S is said to be a k-duo* Γ*-semiring if every one sided k-ideal of S is a two sided k-ideal. That is a* Γ*-semiring* S *is said to be a k-duo* Γ*-semiring if it is both left k-duo and right k-duo.*

Theorem 3.2:- *If* S *is k-regular, then S is left k-duo if and only if for any two left k-ideals* A *and B of* S, $A \cap B = \overline{A\Gamma B}$.

Proof :- Let S be a k-regular Γ-semiring. Suppose that S is a left k-duo Γ-semiring. Let A and B be any two left k-ideals of S. Hence A is a right k-ideal of S. Therefore by Theorem 2.12, $A \cap B = \overline{A\Gamma B}$. Conversely, by assumption $\overline{L\Gamma S} = L \cap$

$S \subseteq L$. This shows that L is a right k-ideal of S. Therefore S is a left k-duo Γ-semiring. ∎

Similar to Theorem 3.2 we have following theorem.

Theorem 3.3:- *If S is k-regular, then S is right k-duo if and only if for any two right k-ideals A and B of S, $A \cap B = \overline{A\Gamma B}$.*

Theorem 3.4:- *If S is k-regular, then S is left k-duo if and only if every k-quasi-ideal of S is a right k-ideal of S.*

Proof :- Let S be a k-regular Γ-semiring and Q be any k-quasi-ideal of S. Suppose that S is left k-duo. Then there exists a right k-ideal R and a left k-ideal L of S such that $Q = R \cap L$. Therefore $Q = R \cap L$ is a right k-ideal of S. Conversely, suppose that every k-quasi-ideal of S is a right k-ideal of S. Let L be a left k-ideal of S. Hence L is a k-quasi-ideal of S. Therefore by assumption L is a right k-ideal of S. Hence S is a left k-duo Γ-semiring. ∎

Proofs of the following theorems are similar to above theorem hence omitted.

Theorem 3.5 :- *If S is k-regular, then S is right k-duo if and only if every k-quasi-ideal of S is a left k-ideal of S.*

Theorem 3.6 :- *A k-regular* Γ*-semiring S is k-duo if and only if every k-quasi-ideal of S is a k-ideal of S.*

Theorem 3.7:- *If S is k-regular, then S is left k-duo if and only if every k-bi-ideal of S is a right k-ideal of S.*

Theorem 3.8:- *If S is k-regular, then S is right k-duo if and only if every k-bi-ideal of S is a left k-ideal of S.*

Theorem 3.9 :- *If S is k-regular, then S is k-duo if and only if every k-bi-ideal of S is a k-ideal of S.*

4. k-REGULAR AND k-DUO Γ-SEMIRING

In this section characterizations of a k-regular and k-duo Γ-semiring are furnished.

Theorem 4.1 :- *Following statements are equivalent in S.*

(1) *S is k-regular and left k-duo.*

(2) *For any k-bi-ideal B and a left k-ideal L of S, $B \cap L = \overline{B\Gamma L}$.*

(3) *For any k-quasi-ideal Q and a left k-ideal L of S, $Q \cap L = \overline{Q\Gamma L}$.*

Proof :- **(1) ⇒ (2)**

Let B be a k-bi-ideal and L be a left k-ideal of S. Then by Theorem 3.7, B is a right k-ideal of S. Therefore $\overline{B\Gamma L} \subseteq B$ and $\overline{B\Gamma L} \subseteq L$. Hence $\overline{B\Gamma L} \subseteq B \cap L$. Let $a \in B \cap L$. Hence $a \in \overline{a\Gamma S\Gamma a}$. Therefore $\overline{a\Gamma S\Gamma a} \subseteq \overline{B\Gamma S\Gamma L} \subseteq \overline{B\Gamma L}$. Thus $B \cap L \subseteq \overline{B\Gamma L}$. Hence we get $B \cap L = \overline{B\Gamma L}$.

(2) ⇒ 3)

Implication holds as every k-quasi-ideal of S is a k-bi-ideal of S.

(3) ⇒ (1)

Let R be a right k-ideal and L be a left k-ideal of S. Then by (3), $R \cap L = \overline{R\Gamma L}$. This shows that S is k-regular by Theorem 2.12. For $L = S$, we have $L \cap S = \overline{L\Gamma S}$. Therefore $L = \overline{L\Gamma S}$. Hence L is a right k-ideal. Thus S is left k-duo. ■

Theorem 4.2 :- *Following statements are equivalent in S.*

(1) *S is k-regular and right k-duo.*

(2) *For any k-bi-ideal B and a right k-ideal R of S, $B \cap R = \overline{R\Gamma B}$.*

(3) *For any k-quasi-ideal Q and a right k-ideal R of S, $Q \cap R = \overline{R\Gamma Q}$.*

Proof :- **(1) ⇒ (2)**

Let B be a k-bi-ideal and R be a right k-ideal of S. Then by Theorem 3.8, B is a left k-ideal of S. Therefore $\overline{R\Gamma B} \subseteq B$ and $\overline{R\Gamma B} \subseteq R$. Hence $\overline{R\Gamma B} \subseteq B \cap R$. Let $a \in B \cap R$. Hence $a \in \overline{a\Gamma S\Gamma a}$. Therefore $\overline{a\Gamma S\Gamma a} \subseteq \overline{R\Gamma S\Gamma B} \subseteq \overline{R\Gamma B}$. Thus $B \cap R \subseteq \overline{R\Gamma B}$. Hence we get $B \cap R = \overline{R\Gamma B}$.

(2) ⇒ 3)

Implication follows as every k-quasi-ideal of S is a k-bi-ideal of S.

(3) ⇒ (1)

Let R be a right k-ideal and L be a left k-ideal of S. Then by (3) , $R \cap L = \overline{R\Gamma L}$. This shows that S is k-regular. For $R = S$, we have $R \cap S = \overline{S\Gamma R}$. Therefore $R = \overline{S\Gamma R}$. Hence R is a left k-ideal. Therefore S is right k-duo. ■

Theorem 4.3:- *In S following conditions are equivalent.*

(1) *S is k-regular and k-duo.*

(2) *For any two k-quasi-ideals Q_1 and Q_2 of S, $Q_1 \cap Q_2 = \overline{Q_1 \Gamma Q_2}$.*

(3) *For a left k-ideal L and a right k-ideal R of S, $L \cap R = \overline{L\Gamma R}$.*

Proof :- **(1) ⇒ (2)**

Let Q_1 and Q_2 be any two k-quasi-ideals of S. Therefore $Q_1 = R_1 \cap L_1$ and $Q_2 = R_2 \cap L_2$, where R_1, R_2 are right k-ideals and L_1, L_2 are left k-ideals of S. Therefore $Q_1 = R_1 \cap L_1$ and $Q_2 = R_2 \cap L_2$ are k-ideals of S. Hence by Theorem 2.12, $Q_1 \cap Q_2 = \overline{Q_1 \Gamma Q_2}$.

(2) ⇒ (3)

Let R be a right k-ideal and L be a left k-ideal of S. Therefore R and L are k-quasi-ideals of S. Hence by (2), we have $L \cap R = \overline{L\Gamma R}$.

(3) ⇒ (1)

For $R = S$, $\overline{L\Gamma S} = L \cap S = L$. This shows that a left k-ideal L is a right k-ideal of S. Similarly we can show that a right k-ideal R is a left k-ideal of S. Thus every one sided k-ideal of S is a k-ideal. Hence S is a k-duo Γ-semiring. Then clearly $R \cap L = \overline{R\Gamma L}$. Hence S is k-regular (see Theorem 2.12). ■

Theorem 4.4 :- *In S following conditions are equivalent.*

(1) *S is k-regular and k-duo.*

(2) $I \cap B = \overline{I\Gamma B\Gamma I}$, *for every k-ideal I and every k-bi-ideal B of S.*

(3) $I \cap Q = \overline{I\Gamma Q\Gamma I}$, *for every k-ideal I and every k-quasi-ideal Q of S.*

Proof :- **(1) ⇒ (2)**

Let I be a k-ideal and B be a k-bi-ideal of S. Hence by Theorem 3.9, B is a k-ideal of S. Therefore $\overline{I\Gamma B\Gamma I} \subseteq I$ and $\overline{I\Gamma B\Gamma I} \subseteq B$. Hence $\overline{I\Gamma B\Gamma I} \subseteq I \cap B$. Take any $a \in I \cap B$. Hence $a \in \overline{a\Gamma S\Gamma a}$. Therefore $\overline{a\Gamma S\Gamma a} \subseteq \overline{a\Gamma S\Gamma(\overline{a\Gamma S\Gamma a})} \subseteq \overline{I\Gamma S\Gamma(\overline{B\Gamma S\Gamma I})} \subseteq \overline{I\Gamma B\Gamma I}$. Thus $I \cap B \subseteq \overline{I\Gamma B\Gamma I}$. Therefore $I \cap B = \overline{I\Gamma B\Gamma I}$.

(2) ⇒ (3)

As every k-quasi-ideal of S is a k-bi-ideal of S, implication holds.

(3) ⇒ (1)

For a left k-ideal L and a right k-ideal R of S, by (3) we have $L = S \cap L = \overline{S\Gamma L\Gamma S}$ and $R = S \cap R = \overline{S\Gamma R\Gamma S}$. Now $\overline{L\Gamma S} = \overline{S\Gamma L\Gamma S\Gamma S} \subseteq \overline{S\Gamma L\Gamma S} = L$ and $\overline{S\Gamma R} =$

$\overline{S\Gamma S\Gamma R\Gamma S} \subseteq \overline{S\Gamma R\Gamma S} = R$. This shows that L is a right k-ideal and R is a left k-ideal of S. Therefore S is a k-duo Γ-semiring. Again by (3), $R \cap L = \overline{R\Gamma L\Gamma R} \subseteq \overline{R\Gamma L}$. But $\overline{R\Gamma L} \subseteq R \cap L$ always. Therefore $R \cap L = \overline{R\Gamma L}$. Hence by Theorem 2.12, S is k-regular. ■

Theorem 4.5:- *S is k-regular and k-duo if and only if $L \cap R = \overline{L\Gamma R\Gamma S}$, for a left k-ideal L and a right k-ideal R of S.*

Proof :- Assume that S is a k-regular and k-duo Γ-semiring. Let R be a right k-ideal and L be a left k-ideal of S. Hence R is a left k-ideal and L is a right k-ideal of S. Therefore $\overline{L\Gamma R\Gamma S} \subseteq L$ and $\overline{L\Gamma R\Gamma S} \subseteq \overline{L\Gamma R} \subseteq R$. Thus we get $\overline{L\Gamma R\Gamma S} \subseteq L \cap R$. Let $a \in L \cap R$. Hence $a \in \overline{a\Gamma S\Gamma a}$. Therefore $\overline{a\Gamma S\Gamma a} \subseteq \overline{\overline{(a\Gamma S\Gamma a)}\Gamma S\Gamma a} \subseteq \overline{(L\Gamma S\Gamma R)\Gamma S} \subseteq \overline{L\Gamma R\Gamma S}$. Hence $L \cap R \subseteq \overline{L\Gamma R\Gamma S}$. Thus we get $L \cap R = \overline{L\Gamma R\Gamma S}$. Conversely, let R be a right k-ideal and L be a left k-ideal of S. Therefore by assumption, $L \cap S = \overline{L\Gamma S\Gamma S}$ and $S \cap R = \overline{S\Gamma R\Gamma S}$. Hence we get $L = \overline{L\Gamma S\Gamma S}$ and $R = \overline{S\Gamma R\Gamma S}$. This shows that L and R are two sided k-ideals of S. Therefore S is a k-duo Γ-semiring. Let I be a k-ideal of S. Hence by assumption, $I \cap S = \overline{I\Gamma S\Gamma S}$, $S \cap I = \overline{S\Gamma I\Gamma S}$, $I \cap I = \overline{I\Gamma I\Gamma S}$. Thus we get $I = \overline{I\Gamma S\Gamma S} = \overline{S\Gamma I\Gamma S} = \overline{I\Gamma I\Gamma S}$. Therefore $\overline{I\Gamma I} = \overline{\overline{(S\Gamma I\Gamma S)}\Gamma\overline{(S\Gamma I\Gamma S)}} = \overline{S\Gamma I}$ and $\overline{I\Gamma I} = \overline{I\Gamma(I\Gamma S\Gamma S)} = \overline{I\Gamma S}$. Hence we get $\overline{I\Gamma S} = \overline{S\Gamma I}$. Now $I = \overline{I\Gamma(I\Gamma S)} = \overline{I\Gamma(S\Gamma I)} = \overline{I\Gamma S\Gamma I}$. Thus $I = \overline{I\Gamma S\Gamma I}$. Therefore S is a k-regular Γ-semiring. ■

Proof of following Theorem is analogous to proof of Theorem 4.5.

Theorem 4.6:- *S is k-regular and k-duo if and only if $L \cap R = \overline{S\Gamma L\Gamma R}$, for a left k-ideal L and a right k-ideal R of S.*

Theorem 4.7:- *S is k-regular and k-duo if and only if $L \cap I = \overline{L\Gamma I}$ and $R \cap I = \overline{I\Gamma R}$, for a left k-ideal L , a right k-ideal R and a k-ideal I of S.*

Proof :- Assume that S is a k-regular and k-duo Γ-semiring. Let R be a right k-ideal and L be a left k-ideal and I be a k-ideal of S. Therefore R is a left k-ideal and L is a right k-ideal of S. Hence $\overline{L\Gamma I} \subseteq L$ and $\overline{L\Gamma I} \subseteq I$. Therefore $\overline{L\Gamma I} \subseteq L \cap I$. Similarly we can show that $\overline{I\Gamma R} \subseteq R \cap I$. Take any $a \in L \cap I$. Hence $a \in \overline{a\Gamma S\Gamma a}$. Therefore $\overline{a\Gamma S\Gamma a} \subseteq \overline{L\Gamma S\Gamma I} \subseteq \overline{L\Gamma I}$. Therefore $L \cap I \subseteq \overline{L\Gamma I}$. Thus we get $L \cap I = \overline{L\Gamma I}$. In the same way we can show that $R \cap I = \overline{I\Gamma R}$. Conversely, let R be a right k-ideal and L be a left k-ideal of S. Hence by assumption, $L \cap S = \overline{L\Gamma S}$ and $S \cap R = \overline{S\Gamma R}$. Therefore $L = \overline{L\Gamma S}$ and $R = \overline{S\Gamma R}$. This shows that L is a right k-ideal and R is a left k-ideal of S. Therefore S is a k-duo Γ-semiring.

Then clearly $R \cap L = \overline{R\Gamma L}$ holds by assumption. Therefore S is a k-regular Γ-semiring (see Theorem 2.12). ■

Theorem 4.8 :- *Following statements are equivalent in* S.

(1) *S is k-regular and k-duo.*

(2) *For any k-bi-ideals A and B of S, $A \cap B = \overline{A\Gamma B\Gamma S}$.*

(3) *For any k-bi-ideals A and B of S, $A \cap B = \overline{S\Gamma A\Gamma B}$.*

(4) *For any k-bi-ideal B and a k-quasi-ideal Q of S, $B \cap Q = \overline{B\Gamma Q\Gamma S}$.*

(5) *For any k-bi-ideal B and a k-quasi-ideal Q of S, $B \cap Q = \overline{S\Gamma B\Gamma Q}$.*

(6) *For any k-bi-ideal B and a k-quasi-ideal Q of S, $B \cap Q = \overline{Q\Gamma B\Gamma S}$.*

(7) *For any k-bi-ideal B and a k-quasi-ideal Q of S, $B \cap Q = \overline{S\Gamma Q\Gamma B}$.*

(8) *For any k-quasi-ideals Q_1 and Q_2 of S, $Q_1 \cap Q_2 = \overline{Q_1\Gamma Q_2\Gamma S}$.*

(9) *For any k-quasi-ideals Q_1 and Q_2 of S, $Q_1 \cap Q_2 = \overline{S\Gamma Q_1\Gamma Q_2}$.*

Proof :- **(1) ⇒ (2)**

Let A and B be any two k-bi-ideals of S. Hence by Theorem 3.9 , both A and B are k-ideals of S. Therefore $\overline{A\Gamma B\Gamma S} \subseteq A$ and $\overline{A\Gamma B\Gamma S} \subseteq \overline{A\Gamma B} \subseteq B$. Hence $\overline{A\Gamma B\Gamma S} \subseteq A \cap B$. Let $a \in A \cap B$. Therefore $a \in \overline{a\Gamma S\Gamma a}$. Hence $\overline{a\Gamma S\Gamma a} \subseteq \overline{\overline{(a\Gamma S\Gamma a)}\,\Gamma S\Gamma a} \subseteq \overline{(A\Gamma S\Gamma B)\Gamma S} \subseteq \overline{A\Gamma B\Gamma S}$. Thus we get $A \cap B \subseteq \overline{A\Gamma B\Gamma S}$. Hence $A \cap B = \overline{A\Gamma B\Gamma S}$.

(2) ⇒ (4) , (4) ⇒ (8), (2) ⇒ (6) , (6) ⇒ (8)

Implications follow as every k-quasi-ideal of S is a k-bi-ideal of S.

(8) ⇒ (1)

Let R be a right k-ideal and L be a left k-ideal of S. Then both R and L are k-quasi-ideals of S. Hence by (8), $L \cap R = \overline{L\Gamma R\Gamma S}$. Therefore by Theorem 4.5, S is a k- regular and k-duo Γ-semiring.

(1) ⇒ (3)

Let A and B be any two k-bi-ideals of S. Therefore by Theorem 3.9, both A and B are k-ideals of S. Then $\overline{S\Gamma A\Gamma B} \subseteq B$ and $\overline{S\Gamma A\Gamma B} \subseteq \overline{A\Gamma B} \subseteq A$. Thus we get $\overline{S\Gamma A\Gamma B} \subseteq A \cap B$. Take any $a \in A \cap B$. Therefore $a \in \overline{a\Gamma S\Gamma a}$. Hence $\overline{a\Gamma S\Gamma a} \subseteq \overline{a\Gamma S\Gamma\overline{(a\Gamma S\Gamma a)}} \subseteq \overline{S\Gamma(A\Gamma S\Gamma B)} \subseteq \overline{S\Gamma A\Gamma B}$. Thus we get $A \cap B \subseteq \overline{S\Gamma A\Gamma B}$. Therefore $A \cap B = \overline{S\Gamma A\Gamma B}$.

(3) ⇒ (5) , (5) ⇒ (9), (3) ⇒ (7) , (7) ⇒ (9)

Clearly implications hold as every k-quasi-ideal of S is a k-bi-ideal of S.

(9) ⇒ (1)

Let R be a right k-ideal and L be a left k-ideal of S. Then both R and L are k-quasi-ideals of S. Therefore by (9), $L \cap R = \overline{S\Gamma L\Gamma R}$. This shows that S is a k-regular and k-duo Γ-semiring by Theorem 4.6. ■

Theorem 4.9:- In S following statements are equivalent.

(1) *S is k-regular and k-duo.*

(2) *For any k-bi-ideals A, B and a k-ideal I of S, $A \cap B \cap I = \overline{A\Gamma B\Gamma I}$.*

(3) *For any k-bi-ideals A, B and a k-ideal I of S, $A \cap B \cap I = \overline{I\Gamma A\Gamma B}$.*

(4) *For any k-bi-ideal B, a k-quasi-ideal Q and a k-ideal I of S,*
$B \cap Q \cap I = \overline{B\Gamma Q\Gamma I}$.

(5) *For any k-bi-ideal B, a k-quasi-ideal Q and a k-ideal I of S,*
$B \cap Q \cap I = \overline{I\Gamma B\Gamma Q}$.

(6) *For any k-bi-ideal B, a k-quasi-ideal Q and a k-ideal I of S,*
$B \cap Q \cap I = \overline{Q\Gamma B\Gamma I}$.

(7) *For any k-bi-ideal B, a k-quasi-ideal Q and a k-ideal I of S,*
$B \cap Q \cap I = \overline{I\Gamma Q\Gamma B}$.

(8) *For any k-quasi-ideals Q_1, Q_2 and a k-ideal I of S, $Q_1 \cap Q_2 \cap I = \overline{Q_1\Gamma Q_2\Gamma I}$.*

(9) *For any k-quasi-ideals Q_1, Q_2 and a k-ideal I of S, $Q_1 \cap Q_2 \cap I = \overline{I\Gamma Q_1\Gamma Q_2}$.*

Proof :- **(1) ⇒ (2)**

Let A , B be any two k-bi-ideals and I be a k-ideal of S. Therefore by Theorem 3.9, both A and B are k-ideals of S. Hence $\overline{A\Gamma B\Gamma I} \subseteq A$ and $\overline{A\Gamma B\Gamma I} \subseteq \overline{A\Gamma B} \subseteq B$. Also $\overline{A\Gamma B\Gamma I} \subseteq I$. Hence we get $\overline{A\Gamma B\Gamma I} \subseteq A \cap B \cap I$. Let $a \in A \cap B \cap I$. Hence $a \in \overline{a\Gamma S\Gamma a}$. Therefore $\overline{a\Gamma S\Gamma a} \subseteq \overline{\overline{(a\Gamma S\Gamma a)}\,\Gamma S\Gamma a} \subseteq \overline{(A\Gamma S\Gamma B)\Gamma S\Gamma I} \subseteq \overline{A\Gamma B\Gamma I}$. Thus we get $A \cap B \cap I \subseteq \overline{A\Gamma B\Gamma I}$. Hence $A \cap B \cap I = \overline{A\Gamma B\Gamma I}$.

(2) ⇒ (4) , (4) ⇒ (8), (2) ⇒ (6) , (6) ⇒ (8)

Implications follow as every k-quasi-ideal of S is a k-bi-ideal of S.

(8) ⇒ (1)

Let R be a right k-ideal and L be a left k-ideal of S. Then both R and L are k-quasi-ideals of S. Therefore by (8), $L \cap R = \overline{L\Gamma R\Gamma S}$. Hence S is a k-regular and k-duo Γ-semiring by Theorem 4.5.

(1) ⇒ (3)

Let A, B be any two k-bi-ideals and I be a k-ideal of S. Hence by Theorem 3.9, both A and B are k-ideals of S. Therefore $\overline{I\Gamma A\Gamma B} \subseteq B$ and $\overline{I\Gamma A\Gamma B} \subseteq A$. Also $\overline{I\Gamma A\Gamma B} \subseteq I$. Therefore $\overline{I\Gamma A\Gamma B} \subseteq A \cap B \cap I$. Take any $a \in A \cap B \cap I$. Hence $a \in \overline{a\Gamma S\Gamma a}$. Therefore $\overline{a\Gamma S\Gamma a} \subseteq \overline{a\Gamma S\Gamma \overline{(a\Gamma S\Gamma a)}} \subseteq \overline{I\Gamma S\Gamma (A\Gamma S\Gamma B)} \subseteq \overline{I\Gamma A\Gamma B}$. Hence $A \cap B \cap I \subseteq \overline{I\Gamma A\Gamma B}$. Thus we get $A \cap B \cap I = \overline{I\Gamma A\Gamma B}$.

(3) ⇒ (5) , (5) ⇒ (9), (3) ⇒ (7) , (7) ⇒ (9)

As every k-quasi-ideal of S is a k-bi-ideal of S, implications hold.

(9) ⇒ (1)

Let R be a right k-ideal and L be a left k-ideal of S. Then both R and L are k-quasi-ideals of S. Therefore by (9), $L \cap R = \overline{S\Gamma L\Gamma R}$. Hence S is a k-regular and k-duo Γ-semiring (see Theorem 4.6). ■

Proof of following theorem is straightforward so omitted

Theorem 4.10:- *In S following statements are equivalent.*

(1) *S is k-regular and k -duo.*

(2) *For every k-bi-ideals A and B of S, $A \cap B = \overline{A\Gamma B}$.*

(3) *For every k-bi-ideal B and a k-quasi-ideal Q of S, $B \cap Q = \overline{B\Gamma Q}$.*

(4) *For every k-bi-ideal B and a right k-ideal R of S, $B \cap R = \overline{B\Gamma R}$.*

(5) *For every k-quasi-ideal Q and a k-bi-ideal B of S, $Q \cap B = \overline{Q\Gamma \mathrm{B}}$.*

(6) *For every k-quasi-ideals Q_1 and Q_2 of S, $Q_1 \cap Q_2 = Q_1\Gamma Q_2$.*

(7) *For every k-quasi-ideal Q and a right k- ideal R of S, $Q \cap R = \overline{Q\Gamma R}$.*

(8) *For every left k- ideal L and a k-bi-ideal B of S, $L \cap B = \overline{L\Gamma B}$.*

(9) *For every left k- ideal L and a right k-ideal R of S, $L \cap R = \overline{L\Gamma R}$.*

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