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Generalization of Uniqueness of Meromorphic Functions Concerning Differential Polynomials

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Abstract

In this paper, we study the uniqueness of meromorphic functions concerning differential polynomials sharing a small function $a(z)$. The results generalize and improve the corresponding results obtained by Jianming Qi and Lei Qiao [7].

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1. INTRODUCTION AND MAIN RESULTS

Let C denote the complex plane. A meromorphic function will mean meromorphic in C. We shall use the standard notations and terms in the Nevanlinna value distribution theory such as $T(r,f), N(r,f), \overline{N}(r,f), m(r,f)$ (see [4, 5, 11, 14]). The notation $S(r,f)$ is defined to be any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$ outside of a possible exceptional set of finite linear measure. A meromorphic function $a(z)$ is called a small function with respect to $f(z)$, provided that $T(r,a) = S(r,f)$.

We denote by $E_{k)}(a,f)$ the set of zeros of $f-a$ with multiplicities at most k, where each zero is counted according to its multiplicity. We denote by $\bar{E}_{k)}(a,f)$ the set of zeros of $f-a$ with multiplicities at most k, where each zero is counted only once. In addition, we denote by $N_{k)}\left(r,\frac{1}{f-a}\right)\left(\bar{N}_{k)}\left(r,\frac{1}{f-a}\right)\right)$ the counting function with respect to the set $E_{k)}(a,f)$ $(\bar{E}_{k)}(a,f))$. We denote by $N_{(k}\left(r,\frac{1}{f-a}\right)$ the counting function of a-points of f (counted with proper multiplicities) whose multiplicities are not less than k, we denote by $\bar{N}_{(k}\left(r,\frac{1}{f-a}\right)$ the corresponding reduced counting function (ignoring multiplicity).

Set

$$N_k\left(r,\frac{1}{f-a}\right)=\bar{N}\left(r,\frac{1}{f-a}\right)+\bar{N}_{(2}\left(r,\frac{1}{f-a}\right)+\cdots+\bar{N}_{(k}\left(r,\frac{1}{f-a}\right).$$

We denote by $N_{11}\left(r,\frac{1}{f-1}\right)$ the counting function for common simple 1-points of both f(z) and g(z) where multiplicity is not counted. $\bar{N}_L(r,v^0{}_{f-1}>v^0{}_{g-1})$ is the counting function for $1-$ points of both f and g about which f has larger multiplicity than g, with multiplicity being not counted. Similarly, we have the notation $\bar{N}_L(r,v^0{}_{g-1}>v^0{}_{f-1})$.

Let f(z) be a nonconstant meromorphic function in the complex plane C, k be a positive integer and a(z) is a small function of f. Define $E(a(z),f)=\{z|f(z)-a(z)=0\}$, where a zero point with multiplicity m is counted m times in the set. If there zero points are only counted once, then we denote the set by $\bar{E}(a(z),f)$.

Define, $E_k(a(z),f)=\{z|f(z)-a(z)=0,\exists i,1\le i\le k, s.t.\, f^{(i)}(z)-a(z)\neq 0\}$, where each zero of $f(z)-a(z)$ with multiplicity m is counted m times where $m\le k$. We say f and g share a CM (IM), if $E(a,f)=E(a,g)$ $(\bar{E}(a,f)=\bar{E}(a,g))$. Similarly, we define that f and g share a small function $a(z)(\not\equiv 0,\infty)$ CM (IM), if $E(a(z),f)=E(a(z),g)\left(\bar{E}(a(z),f)=\bar{E}(a(z),g)\right)$. If $E_k(a(z),f)=E_k(a(z),g)$, then we say that $f-a(z)$ and $g-a(z)$ have the same zeros with multiplicities atmost k.

In 1997, Yang and Hua Studied meromorphic functions sharing one value. They proved the following result

Theorem A [13]. Let f and g be two nonconstant meromorphic functions, and $n(\ge 11)$ be a positive integer. If f^nf' and g^ng'share 1CM, then either $f=c_1e^{cz}$, $g=$

$c_2 e^{-cz}$, where c_1, c_2 and c are three constants, satisfying $(c_1 c_2)^{n+1} c^2 = -1$, or $f = tg$ for a constant t such that $t^{n+1} = 1$.

Later, Fang and Qiu investigated meromorphic functions sharing fixed points, which is an improvement of Theorem A

Theorem B [3]. Let f and g be two nonconstant meromorphic (entire) functions, and $n \geq 11 (n \geq 6)$ be a positive integer. If $f^n f'$ and $g^n g'$ share z CM, then either $f = c_1 e^{cz^2}$, $g = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants, satisfying $4(c_1 c_2)^{n+1} c^2 = -1$, or $f = tg$ for a constant t such that $t^{n+1} = 1$.

Corresponding to transcendental meromorphic functions, Wang and Gao further extended the above results as follows

Theorem C [8]. Let f and g be two transcendental meromorphic functions, and let $a(z) (\not\equiv 0)$ be a common small function with respect to them, and let $n (\geq 11)$ be a positive integer. If $f^n f'$ and $g^n g'$ share $a(z)$ CM, then either $f^n f' g^n g' \equiv a^2(z)$, or $f = tg$ for a constant t such that $t^{n+1} = 1$.

Corresponding to nonconstant meromorphic functions, in 2015, Jianming Qi and Lei Qiao [7], obtained the following results

Theorem D. Let f and g be two nonconstant meromorphic functions and let $n (\geq 11)$ and $k (\geq 3)$, be two positive integers. If $E_k(a(z), f^n f') = E_k(a(z), g^n g')$ then either $f^n f' g^n g' \equiv a^2(z)$, or $f = tg$ for a constant t such that $t^{n+1} = 1$, where $a(z)$ is a meromorphic function such that $a(z) \not\equiv 0, \infty$ and $a(z)$ is a small function of f and g.

Theorem E. Let f and g be two nonconstant meromorphic functions and $n (\geq 13)$ be a positive integer. If $E_2(a(z), f^n f') = E_2(a(z), g^n g')$ then either $f^n f' g^n g' \equiv a^2(z)$, or $f = tg$ for a constant t such that $t^{n+1} = 1$, where $a(z)$ is a meromorphic function such that $a(z) \not\equiv 0, \infty$ and $a(z)$ is a small function of f and g.

Theorem F. Let f and g be two nonconstant meromorphic functions and $n (\geq 19)$ be a positive integer. If $E_1(a(z), f^n f') = E_1(a(z), g^n g')$ then either $f^n f' g^n g' \equiv a^2(z)$, or $f = tg$ for a constant t such that $t^{n+1} = 1$, where $a(z)$ is a meromorphic function such that $a(z) \not\equiv 0, \infty$ and $a(z)$ is a small function of f and g.

Theorem G. Let f and g be two nonconstant meromorphic functions and $n (\geq 23)$ be a positive integer. If $\bar{E}(a(z), f^n f') = \bar{E}(a(z), g^n g')$ then either $f^n f' g^n g' \equiv a^2(z)$, or $f = tg$ for a constant t such that $t^{n+1} = 1$, where $a(z)$ is a meromorphic function such that $a(z) \not\equiv 0, \infty$ and $a(z)$ is a small function of f and g.

This paper is motivated by the following question

Question. What will happens in Theorems D - G if $f^n f'$ and $g^n g'$ is replaced by $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ respectively.

We will concentrate our attention to the above question and provide an affirmative answer in this direction.

We now state our main results.

Theorem 1. Let f and g be two nonconstant meromorphic functions and let $n(> 3k + m + 8)$ and $k(\geq 3)$, be two positive integers. If $E_k\left(a(z), [f^n P(f)]^{(k)}\right) = E_k\left(a(z), [g^n P(g)]^{(k)}\right)$, then

(i) when $P(w) \equiv c_0$, $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$.

(ii) when $P(w) = a_m w^m + a_{m-1} w^{m-1} + \cdots + a_1 w + a_0$, where $a_0 \neq 0, a_1, \ldots, a_{m-i}, a_m \neq 0$, then either $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = GCD(n+m, \ldots, n+m-i, \ldots, n)$, $(a_{m-i} \neq 0)$, for some $i = 0,1, \ldots, m$, or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(w_1, w_2) = {w_1}^n (a_m {w_1}^m + a_{m-1} {w_1}^{m-1} + \cdots + a_0) - {w_2}^n (a_m {w_2}^m + a_{m-1} {w_2}^{m-1} + \cdots + a_0)$.

(iii) $[f^n P(f)]^k [g^n P(g)]^k \equiv a^2(z)$.

Theorem 2. Let f and g be two nonconstant meromorphic functions and $n \left(> 4k + \frac{3m}{2} + 9\right)$ be a positive integer. If $E_2\left(a(z), [f^n P(f)]^{(k)}\right) = E_2\left(a(z), [g^n P(g)]^{(k)}\right)$, then the conclusion of theorem 1 holds.

Theorem 3. Let f and g be two nonconstant meromorphic functions and $n(> 7k + 3m + 12)$ be a positive integer. If $E_1\left(a(z), [f^n P(f)]^{(k)}\right) = E_1\left(a(z), [g^n P(g)]^{(k)}\right)$, then the conclusion of theorem 1 holds.

Theorem 4. Let f and g be two nonconstant meromorphic functions and $n(> 9k + 4m + 14)$ be a positive integer. If $\bar{E}\left(a(z), [f^n P(f)]^{(k)}\right) = \bar{E}\left(a(z), [g^n P(g)]^{(k)}\right)$, then the conclusion of theorem 1 holds.

2. Some Lemmas

In this section, we present some lemmas which will be needed in the sequel .

Lemma 1 ([14]). Let f be a non constant meromorphic function and k be a positive integer, then

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k\bar{N}(r, f) + S(r, f).$$

Lemma 2 ([10]). Let f be a non-constant meromorphic function and $P(f) = a_n f^n + a_{n-1}f^{n-1} + \cdots + a_1 f' + a_0$, where $a_0, a_1, \ldots, a_n$ are constants and $a_n \neq 0$. Then

$$T\big(r, P(f)\big) = nT(r,f) + S(r,f).$$

Lemma 3 ([9]). Let f, g be two nonconstant meromorphic functions, k, $n > 2k + 1$ be two positive integers. If $[f^n]^{(k)} = [g^n]^{(k)}$, then $f = tg$ for a constant t such that $t^n = 1$.

Lemma 4 ([9]). Let f and g be two nonconstant meromorphic functions. Let $P(w) = a_m w^m + a_{m-1}w^{m-1} + \cdots + a_1 w + a_0$ or $P(w) \equiv c_0$, where $a_0 \neq 0, a_1, \ldots, a_{m-i}, a_m \neq 0, c_0 \neq 0$ are complex constants, and $n > 0, k > 0$, and $m \geq 0$ be three integers with $n > 2k + m + 1$. If $[f^n P(f)]^{(k)} = [g^n P(g)]^{(k)}$, then $f^n P(f) = g^n P(g)$.

Lemma 5. Let f and g be two nonconstant meromorphic functions, $F = \frac{[f^n P(f)]^{(k)}}{a(z)}$ and $G = \frac{[g^n P(g)]^{(k)}}{a(z)}$, where $n \geq 2k + m + 1$ is a positive integer. If $F \equiv G$, then

(i) when $P(w) \equiv c_0$, $f \equiv tg$ for a constant t such that $t^n = 1$.

(ii) when $P(w) = a_m w^m + a_{m-1}w^{m-1} + \cdots + a_1 w + a_0$, where $a_0 \neq 0, a_1, \ldots, a_{m-i}, a_m \neq 0$, then either $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = GCD(n+m, \ldots, n+m-i, \ldots, n), (a_{m-i} \neq 0)$, for some $i = 0, 1, \ldots, m$, or f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(w_1, w_2) = {w_1}^n(a_m {w_1}^m + a_{m-1}{w_1}^{m-1} + \cdots + a_0) - {w_2}^n(a_m {w_2}^m + a_{m-1}{w_2}^{m-1} + \cdots + a_0)$.

Proof. If $F \equiv G$, then by Lemma 4, we have

$$f^n P(f) \equiv g^n P(g). \tag{2.1}$$

(i) when $P(w) \equiv c_0$, then from (2.1), we get $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$.

(ii) when $P(w) = a_m w^m + a_{m-1}w^{m-1} + \cdots + a_1 w + a_0$, then from (2.1), we get

$$f^n(a_m f^m + a_{m-1}f^{m-1} + \cdots + a_1 f + a_0) \equiv g^n(a_m g^m + a_{m-1}g^{m-1} + \cdots + a_1 g + a_0). \tag{2.2}$$

Let $h = \frac{f}{g}$, if h is a constant, then substituting $f = gh$ into (2.2), we deduce

$$a_m g^{n+m}(h^{n+m}-1)+a_{m-1}g^{n+m-1}(h^{n+m-1}-1)+\cdots+a_0 g(h^n-1)\equiv 0, \tag{2.3}$$

which implies $h^d=1$, where $d=GCD(n+m,n+m-i,\dots,n), a_{m-i}\neq 0$ for some $i=0,1,\dots,m$. Thus f(z) ≡ tg(z) for a constant t such that $t^d=1$. If h is not a constant, then we know by (2.2) that f and g satisfy the algebraic equation $R(f,g)\equiv 0$, where $R(w_1,w_2)={w_1}^n(a_m{w_1}^m+a_{m-1}{w_1}^{m-1}+\cdots+a_0)-{w_2}^n(a_m{w_2}^m+a_{m-1}{w_2}^{m-1}+\cdots+a_0)$. So Lemma 5 is proved.

Lemma 6 [2]. Let f and g be two meromorphic functions, and let k be be a positive integer. If $E_k(1,f)=E_k(1,g)$, then one of the following occurs:

(i) $$T(r,f)+T(r,g)\le \overline{N}_2(r,f)+\overline{N}_2\left(r,\frac{1}{f}\right)+\overline{N}_2(r,g)+\overline{N}_2\left(r,\frac{1}{g}\right)$$
$$+\overline{N}\left(r,\frac{1}{f-1}\right)+\overline{N}\left(r,\frac{1}{g-1}\right)-N_{11}\left(r,\frac{1}{f-1}\right)+\overline{N}_{(k+1}\left(r,\frac{1}{f-1}\right)$$
$$+\overline{N}_{(k+1}\left(r,\frac{1}{g-1}\right)\quad+S(r,f).$$

(ii) $f=\frac{(b+1)g+(a-b-1)}{bg+(a-b)}$, where $a\neq 0$, b are two constants.

Lemma 7 [1]. Let f and g be two meromorphic functions. If $\bar{E}(1,f)=\bar{E}(1,g)$, then one of the following occurs:

(i) $$T(r,f)+T(r,g)\le 2\left[N_2(r,f)+N_2\left(r,\frac{1}{f}\right)+N_2(r,g)+N_2\left(r,\frac{1}{g}\right)\right]$$
$$+3\overline{N}_L\left(r,{v^0}_{f-1}>{v^0}_{g-1}\right)+3\overline{N}_L\left(r,{v^0}_{g-1}>{v^0}_{f-1}\right)$$
$$+S(r,f)+S(r,g).$$

(ii) $f=\frac{(b+1)g+(a-b-1)}{bg+(a-b)}$, where $a\neq 0$, b are two constants.

Lemma 8. Let f and g be two meromorphic functions, $n(>3k+m+3)$ be a positive integer and let F and G be defined as in Lemma 5. If

$$F=\frac{(b+1)G+(a-b-1)}{bG+(a-b)}, \tag{2.4}$$

where $a(\neq 0)$, b are two constants, then

(i) when $P(w)\equiv c_0$, f ≡ tg for a constant t such that $t^{n+1}=1$.

(ii) when $P(w)=a_m w^m+a_{m-1}w^{m-1}+\cdots+a_1 w+a_0$, where $a_0\neq 0, a_1,\dots,a_{m-i},a_m\neq 0$, then either f ≡ tg for a constant t such that $t^d=1$, where

$d = GCD(n+m, \dots, n+m-i, \dots, n), (a_{m-i} \neq 0)$, for some $i = 0,1, \dots, m$, or f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(w_1, w_2) = {w_1}^n(a_m {w_1}^m + a_{m-1}{w_1}^{m-1} + \dots + a_0) - {w_2}^n(a_m {w_2}^m + a_{m-1}{w_2}^{m-1} + \dots + a_0)$.

(iii) $[f^n P(f)]^{(k)}[g^n P(g)]^{(k)} \equiv \mathrm{a}^2(\mathrm{z})$.

Proof. By Lemma 2, we get

$$\begin{aligned} T(r,F) &= T(r,[f^n P(f)]^k) \\ &\leq (n+m)T(r,f) + k\overline{N}(r,f) + S(r,f) \\ &\leq (n+m+k)T(r,f) + S(r,f). \end{aligned} \tag{2.5}$$

On the other hand, we have

$$\begin{aligned} (n+m)T(r,f) &\leq T\big(r, f^n P(f)\big) + S(r,f) \\ &\leq N\big(r, f^n P(f)\big) + m\big(r, f^n P(f)\big) + S(r,f) \\ &\leq N\big(r, [f^n P(f)]^{(k)}\big) - k\overline{N}(r,f) + m\left(r, \frac{f^n P(f)}{[f^n P(f)]^{(k)}}\right) \\ &\quad + m\big(r, [f^n P(f)]^{(k)}\big) + S(r,f) \end{aligned}$$

$$(n+m)T(r,f) \leq T(r,F) - k\overline{N}(r,f) + S(r,f).$$

So

$$T(r,F) \geq (n+m+k)T(r,f) + S(r,f). \tag{2.6}$$

Thus, by Eqs. (2.5) and (2.6), and $n > 3k+m+3$, we get $S(r,F) = S(r,f)$.

Similarly, we get

$$T(r,G) \geq (n+m+k)T(r,g) + S(r,g), \tag{2.7}$$

and $S(r,G) = S(r,g)$.

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,f) \leq T(r,g), r \in I$. Next, we consider three cases.

Case 1. $\mathrm{b} \neq -1, 0$. If $a-b-1 \neq 0$, then by Eq. (2.4) we have

$$\overline{N}\left(r, \frac{1}{G + \frac{a-b-1}{b+1}}\right) = \overline{N}\left(r, \frac{1}{F}\right).$$

By Nevanlinna second fundamental theorem and Lemma 1 we get

$$T(r,G) \leq \overline{N}(r,G) + \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}\left(r,\frac{1}{G+\frac{a-b-1}{b+1}}\right) + S(r,G)$$

$$= \overline{N}(r,G) + \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}\left(r,\frac{1}{F}\right) + S(r,g)$$

$$\leq \overline{N}(r,g) + \overline{N}\left(r,\frac{1}{[g^nP(g)]^{(k)}}\right) + \overline{N}\left(r,\frac{1}{[f^nP(f)]^{(k)}}\right) + S(r,g)$$

$$\leq \overline{N}(r,g) + (k+1)\overline{N}\left(r,\frac{1}{g}\right) + mN\left(r,\frac{1}{g}\right) + k\overline{N}(r,g)$$

$$+ (k+1)\overline{N}\left(r,\frac{1}{f}\right) + mN\left(r,\frac{1}{f}\right) + k\overline{N}(r,f) + S(r,g)$$

$$\leq (4k+2m+3)T(r,g) + S(r,g).$$

Hence, by $n > 3k+m+3$ and Eq. (2.7), we have $T(r,g) \leq S(r,g), r \in I$. This is impossible.

If $a-b-1=0$, then by Eq. (2.4) we have

$$\overline{N}\left(r,\frac{1}{G+\frac{1}{b}}\right) = \overline{N}(r,F).$$

By Nevanlinna second fundamental theorem and Lemma 1 we get

$$T(r,G) \leq \overline{N}(r,G) + \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}\left(r,\frac{1}{G+\frac{1}{b}}\right) + S(r,G)$$

$$= \overline{N}(r,G) + \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}(r,F) + S(r,g)$$

$$\leq \overline{N}(r,g) + \overline{N}\left(r,\frac{1}{[g^nP(g)]^{(k)}}\right) + \overline{N}(r,f) + S(r,g)$$

$$\leq \overline{N}(r,g) + (k+1)\overline{N}\left(r,\frac{1}{g}\right) + mN\left(r,\frac{1}{g}\right) + k\overline{N}(r,g) + \overline{N}(r,f) + S(r,g)$$

$$\leq (2k+m+3)T(r,g) + S(r,g).$$

Thus by $n > 3k+m+3$ and Eq. (2.7), we know $T(r,g) \leq S(r,g), r \in I$, which is a contradiction.

Case 2. Assume that $b=-1$. Then by Eq. (2.4) we have $F=\frac{a}{a+1-G}$.

If $a+1\neq 0$, then $\overline{N}\left(r,\frac{1}{G-a-1}\right)=\overline{N}(r,F)$. Proceeding as the proof of case 1, we get a contradiction.

If $a+1=0$, then $FG\equiv 1$, that is ,

$$[f^nP(f)]^{(k)}[g^nP(g)]^{(k)}\equiv \mathrm{a}^2(\mathrm{z}).$$

Case 3. Assume that $b=0$. Then by Eq. (2.4) we have $F=\frac{G+a-1}{a}$.

If $a-1\neq 0$, then $\overline{N}\left(r,\frac{1}{G+a-1}\right)=\overline{N}\left(r,\frac{1}{f}\right)$. Proceeding as the proof of case 1, we get a contradiction.

If $a-1=0$, then $F\equiv G$. By Lemma 5, we obtain the remaining (i) and (ii) conclusions of this Lemma.

3. Proof of the Theorems

F and G be defined as in Lemma 5.

Proof of Theorem 1. Since $k\geq 3$, we have

$$\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)-N_{11}\left(r,\frac{1}{F-1}\right)+\overline{N}_{(k+1}\left(r,\frac{1}{F-1}\right)+\overline{N}_{(k+1}\left(r,\frac{1}{G-1}\right)$$
$$\leq\frac{1}{2}N\left(r,\frac{1}{F-1}\right)+\frac{1}{2}N\left(r,\frac{1}{G-1}\right)$$
$$\leq\frac{1}{2}T(r,F)+\frac{1}{2}T(r,G)+S(r,f)+S(r,g). \tag{3.1}$$

Then (i) in Lemma 6 becomes

$$T(r,F)+T(r,G)$$
$$\leq 2\left\{N_2\left(r,\frac{1}{F}\right)+N_2(r,F)+N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\right\}+\mathrm{S}(\mathrm{r},\mathrm{f})+\mathrm{S}(\mathrm{r},\mathrm{g}). \tag{3.2}$$

By logarithmic derivatives and the first main theorem of Nevanlinna, we have

$$N_2(r,F)+N_2\left(r,\frac{1}{F}\right)\leq N_2\left(r,\frac{[f^nP(f)]^{(k)}}{a(z)}\right)+N_2\left(r,\frac{a(z)}{[f^nP(f)]^{(k)}}\right)$$
$$\leq (k+2)\overline{N}\left(r,\frac{1}{f}\right)+m\,N\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+2\overline{N}(r,f). \tag{3.3}$$

Similarly, we get

$$N_2(r,G) + N_2\left(r,\frac{1}{G}\right) \leq (k+2)\overline{N}\left(r,\frac{1}{g}\right) + m\,N\left(r,\frac{1}{g}\right) + k\overline{N}(r,g) + 2\overline{N}(r,g). \tag{3.4}$$

Suppose that Eq. (3.2) holds. By Lemma 1, Eqs. (3.3) and (3.4), we have

$$\begin{aligned} T(r,F) + T(r,G) &\leq 2\left[(k+2)\overline{N}\left(r,\frac{1}{f}\right) + m\,N\left(r,\frac{1}{f}\right) + k\overline{N}(r,f) + 2\overline{N}(r,f)\right] \\ &+ 2\ \left[(k+2)\overline{N}\left(r,\frac{1}{g}\right) + m\,N\left(r,\frac{1}{g}\right) + k\overline{N}(r,g) + 2\overline{N}(r,g)\right] \\ &+\ S(r,f) + S(r,g) \\ &\leq (4k+2m+8)T(r,f) + (4k+2m+8)T(r,g) \\ &+ S(r,f) + S(r,g) \end{aligned}$$

$$(n+m+k)[T(r,f) + T(r,g)] \leq\ (4k+2m+8)[T(r,f) + T(r,g)] + S(r,f) + S(r,g)$$

$$(n-3k-m-8)\ \left(T(r,f) + T(r,g)\right)\ \leq S(r,f) + S(r,g).$$

By $n > 3k+m+8$, Eqs. (2.6) and (2.7), we obtain a contradiction. Thus by Lemma 6, $F = \frac{(b+1)G+(a-b-1)}{bG+(a-b)}$, where $a(\neq 0)$, b are two constants. Then by Lemma 8, we can prove Theorem 1.

Proof of Theorem 2. Obviously, we have

$$\begin{aligned} &\overline{N}\left(r,\frac{1}{F-1}\right) + \overline{N}\left(r,\frac{1}{G-1}\right) - N_{11}\left(r,\frac{1}{F-1}\right) + \frac{1}{2}\overline{N}_{(3}\left(r,\frac{1}{F-1}\right) + \frac{1}{2}\overline{N}_{(3}\left(r,\frac{1}{G-1}\right) \\ &\leq \frac{1}{2}N\left(r,\frac{1}{F-1}\right) + \frac{1}{2}N\left(r,\frac{1}{G-1}\right) \\ &\leq \frac{1}{2}T(r,F) + \frac{1}{2}T(r,G) + S(r,f) + S(r,g). \end{aligned} \tag{3.5}$$

Then (i) in Lemma 6 becomes

$$\begin{aligned} T(r,F) + T(r,G) \leq 2\left\{N_2(r,F) + N_2\left(r,\frac{1}{F}\right) + N_2(r,G) + N_2\left(r,\frac{1}{G}\right)\right\} + \overline{N}_{(3}\left(r,\frac{1}{F-1}\right) \\ + \overline{N}_{(3}\left(r,\frac{1}{G-1}\right) + \mathrm{S(r,f)} + \mathrm{S(r,g)}. \end{aligned} \tag{3.6}$$

Consider

$$\overline{N}_{(3}\left(r,\frac{1}{F-1}\right) \le \frac{1}{2}N\left(r,\frac{F}{F'}\right)$$
$$\le \frac{1}{2}N\left(r,\frac{F'}{F}\right)+S(r,f)$$
$$\le \frac{1}{2}\overline{N}(r,F)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right)+s(r,f)$$
$$\le \frac{1}{2}\left[N_1\left(r,\frac{1}{[f^nP(f)]^{(k)}}\right)+\overline{N}(r,f)\right]+S(r,f)$$
$$\le \frac{1}{2}\left[(k+1)\overline{N}\left(r,\frac{1}{f}\right)+mN\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+\overline{N}(r,f)\right]+S(r,f)$$
$$\le \left[\frac{2k+m+2}{2}\right]T(r,f)+S(r,f). \tag{3.7}$$

Similarly, we get

$$\overline{N}_{(3}\left(r,\frac{1}{G-1}\right) \le \left[\frac{2k+m+2}{2}\right]T(r,g)+S(r,g). \tag{3.8}$$

Suppose that Eq. (3.6) holds. Combining Eqs. (3.3), (3.4), (3.7), and (3.8), we have

$$T(r,F)+T(r,G) \le 2\left[N_2\left(r,\frac{1}{[f^nP(f)]^{(k)}}\right)+N_2\left(r,[f^nP(f)]^{(k)}\right)\right]$$
$$+2\left[N_2\left(r,\frac{1}{[g^nP(g)]^{(k)}}\right)+N_2\left(r,[g^nP(g)]^{(k)}\right)\right]$$
$$+\left[\frac{2k+m+2}{2}\right]T(r,f)+\left[\frac{2k+m+2}{2}\right]T(r,g)+S(r,f)$$
$$+S(r,g)$$

$$(n+m+k)[T(r,f)+T(r,g)] \le \left(\frac{10k+5m+18}{2}\right)[T(r,f)+T(r,g)]$$
$$+S(r,f)+S(r,g)$$

$$\left(n-\frac{(8k+3m+18)}{2}\right)[T(r,f)+T(r,g)] \le \mathrm{S(r,f)}+\mathrm{S(r,g)}. \tag{3.9}$$

By $n>\frac{8k+3m+18}{2}$, Eqs. (2.6), (2.7) and (3.9), we get a contradiction. Thus, by Lemma 6, $F=\frac{(b+1)G+(a-b-1)}{bG+(a-b)}$, where $a(\neq 0)$, b are two constants. Then by Lemma 8, we can prove Theorem 2.

Proof of Theorem 3. Similarly as in proof of Theorem 2, we have

$$\overline{N}\left(r,\frac{1}{F-1}\right)+\ \overline{N}\left(r,\frac{1}{G-1}\right)-N_{11}\left(r,\frac{1}{F-1}\right)$$
$$\leq \frac{1}{2}N\left(r,\frac{1}{F-1}\right)+\frac{1}{2}N\left(r,\frac{1}{G-1}\right)$$
$$\leq \frac{1}{2}T(r,F)+\frac{1}{2}T(r,G)+S(r,f)+S(r,g). \qquad (3.10)$$

Then (i) in Lemma 6 becomes

$$T(r,F)+T(r,G)$$
$$\leq 2\left\{N_2(r,F)+N_2\left(r,\frac{1}{F}\right)+N_2(r,G)+N_2\left(r,\frac{1}{G}\right)+\overline{N}_{(2}\left(r,\frac{1}{F-1}\right)\right.$$
$$\left.+\ \overline{N}_{(2}\left(r,\frac{1}{G-1}\right)\right\}+\mathrm{S(r,f)}+\mathrm{S(r,g)}. \qquad (3.11)$$

Consider

$$\overline{N}_{(2}\left(r,\frac{1}{F-1}\right)\leq N\left(r,\frac{F}{F'}\right)$$
$$\leq N\left(r,\frac{F'}{F}\right)+S(r,f)$$
$$\leq \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+s(r,f)$$
$$\leq \overline{N}(r,f)+(k+1)\overline{N}\left(r,\frac{1}{f}\right)+mN\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)$$
$$+\ S(r,f)$$
$$\leq (2k+m+2)T(r,f)+S(r,f) \qquad (3.12)$$

Similarly, we get

$$\overline{N}_{(2}\left(r,\frac{1}{G-1}\right)\ \leq (2k+m+2)T(r,g)+S(r,g). \qquad (3.13)$$

Suppose that Eq. (3.11) holds. Combining Eqs. (3.3), (3.4), (3.12), and (3.13), we have

$$T(r,F)+T(r,G)\leq 2\left[N_2\left(r,\frac{1}{[f^nP(f)]^{(k)}}\right)+N_2\left(r,[f^nP(f)]^{(k)}\right)\right]$$
$$+2\left[N_2\left(r,\frac{1}{[g^nP(g)]^{(k)}}\right)+N_2\left(r,[g^nP(g)]^{(k)}\right)\right]$$

$$+\ [4k+2m+4]T(r,f)+[4k+2m+4]T(r,g)$$
$$+S(r,f)+S(r,g).$$
$$(n+m+k)[T(r,f)+T(r,g)]\leq\ (8k+4m+12)[T(r,f)+T(r,g)]$$
$$+\ S(r,f)+S(r,g)$$
$$(n-7k-3m-12)[T(r,f)+T(r,g)]\leq \mathrm{S(r,f)+S(r,g)}. \tag{3.14}$$

From Eqs. (2.6), (2.7) , (3.14), and $n>7k+3m+12,$ we get a contradiction. Thus, by Lemma 6, $F=\frac{(b+1)G+(a-b-1)}{bG+(a-b)}$, where $a(\neq 0)$, b are two constants. Then by Lemma 8, we can prove Theorem 3.

Proof of Theorem 4. Obviously, we get

$$\overline{N}_L\left(r,\frac{1}{F-1}\right)\leq N\left(r,\frac{F}{F'}\right)\leq N\left(r,\frac{F'}{F}\right)+S(r,f)$$
$$\leq \overline{N}(r,F)+\overline{N}\left(r,\tfrac{1}{F}\right)+s(r,f)$$
$$\leq (2k+m+2)T(r,f)+S(r,f). \tag{3.15}$$

Similarly, we have

$$\overline{N}_L\left(r,\tfrac{1}{G-1}\right)\leq (2k+m+2)T(r,g)+S(r,f). \tag{3.16}$$

Suppose that F and G satisfied (i) in Lemma 7, we get

$$T(r,F)+T(r,G)$$
$$\leq 2\left\{N_2(r,F)+N_2\left(r,\frac{1}{F}\right)+N_2(r,G)+N_2\left(r,\frac{1}{G}\right)\right\}+3\overline{N}_L\left(r,\frac{1}{F-1}\right)$$
$$+3\overline{N}_L\left(r,\tfrac{1}{G-1}\right)+\mathrm{S(r,f)+S(r,g)}. \tag{3.17}$$

Combining Eqs. (3.3), (3.4), (3.15), (3.16), and (3.17), we have

$$T(r,F)+T(r,G)\leq (4k+2m+8)[T(r,f)+T(r,g)]+3[2k+m+2]T(r,f)$$
$$+3[2k+m+2]T(r,g)+S(r,f)+S(r,g)$$
$$\leq (4k+2m+8)[T(r,f)+T(r,g)]+(6k+3m+6)$$
$$[T(r,f)+T(r,g)]+S(r,f)+S(r,g)$$
$$(n+m+k)[T(r,f)+T(r,g)]\leq\ (10k+5m+14)[T(r,f)+T(r,g)]$$

$$+ S(r,f) + S(r,g)$$

$$(n-9k-4m-14)[T(r,f)+T(r,g)] \leq S(r,f) + S(r,g). \tag{3.18}$$

From Eqs. (2.6), (2.7) , (3.18), and $n > 9k + 4m + 14$, we get a contradiction. Thus, by Lemma 7, $F = \frac{(b+1)G+(a-b-1)}{bG+(a-b)}$, where $a(\neq 0)$, b are two constants. Then by Lemma 8, we can prove Theorem 4.

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