Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 9 (2017), pp. 5235–5244

http://www.ripublication.com/gjpam.htm

Subclasses of spirallike functions defined by certain fractional integral operator

Jae Ho Choi
*Department of Mathematics Education,
Daegu National University of Education,
219 Jungangdaero, Namgu, Daegu 42411, Korea.*

Abstract

In this paper, we introduce two subclasses of analytic functions which are defined by means of general fractional integral operator and investigate some convolution properties and coefficient estimates for these classes. Furthermore, several inclusion properties and relevant connections of the results presented here with those obtained in earlier works are also considered.

AMS subject classification: 30C45, 30C50.
Keywords: Analytic functions, Hadamard product, Subordination, Gaussian hypergeometric function, Fractional integral operator.

1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also let f and g be analytic in $\mathbb{U}$ with $f(0) = g(0)$. Then we say that f is *subordinate* to g in $\mathbb{U}$, written $f \prec g$ or $f(z) \prec g(z)$, if there exists the Schwarz function w, analytic in $\mathbb{U}$ such that $w(0) = 0$, $|w(z)| < 1$ and $f(z) = g(w(z))$ $(z \in \mathbb{U})$. We also observe that

$$f(z) \prec g(z) \quad \text{in } \mathbb{U}$$

if and only if

$$f(0) = g(0) \qquad \text{and} \qquad f(\mathbb{U}) \subset g(\mathbb{U})$$

whenever g is univalent in $\mathbb{U}$.

For functions $f_j(z) \in \mathcal{A}$, given by

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k \qquad (j = 1, 2),$$

we define the *Hadamard product (or convolution)* of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = z + \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 * f_1)(z) \qquad (z \in \mathbb{U}).$$

Making use of the principle of subordination between analytic functions, Bhoosnurmath and Devadas [1] considered the subclasses $\mathcal{S}^{\alpha}[A, B]$ and $\mathcal{K}^{\alpha}[A, B]$ of the class $\mathcal{A}$ for $|\alpha| < \frac{\pi}{2}$ and $-1 \leq B < A \leq 1$ as following (see also [2] and [3]):

$$\mathcal{S}^{\alpha}[A, B] = \left\{ f \in \mathcal{A} : e^{i\alpha} \frac{zf'(z)}{f(z)} \prec \cos\alpha \left(\frac{1 + Az}{1 + Bz} \right) + i \sin\alpha \;\; (z \in \mathbb{U}) \right\}, \tag{1.2}$$

and

$$\mathcal{K}^{\alpha}[A, B] = \left\{ f \in \mathcal{A} : e^{i\alpha} \frac{\left(zf'(z)\right)'}{f(z)} \prec \cos\alpha \left(\frac{1 + Az}{1 + Bz} \right) + i \sin\alpha \;\; (z \in \mathbb{U}) \right\}. \tag{1.3}$$

We note that

$$\mathcal{S}^{0}[A, B] = \mathcal{S}[A, B], \quad \mathcal{K}^{0}[A, B] = \mathcal{K}[A, B] \quad (-1 \leq B < A \leq 1),$$

where the classes $\mathcal{S}[A, B]$ and $\mathcal{S}[A, B]$ are introduced and studied by many authors (see [4], [5], [6] and [7]).

Let a, b and c be complex numbers with $c \neq 0, -1, -2, \cdots$. Then the *Gaussian/classical hypergeometric function* ${}_2F_1(a, b; c; z)$ is defined by

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \tag{1.4}$$

where $(\eta)_k$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\eta)_k = \frac{\Gamma(\eta + k)}{\Gamma(\eta)} = \begin{cases} 1 & (k = 0) \\ \eta(\eta + 1) \cdots (\eta + k - 1) & (k \in \mathbb{N}). \end{cases}$$

The hypergeometric function ${}_2F_1(a, b; c; z)$ is analytic in $\mathbb{U}$ and if a or b is a negative integer, then it reduces to a polynomial.

Many essentially equivalent definitions of fractional calculus have been given in the literature (*cf., e.g.* [8], [9]). We state here the following definition due to Saigo [10] (see also [11] and [12]).

Definition 1.1. For $\lambda > 0$, μ, $\nu \in \mathbb{R}$, the fractional integral operator $\mathcal{I}_{0,z}^{\lambda,\mu,\nu}$ is defined by

$$\mathcal{I}_{0,z}^{\lambda,\mu,\nu} f(z) = \frac{z^{-\lambda-\mu}}{\Gamma(\lambda)} \int_0^z (z-\zeta)^{\lambda-1} {}_2F_1\left(\lambda+\mu, -\nu; \lambda; 1-\frac{\zeta}{z}\right) f(\zeta)\, d\zeta, \tag{1.5}$$

where ${}_2F_1$ is the Gaussian hypergeometric function defined by (1.4) and $f(z)$ is taken to be an analytic function in a simply-connected region of the z-plane containing the origin with the order

$$f(z) = \mathcal{O}(|z|^{\epsilon}) \quad (z \to 0)$$

for $\epsilon > \max\{0, \mu - \nu\} - 1$, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring that $\log(z-\zeta)$ to be real when $z - \zeta > 0$.

The definition (1.5) is an interesting extension of both the Riemann-Liouville and Erdélyi-Kober fractional operators in terms of Gauss' hypergeometric functions.

With the aid of the above definition, Owa *et al.* [12] defined a modification of the fractional integral operator $\mathcal{J}_{0,z}^{\lambda,\mu,\nu}$ by

$$\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z) = \frac{\Gamma(2-\mu)\Gamma(2+\lambda+\nu)}{\Gamma(2-\mu+\nu)} z^{\mu} \mathcal{I}_{0,z}^{\lambda,\mu,\nu} f(z) \tag{1.6}$$

for $f(z) \in \mathcal{A}$ and $\mu - \nu < 2$. Then it is observed that $\mathcal{J}_{0,z}^{\lambda,\mu,\nu}$ also maps $\mathcal{A}$ onto itself as follows:

$$\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z) = z + \sum_{k=2}^{\infty} C_k(\lambda, \mu, \nu) a_k z^k, \tag{1.7}$$

where

$$C_k(\lambda, \mu, \nu) = \frac{(2-\mu+\nu)_{k-1}(2)_{k-1}}{(2-\mu)_{k-1}(\lambda+\nu+2)_{k-1}} \tag{1.8}$$
$$(f \in \mathcal{A}; \lambda > 0; \mu - \nu < 2).$$

We note that

$$\mathcal{J}_{0,z}^{0,0,\nu} f(z) = f(z)$$
$$\mathcal{J}_{0,z}^{\gamma-\delta+1,0,\delta-2} f(z) = \mathcal{I}_{\gamma,\delta} f(z) \quad (f \in \mathcal{A}; \gamma + 1 > \delta > 0)$$
$$\mathcal{J}_{0,z}^{\gamma,0,\delta-1} f(z) = \mathcal{Q}_{\delta}^{\gamma}(f)(z) \quad (f \in \mathcal{A}; \gamma \geq 0; \delta > -1),$$

where $\mathcal{I}_{\gamma,\delta}$ and $\mathcal{Q}_{\delta}^{\gamma}(f)$ are the integral operators introduced by Choi *et al.* [13] and Liu [14].

It is easily verified from (1.7) that

$$z\left(\mathcal{J}_{0,z}^{\lambda+1,\mu,\nu}f(z)\right)' = (\lambda+\nu+2)\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f(z)-(\lambda+\nu+1)\mathcal{J}_{0,z}^{\lambda+1,\mu,\nu}f(z) \tag{1.9}$$
$$(f\in\mathcal{A};\ \lambda>0;\ \mu-\nu<2).$$

Next, by using the integral operator $\mathcal{J}_{0,z}^{\lambda,\mu,\nu}$, we introduce the following new classes of analytic functions for $\lambda>0$, $\mu-\nu<2$, $|\alpha|<\dfrac{\pi}{2}$ and $-1\leq B<A\leq 1$:

$$\mathcal{S}^{\alpha}_{\lambda,\mu,\nu}[A,B]:=\left\{f\in\mathcal{A}:\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f(z)\in\mathcal{S}^{\alpha}[A,B]\ \ (z\in\mathbb{U})\right\} \tag{1.10}$$

and

$$\mathcal{K}^{\alpha}_{\lambda,\mu,\nu}[A,B]:=\left\{f\in\mathcal{A}:\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f(z)\in\mathcal{K}^{\alpha}[A,B]\ \ (z\in\mathbb{U})\right\}. \tag{1.11}$$

It follows from the definitions (1.10) and (1.11) that

$$f(z)\in\mathcal{K}^{\alpha}_{\lambda,\mu,\nu}[A,B]\Leftrightarrow zf'(z)\in\mathcal{S}^{\alpha}_{\lambda,\mu,\nu}[A,B] \qquad (z\in\mathbb{U}). \tag{1.12}$$

In particular, we set

$$\mathcal{S}^{0}_{\gamma-\delta+1,0,\delta-2}[A,B]=\mathcal{S}^{*}_{\gamma,\delta}[A,B]$$

and

$$\mathcal{K}^{0}_{\gamma-\delta+1,0,\delta-2}[A,B]=\mathcal{K}_{\gamma,\delta}[A,B]$$

for $-1\leq B<A\leq 1$ and $\gamma+1>\delta>0$. Furthermore, the subclasses $\mathcal{S}^{*}_{\gamma,\delta}[A,B]$ and $\mathcal{K}_{\gamma,\delta}[A,B]$ of $\mathcal{A}$ was considered by Choi *et al.* [13].

In this paper, we investigate some convolution properties and coefficient estimates for the classes $\mathcal{S}^{\alpha}_{\lambda,\mu,\nu}[A,B]$ and $\mathcal{K}^{\alpha}_{\lambda,\mu,\nu}[A,B]$. Furthermore, several inclusion properties and relevant connections of the results presented here with those obtained in earlier works are also discussed.

2. Main Results

Unless otherwise mentioned, we will assume in the reminder of this paper that $-1\leq B<A\leq 1$, $|\alpha|<\dfrac{\pi}{2}$, $|\zeta|=1$ and $C_k(\lambda,\mu,\nu)$ is given by (1.8). In order to establish our convolution properties, we shall need the following lemmas due to Bhoosnurnath and Devadas [1, 2].

Lemma 2.1. **([1])** The function $f(z)$ defined by (1.1) is in the class $\mathcal{S}^{\alpha}[A,B]$ if and only if

$$\frac{1}{z}\left\{f(z)*(1-Mz)\frac{z}{(1-z)^2}\right\}\neq 0 \quad (z\in\mathbb{U}), \tag{2.1}$$

where

$$M = \frac{e^{i\alpha} + (A\cos\alpha + iB\sin\alpha)\zeta}{(A-B)\zeta\cos\alpha}. \tag{2.2}$$

Lemma 2.2. ([2] Lemma 2.7 with n=1) The function $f(z)$ defined by (1.1) is in the class $\mathcal{K}^{\alpha}[A, B]$ if and only if

$$\frac{1}{z}\left\{f(z) * (1 - Nz)\frac{z}{(1-z)^3}\right\} \neq 0 \quad (z \in \mathbb{U}), \tag{2.3}$$

where

$$N = \frac{2e^{i\alpha} + [(A+B)\cos\alpha + i2B\sin\alpha]\zeta}{(A-B)\zeta\cos\alpha}. \tag{2.4}$$

We begin by proving the following theorem.

Theorem 2.3. Let $\lambda > 0$ and $\mu - \nu < 2$. The function $f(z)$ defined by (1.1) is in the class $\mathcal{S}^{\alpha}_{\lambda,\mu,\nu}[A, B]$ if and only if

$$1 - \sum_{k=2}^{\infty} \frac{(k - 1 + kB\zeta)e^{i\alpha} - (A\cos\alpha + iB\sin\alpha)\zeta}{(A-B)\zeta\cos\alpha} C_k(\lambda, \mu, \nu)a_k z^{k-1} \neq 0 \ \ (z \in \mathbb{U}).$$

Proof. From Lemma 2.1, we find that $f(z) \in \mathcal{S}^{\alpha}_{\lambda,\mu,\nu}[A, B]$ if and only if

$$\frac{1}{z}\left[\mathcal{J}^{\lambda,\mu,\nu}_{0,z} f(z) * (1 - Mz)\frac{z}{(1-z)^2}\right] \neq 0 \quad (z \in \mathbb{U}), \tag{2.5}$$

where M is given by (2.2). Then, by applying (1.7), the left hand side of (2.5) becomes

$$\begin{aligned}
& \frac{1}{z}\left[\mathcal{J}^{\lambda,\mu,\nu}_{0,z} f(z) * \left(\frac{z}{(1-z)^2} - \frac{Mz}{(1-z)^2}\right)\right] \\
= \ & \frac{1}{z}\left[z\left(\mathcal{J}^{\lambda,\mu,\nu}_{0,z} f(z)\right)' - M\left\{z\left(\mathcal{J}^{\lambda,\mu,\nu}_{0,z} f(z)\right)' - \mathcal{J}^{\lambda,\mu,\nu}_{0,z} f(z)\right\}\right] \\
= \ & 1 - \sum_{k=2}^{\infty} \frac{(k - 1 + kB\zeta)e^{i\alpha} - (A\cos\alpha + iB\sin\alpha)\zeta}{(A-B)\zeta\cos\alpha} C_k(\lambda, \mu, \nu)a_k z^{k-1},
\end{aligned}$$

which completes the proof of Theorem 2.3. ■

Theorem 2.4. Let $\lambda > 0$ and $\mu - \nu < 2$. The function $f(z)$ defined by (1.1) is in the class $\mathcal{K}^{\alpha}_{\lambda,\mu,\nu}[A, B]$ if and only if

$$1 - \sum_{k=2}^{\infty} k\frac{(k-1)e^{i\alpha} - [(A - kB)\cos\alpha - i(k-1)B\sin\alpha]\zeta}{(A-B)\zeta\cos\alpha} C_k(\lambda, \mu, \nu)a_k z^{k-1} \neq 0 \ (z \in \mathbb{U}).$$

Proof. From Lemma 2.2, we observe that $f(z) \in \mathcal{K}^{\alpha}_{\lambda,\mu,\nu}[A,B]$ if and only if

$$\frac{1}{z}\left[\mathcal{J}^{\lambda,\mu,\nu}_{0,z} f(z) * (1-Nz)\frac{z}{(1-z)^3}\right] \neq 0 \quad (z \in \mathbb{U}), \tag{2.6}$$

where N is given by (2.4). Then, by using (1.7), the left hand side of (2.6) may be written as

$$\begin{aligned}
&\frac{1}{z}\left[\mathcal{J}^{\lambda,\mu,\nu}_{0,z} f(z) * \left(\frac{z}{(1-z)^3} - \frac{Nz^2}{(1-z)^3}\right)\right] \\
= \; &\frac{1}{z}\left[\frac{1}{2}z\left(z\mathcal{J}^{\lambda,\mu,\nu}_{0,z} f(z)\right)'' - N\left\{\frac{1}{2}z\left(z\mathcal{J}^{\lambda,\mu,\nu}_{0,z} f(z)\right)'' - z\left(\mathcal{J}^{\lambda,\mu,\nu}_{0,z} f(z)\right)'\right\}\right] \\
= \; &1 - \sum_{k=2}^{\infty} k \frac{(k-1)e^{i\alpha} - [(A-kB)\cos\alpha - i(k-1)B\sin\alpha]\zeta}{(A-B)\zeta\cos\alpha} C_k(\lambda,\mu,\nu)a_k z^{k-1},
\end{aligned}$$

which evidently proves Theorem 2.4. ∎

Next, we determine coefficients estimates for a function of the form (1.1) to be in the classes $\mathcal{S}^{\alpha}_{\lambda,\mu,\nu}[A,B]$ and $\mathcal{K}^{\alpha}_{\lambda,\mu,\nu}[A,B]$.

Theorem 2.5. Let $\lambda > 0$ and $\max\{\mu, \mu-\nu, -\lambda-\nu\} < 2$. The function $f(z)$ defined by (1.1) is in the class $\mathcal{S}^{\alpha}_{\lambda,\mu,\nu}[A,B]$ if its coefficients satisfy the condition

$$\sum_{k=2}^{\infty}\left(k-1+|A\cos\alpha + iB\sin\alpha - kBe^{i\alpha}|\right)C_k(\lambda,\mu,\nu)|a_k| \leq (A-B)\cos\alpha.$$

Proof. Since

$$\begin{aligned}
&\left|1 - \sum_{k=2}^{\infty}\frac{(k-1+kB\zeta)e^{i\alpha} - (A\cos\alpha + iB\sin\alpha)\zeta}{(A-B)\zeta\cos\alpha}C_k(\lambda,\mu,\nu)a_k z^{k-1}\right| \\
&\geq 1 - \sum_{k=2}^{\infty}\left|\frac{(k-1+kB\zeta)e^{i\alpha} - (A\cos\alpha + iB\sin\alpha)\zeta}{(A-B)\zeta\cos\alpha}\right|C_k(\lambda,\mu,\nu)|a_k|,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{(k-1+kB\zeta)e^{i\alpha} - (A\cos\alpha + iB\sin\alpha)\zeta}{(A-B)\zeta\cos\alpha}\right| \\
&= \frac{\left|(k-1)e^{i\alpha} - (A\cos\alpha + iB\sin\alpha - kBe^{i\alpha})\zeta\right|}{(A-B)\cos\alpha} \\
&\leq \frac{(k-1) + \left|A\cos\alpha + iB\sin\alpha - kBe^{i\alpha}\right|}{(A-B)\cos\alpha},
\end{aligned}$$

by virtue of Theorem 2.3, we conclude that $f(z) \in \mathcal{S}^{\alpha}_{\lambda,\mu,\nu}[A, B]$. Thus, the proof proof of Theorem 2.5 is completed. ■

By using arguments similar to those above with Theorem 2.5, we can prove the following theorem.

Theorem 2.6. Let $\lambda > 0$ and $\max\{\mu, \mu - \nu, -\lambda - \nu\} < 2$. The function $f(z)$ defined by (1.1) is in the class $\mathcal{K}^{\alpha}_{\lambda,\mu,\nu}[A, B]$ if its coefficients satisfy the condition

$$\sum_{k=2}^{\infty} k\,\{k - 1 + |(A - kB)\cos\alpha - i(k-1)B\sin\alpha|\}\, C_k(\lambda, \mu, \nu)|a_k| \leq (A - B)\cos\alpha.$$

Next, we investigate two inclusion properties for a function of the form (1.1) to be in the classes $\mathcal{S}^{\alpha}_{\lambda,\mu,\nu}[A, B]$ and $\mathcal{K}^{\alpha}_{\lambda,\mu,\nu}[A, B]$. To prove these results we shall require the following lemma due to Eenigenburg *et al.* [15].

Lemma 2.7. ([15]) Let $h(z)$ be convex univalent in $\mathbb{U}$ with $h(0) = 1$ and $\mathrm{Re}\{\beta h(z) + \nu\} > 0 \quad (\beta, \nu \in \mathbb{C})$. If $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$, then

$$p(z) + \frac{zp'(z)}{\beta p(z) + \nu} \prec h(z) \qquad (z \in \mathbb{U})$$

implies that $p(z) \prec h(z)$ $(z \in \mathbb{U})$.

By using Lemma 2.7, we prove

Theorem 2.8. Let $\lambda > 0$, $\lambda + \nu > 2$ and $\mu - \nu < 2$. If

$$\mathrm{Re}\left\{e^{-i\alpha}\frac{z}{1 + Bz}\right\} > -\frac{\lambda + \nu + 2}{(A - B)\cos\alpha} \qquad (z \in \mathbb{U}), \tag{2.7}$$

then

$$\mathcal{S}^{\alpha}_{\lambda,\mu,\nu}[A, B] \subset \mathcal{S}^{\alpha}_{\lambda+1,\mu,\nu}[A, B].$$

Proof. Let $f(z) \in \mathcal{S}^{\alpha}_{\lambda,\mu,\nu}[A, B]$ for $\lambda > 0$, $\lambda + \nu > 2$ and $\mu - \nu < 2$, and set

$$p(z) = e^{i\alpha}\frac{z(J^{\lambda+1,\mu,\nu}_{0,z} f(z))'}{J^{\lambda+1,\mu,\nu}_{0,z} f(z)} \qquad (z \in \mathbb{U}), \tag{2.8}$$

where $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = e^{i\alpha}$. By applying the identity (1.9), we obtain

$$e^{-i\alpha}p(z) - (\lambda + \nu + 1) = (\lambda + \nu + 2)\frac{J^{\lambda,\mu,\nu}_{0,z} f(z)}{J^{\lambda+1,\mu,\nu}_{0,z} f(z)}. \tag{2.9}$$

Making use of the logarithmic differentiation on both side in (2.9), we have

$$p(z) + \frac{zp'(z)}{e^{-i\alpha}p(z) - \mu + \nu + 1} \prec \cos\alpha\left(\frac{1 + Az}{1 + Bz}\right) + i\sin\alpha = h(z). \tag{2.10}$$

Since the function $h(z)$ is convex univalent in $\mathbb{U}$ with $h(0) = e^{i\alpha}$, from (2.7) we see that

$$\mathrm{Re}\{e^{-i\alpha}h(z) + \lambda + \nu + 1\} > 0 \qquad (z \in \mathbb{U}).$$

Thus, by using Lemma 2.7 and (2.10), we observe that $p(z) \prec h(z)$ in $\mathbb{U}$, so that $f(z) \in \mathcal{S}^{\alpha}_{\lambda,\mu,\nu}[A, B]$. This completes the proof of theorem 2.8. ■

Remark 2.9. Taking $\lambda = \nu + 1 = n$ $(n \in \mathbb{N})$ and $\mu = 0$ in Theorem 2.3, 2.5 and 2.8, respectively, we have the ruselts obtained by Bhoosnurmath and Devadas [3, Theorem 2.3, 2.5 and 2.6, respectively].

Theorem 2.10. Let $\lambda > 0$ and $\max\{-\lambda - \mu, \mu - \nu\} < 2$. Suppose that (2.7) holds for all $z \in \mathbb{U}$. Then

$$\mathcal{K}^{\alpha}_{\lambda,\mu,\nu}[A, B] \subset \mathcal{K}^{\alpha}_{\lambda+1,\mu,\nu}[A, B].$$

Proof. Applying (1.12) and Theorem 2.8, we observe that

$$\begin{aligned} f(z) \in \mathcal{K}^{\alpha}_{\lambda,\mu,\nu}[A, B] \ &\Leftrightarrow \ zf'(z) \in \mathcal{S}^{\alpha}_{\lambda,\mu,\nu}[A, B] \\ &\Rightarrow \ zf'(z) \in \mathcal{S}^{\alpha}_{\lambda+1,\mu,\nu}[A, B] \\ &\Leftrightarrow \ f(z) \in \mathcal{K}^{\alpha}_{\lambda+1,\mu,\nu}[A, B], \end{aligned}$$

which evidently proves Theorem 2.10. ■

Putting $\lambda = \gamma - \delta + 1$, $\mu = 0$ and $\nu = \delta - 2$ in Theorem 2.8 and 2.10, we have the following corollary.

Corollary 2.11. Suppose that $\gamma + 1 > \delta > 0$ and

$$\mathrm{Re}\left\{e^{-i\alpha}\frac{z}{1 + Bz}\right\} > -\frac{\gamma + 1}{(A - B)\cos\alpha} \qquad (z \in \mathbb{U}).$$

Then

$$\mathcal{S}^{*}_{\gamma,\delta}[A, B] \subset \mathcal{S}^{*}_{\gamma+1,\delta}[A, B]$$

and

$$\mathcal{K}_{\gamma,\delta}[A, B] \subset \mathcal{K}_{\gamma+1,\delta}[A, B].$$

Finally, we consider the generalized Bernardi-Libera-Livingston integral operator $\mathcal{L}_{\sigma}(f)$ defined by (*cf.* [16], [17], and [18])

$$\mathcal{L}_{\sigma}(f) \equiv \mathcal{L}_{\sigma}(f)(z) := \frac{\sigma + 1}{z^{\sigma}} \int_0^z t^{\sigma - 1} f(t)dt \qquad (f \in \mathcal{A}; \sigma > -1). \tag{2.11}$$

Theorem 2.12. Let $\lambda > 0$, $\mu - \nu < 2$ and $\sigma > -1$. Suppose that

$$\mathrm{Re}\left\{e^{-i\alpha}\frac{z}{1 + Bz}\right\} > -\frac{\sigma + 1}{(A - B)\cos\alpha} \qquad (z \in \mathbb{U}). \tag{2.12}$$

If $f(z) \in \mathcal{S}^{\alpha}_{\lambda,\mu,\nu}[A, B]$, then $\mathcal{L}_{\sigma}(f)(z) \in \mathcal{S}^{\alpha}_{\lambda,\mu,\nu}[A, B]$.

Proof. If we set

$$p(z) = e^{i\alpha} \frac{z(J^{\lambda,\mu,\nu}_{0,z} \mathcal{L}_{\sigma}(f)(z))'}{J^{\lambda,\mu,\nu}_{0,z} \mathcal{L}_{\sigma}(f)(z)} \qquad (z \in \mathbb{U}), \tag{2.13}$$

where $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = e^{i\alpha}$. By virtue of (2.11), we observe that

$$z\left(J^{\lambda,\mu,\nu}_{0,z} \mathcal{L}_{\sigma}(f)(z)\right)' = (\sigma + 1) J^{\lambda,\mu,\nu}_{0,z} f(z) - \sigma J^{\lambda,\mu,\nu}_{0,z} \mathcal{L}_{\sigma}(f)(z) \quad (z \in \mathbb{U}). \tag{2.14}$$

In view of (2.13) and (2.14), we have

$$e^{-i\alpha} p(z) + \sigma = (\sigma + 1) \frac{J^{\lambda,\mu,\nu}_{0,z} f(z)}{J^{\lambda,\mu,\nu}_{0,z} \mathcal{L}_{\sigma}(f)(z)}.$$

By using same argument as in the proof of Theorem 2.8 with (2.12), we conclude that $\mathcal{L}_{\sigma}(f)(z) \in \mathcal{S}^{\alpha}_{\lambda,\mu,\nu}[A, B]$. This evidently completes the proof of Theorem 2.12. ■

Theorem 2.13. Let $\lambda > 0$, $\mu - \nu < 2$ and $\sigma > -1$. Suppose that (2.12) holds for all $z \in \mathbb{U}$. If $f(z) \in \mathcal{K}^{\alpha}_{\lambda,\mu,\nu}[A, B]$, then $\mathcal{L}_{\sigma}(f)(z) \in \mathcal{K}^{\alpha}_{\lambda,\mu,\nu}[A, B]$.

Proof. By using Theorem 2.12, it follows that

$$\begin{aligned} f(z) \in \mathcal{K}^{\alpha}_{\lambda,\mu,\nu}[A, B] \;&\Leftrightarrow\; zf'(z) \in \mathcal{S}^{\alpha}_{\lambda,\mu,\nu}[A, B] \\ &\Rightarrow\; \mathcal{L}_{\sigma}\left(zf'(z)\right) \in \mathcal{S}^{\alpha}_{\lambda,\mu,\nu}[A, B] \\ &\Leftrightarrow\; z(\mathcal{L}_{\sigma}(f)(z))' \in \mathcal{S}^{\alpha}_{\lambda,\mu,\nu}[A, B] \\ &\Rightarrow\; \mathcal{L}_{\sigma}(f)(z) \in \mathcal{K}^{\alpha}_{\lambda,\mu,\nu}[A, B], \end{aligned}$$

which completes the proof of Theorem 2.13. ■

Acknowledgements

This work was supported by Daegu National University of Education Research grant in 2016.

References

[1] S.S. Bhoosnurmath and M. V. Devadas, Subclasses of spirallike functions defined by subordination, *J. of Analysis Madras* **4** (1996), 173–183.

[2] S.S. Bhoosnurmath and M. V. Devadas, Subclasses of spirallike functions defined by Ruschweyh derivatives, *Tamkang J. Math.* **28** (1997), 59–65.

[3] T. M. Seoudy, Classes of spirallike functions defined by the Dziok-Srivastava operator, *Le Mathematiche* **69** (2014), 93–102.

[4] O. P. Ahuja, Families of analytic functions related to Ruscheweyh derivatives and subordinate to convex functions, *Yokohama Math. J.* **41** (1993), 39–50.

[5] R. M. Goel and B. S. Mehrok, On the coefficients of a subclass of starlike functions, *Indian J. Pure Appl. Math.* **12** (1981), 634–647.

[6] W. Janowski, Some extremal problems for certain families of analytic functions, *Annl. Polon. Math.* **28** (1973), 297–326.

[7] H. Silverman and E. M. Silvia, Subclasses of starlike functions subordinate to convex functions, *Canad. Math. J.* **1** (1985), 48–61.

[8] H.M. Srivastava and R.G. Buschman, *Theory and Applications of Convolution Integral Equations*, Kluwer Academic Publishers, dordrecht, Boston and London, 1992.

[9] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integral and Derivatives, Theory and Applications*, Gordon and Breach, New York, Philadelphia, London, Paris, Montreux, Toronto and Melbourne, 1993.

[10] M. Saigo, A remark on integral operators involving the Gauss hypergeometric functions, *Math. Rep. Kyushu Univ.* **11**(2)1977/78, 135–143.

[11] J. H. Choi, Note on differential subordination associated with fractional integral operator, *Far East J. Math. Sci.* **26** (2007), 499–511.

[12] S. Owa, M. Saigo and H.M. Srivastava, Some characterization theorems for starlike and convex functions involving a certain fractional integral operator, *J. Math. Anal. Appl.* **140** (1989), 419–426.

[13] J. H. Choi, M. Saigo and H. M. Srivastava, Some inclusion properties of a certain family of integral operators, *J. Math. Anal. Appl.* **276** (2002), 432–445.

[14] J.-L. Liu, Notes on Jung-Kim-Srivastava integral operator, *J. Math. Anal. Appl.* **294** (2004), 96–103.

[15] P. Eenigenburg, S. S. Miller, P. T. Mocanu and M. O. Reade, On a Briot-Bouquet differential subordination, in *General Inequalities* 3, International Series of Numerical Mathematics, Vol. **64**, Birkhäuser Verlag, Basel, 1983, pp. 339–348.

[16] S. D. Bernardi, Convex and starlike univalent functions, *Trans. Amer. Math. Soc.* **135** (1969), 429–446.

[17] R. J. Libera, Some classes of regular univalent functions, *Proc. Amer. Math. Soc.* **16** (1965), 755–758.

[18] H. M. Srivastava and S. Owa (Editors), *Current Topics in Analytic Function Theory*, World Scientific Publishing Company, Singapore, New Jersey, London, and Hong Kong, 1992.