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Roughness in G-modules

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Abstract

Rough set theory is a generalisation of classical set theory. It is an effective mathematical approach to deal with vagueness and ambiguity in information systems. Combining this theory with rough algebraic structures is a recent trend in the area of mathematical research. In this paper we consider G-module as the universal set and we introduce the notion of rough G-module with respect to a G-submodule of a G-module.

Keywords: Full congruence relation, approximation space, rough set, lower rough G-module, upper rough G-module, rough G-module

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1. INTRODUCTION

The theory of group representation was developed by G.Frobenius in the 19th century. The works of Emmy Noether on representation theory led to the absorption of the theory of group representations into the study of modules over rings and algebra. Module theoretic approach especially G-module structure has been extensively used for the study of group representation.

The rough set theory proposed by Z.Pawlak [17] is a powerful mathematical tool to deal with vague, uncertain and imperfect knowledge. The key concept is an equivalence relation and the equivalence classes are the building blocks for the construction of lower and upper approximations in terms of which a rough set is defined. A lot of research work has been carried out by combining rough set theory with abstract algebra. Some papers directly introduced the concepts of rough algebraic structures into an approximation space. On the other hand some papers substituted an algebraic structure for the universal set, and investigated the roughness in algebraic structures.

Biswas and Nanda[2] proposed the notion of rough subgroups. Kuroki and Mordeson [13, 14] studied the structure of rough sets and rough groups. B. Davvaz studied roughness in several algebraic structures like rings and modules and introduced the notion of rough ideals and rough subrings with respect to an ideal of a ring [4, 5, 6, 7]. The roughness in vector spaces and BCK algebras are also studied [12, 15, 16]. Qun-Fen Zhang proposed the concept of rough modules in an approximation space and investigated their properties [19]. P.Isaac and P.Ursala introduced the concept of rough G-module into an approximation space [11]. They also introduced rough fuzzy G-modules and studied about its properties in [18]. The properties of fuzzy G-modules are studied in [8].

In this paper we go with the second approach ie, to study the applications of rough sets in the algebraic structure of G-modules. We consider a G-module as the universal set and based on this G-submodule the lower and upper approximations and the notion of rough G module are defined.

In section 2 we recall some of the basics of algebra and rough set theory. In section 3 we discuss the properties of full congruence relation on G-modules. Then in section 4, we introduce rough G-module and prove some related results. We conclude in section 5 with possible future work in the area of rough G-modules.

2. PRELIMINARIES

In this section, we see some basic definitions that will be needed in the sequel. For crisp algebraic concepts one may refer the books by Artin [1], Curties [3], Fraleigh [9] or Gallian [10].

Definition 2.1 [3] Let G be a finite group. A vector space M over a field K is called a G-*module* if for every $g \in G$ and $x \in M$, there exists a product $g \cdot x \in M$ (called the action of G on M) satisfying the following axioms:

(1)$1_G \cdot x = x, \forall x \in M$ (1_G being the identity element in G)

(2)$(g \cdot h) \cdot x = g \cdot (h \cdot x), \forall g, h \in G; x \in M$ and

(3)$g \cdot (k_1 x_1 + k_2 x_2) = k_1 (g \cdot x_1) + k_2 (g \cdot x_2), \forall k_1, k_2 \in K; g \in G; x \in M$

Example 2.2 [8] Consider the group $G = \{1, -1, i, -i\}$ under multiplication and let $M = \mathbb{C}^n$ and $K = \mathbb{C}$. Then M is a G-module over K under the usual addition and multiplication of complex numbers.

Definition 2.3 [3] A subspace N of M is said to be a G*-submodule* if N is a G-module under the same action of G.

Example 2.4 [8] Let $\mathbb{Q}$ be the field of rational numbers, $G = \{1, -1\}$ and $M = \mathbb{R}$. Then M is a G-module over $\mathbb{Q}$. For each irrational number r, $N = \mathbb{Q}(r)$ is a G-submodule of M.

Definition 2.5 [18] An equivalence relation θ on M is called a *full congruence relation* if $(a,b) \in \theta$ implies $(a+x, b+x), (ka, kb), (g \cdot a, g \cdot b) \in \theta$ for all $a, b, x \in M; k \in K; g \in G$.

Definition 2.6 [17] A pair (U, θ) where $U \neq \phi$ and θ is an equivalence relation on U is called an *approximation space*.

Definition 2.7 [17] For an approximation space (U, θ) and a subset X of U, the sets

$$\overline{X} = \{x \in \mathrm{U} \mid [x]_\theta \cap \mathrm{X} \neq \phi\}$$

$$\underline{X} = \{x \in \mathrm{U} \mid [x]_\theta \subseteq \mathrm{X}\}$$

$BN(X) = \overline{X} - \underline{X}$ are called respectively the *upper approximation, lower approximation* and *boundary region* of X in (U, θ). X is called a *rough set* if $BN(X)$ is non empty.

3 FULL CONGRUENCE RELATION ON A G-MODULE

In this section, we discuss the properties of full congruence relation on a G-module.

Definition 3.1 Let X and Y be non empty subsets of a G-module M. The sum of X and Y is defined by $X + Y = \{x + y \mid x \in X, y \in Y\}$ and the product of X and an element k of K is defined by $kX = \{kx \mid x \in X\}$ and the action of an element $g \in G$ on X as $g \cdot X = \{g \cdot x \mid x \in X\}$.

Theorem 3.2 Let θ be a full congruence relation on M. Then the congruence class $[0]_\theta$ where 0 denotes the zero element of M, is a G-submodule of M and $[a]_\theta = a + [0]_\theta$ for all $a \in M$.

Proof. Suppose $x, y \in [0]_\theta$ then $(x,0) \in \theta$ and $(y,0) \in \theta$. Since θ is a full congruence relation on M we have $(x + y,0) \in \theta \Rightarrow x + y \in [0]_\theta$. Also $x \in [0]_\theta \Rightarrow (x,0) \in \theta \Rightarrow (kx,k0) \in \theta \Rightarrow (kx,0) \in \theta \Rightarrow kx \in [0]_\theta \forall k \in K$ Also $x \in [0]_\theta \Rightarrow (x,0) \in \theta \Rightarrow (g \cdot x, g \cdot 0) \in \theta \Rightarrow (g \cdot x,0) \in \theta \Rightarrow g \cdot x \in [0]_\theta \forall g \in G$. Thus $[0]_\theta$ is a G-submodule of M.

Moreover let $x \in [a]_\theta$. Then $(x,a) \in \theta \Rightarrow (x - a, a - a) \in \theta \Rightarrow (x - a,0) \in \theta \Rightarrow x - a \in [0]_\theta \Rightarrow x \in a + [0]_\theta$. On the other hand for any $x \in a + [0]_\theta$ there exists $y \in [0]_\theta$ such that $x = a + y$. Since $(y,0) \in \theta$ we have $(x - a,0) \in \theta \Rightarrow (x,a) \in \theta \Rightarrow x \in [a]_\theta . \therefore [a]_\theta = a + [0]_\theta$.

Theorem 3.3 Let N be a G-submodule of M and $\theta_N = \{(x, y) \mid x, y \in M; x - y \in N\}$. Then θ_N is a full congruence relation on M and $[0]_{\theta_N} = N$.

Proof. Let $(x, y) \in \theta_N$. Then $x - y \in N$. Let $z \in M$. Then $(x + z) - (y + z) = x - y \in N \Rightarrow (x + z, y + z) \in \theta_N$. Let $k \in K$. Then $kx - ky = k(x - y) \in N \Rightarrow (kx, ky) \in \theta_N$. Let $g \in G$. Then $g \cdot x - g \cdot y = g \cdot (x - y) \in N \Rightarrow (g \cdot x, g \cdot y) \in \theta_N$. Moreover if $x \in [0]_{\theta_N}$ then $(x,0) \in \theta_N \Rightarrow x - 0 \in N \Rightarrow x \in N$. On the other hand if $x \in N$ then $x = x - 0 \in N \Rightarrow (x,0) \in \theta_N \Rightarrow x \in [0]_{\theta_N}$.

Corollary 3.4 Let N be a G-submodule of M. Then $[a]_{\theta_N} = a + N$.

From the above two theorems we can see that there exists a one to one correspondence between the set of all full congruence relations on M and the set of all G-submodules of M.

Throughout this paper θ_N denotes the full congruence relation determined by a G

-submodule N and $\theta_N(a)$ denotes the congruence class $[a]_{\theta_N}$ where $a \in M$ ie,. $\theta_N(a) = a + N$

Theorem 3.5 Let N be a G-submodule of M. Then for all $x, y \in M; k \in K; g \in G;$ we have

1) $\theta_N(x) + \theta_N(y) = \theta_N(x+y)$

2) $\theta_N(kx) = k\theta_N(x)$

3) $\theta_N(g \cdot x) = g \cdot \theta_N(x)$

Proof. $\theta_N(x) + \theta_N(y) = (x+N) + (y+N) = (x+y) + N = \theta_N(x+y)$

$\theta_N(kx) = kx + N = kx + kN = k(x+N) = k\theta_N(x)$

$\theta_N(g \cdot x) = g \cdot x + N = g \cdot x + g \cdot N = g \cdot (x+N) = g \cdot \theta_N(x)$

Theorem 3.6 Let N_1 and N_2 be G-submodules of M. Then for all $x, y \in M;$ we have

1) $\theta_{N_1+N_2}(x+y) = \theta_{N_1}(x) + \theta_{N_2}(y)$

2) $\theta_{N_1 \cap N_2}(x) = \theta_{N_1}(x) \cap \theta_{N_2}(x)$

Proof. $\theta_{N_1+N_2}(x+y) = (x+y) + (N_1 + N_2) = (x+N_1) + (y+N_2) = \theta_{N_1}(x) + \theta_{N_2}(y)$. $\theta_{N_1 \cap N_2}(x) = x + (N_1 \cap N_2) \subseteq (x+N_1) \cap (x+N_2) = \theta_{N_1}(x) \cap \theta_{N_2}(x)$. On the other hand, for any $y \in \theta_{N_1}(x) \cap \theta_{N_2}(x)$ there exists $n_1 \in N_1$ and $n_2 \in N_2$ such that $y = x + n_1 = x + n_2$. Then $n_1 = n_2 \in N_1 \cap N_2$ so that $y \in x + N_1 \cap N_2 = \theta_{N_1 \cap N_2}(x)$.

4 LOWER AND UPPER APPROXIMATIONS IN A G-MODULE

Definition 4.1 Let N be a G-submodule of M and $X \subseteq M$. The sets $\underline{N}(X) = \{x \in M \mid x + N \subseteq X\}$ and $\overline{N}(X) = \{x \in M \mid (x+N) \cap X \neq \phi\}$ are respectively called the *lower* and *upper approximations* of a set X with respect to the G-submodule N and (M, N) is called the *approximation space*.

Theorem 4.2 Let N be a G-submodule of M and X and Y be non empty subsets of M. Then

1)$\underline{N}(X) \subseteq X \subseteq \overline{N}(X)$

2)$\underline{N}(X \cap Y) = \underline{N}(X) \cap \underline{N}(Y)$

3)$\overline{N}(X \cup Y) = \overline{N}(X) \cup \overline{N}(Y)$

4)$\underline{N}(X \cup Y) \supseteq \underline{N}(X) \cup \underline{N}(Y)$

5)$\overline{N}(X \cap Y) \subseteq \overline{N}(X) \cap \overline{N}(Y)$

6)If $X \subseteq Y$ then $\underline{N}(X) \subseteq \underline{N}(Y), \overline{N}(X) \subseteq \overline{N}(Y)$

7)$\overline{N}(X) = X + N$

8)If $N \subseteq X$ then $N \subseteq \underline{N}(X)$ and $\underline{N}(X) \neq \phi$

9)$\overline{N}(X + Y) = \overline{N}(X) + \overline{N}(Y)$

10)$\underline{N}(X + Y) \supseteq \underline{N}(X) + \underline{N}(Y)$

11)$\underline{N}(kX) = k\underline{N}(X)$ for any $k \in K$

12)$\overline{N}(kX) = k\overline{N}(X)$ for any $k \in K$

13)$\underline{N}(g \cdot X) = g \cdot \underline{N}(X)$ for any $g \in G$

14)$\overline{N}(g \cdot X) = g \cdot \overline{N}(X)$ for any $g \in G$

Proof. We prove only (13) and (14). The proof of other conclusions are similar to the conclusions in [15]. Let $x \in g \cdot \underline{N}(X) \Rightarrow \exists y \in \underline{N}(X)$ such that $x = g \cdot y$. Since $(y + N) \subseteq X$ we have
$g \cdot (y + N) \subseteq g \cdot X \Rightarrow g \cdot y + g \cdot N \subseteq g \cdot X \Rightarrow x + N \subseteq g \cdot X \Rightarrow x \in \underline{N}(g \cdot X)$.
Conversely if $x \in \underline{N}(g \cdot X)$ then $x + N \subseteq g \cdot X$. Now
$x + N = g \cdot (g^{-1} \cdot x) + N = g \cdot (g^{-1} \cdot x + N) \subseteq g \cdot X \Rightarrow g^{-1} \cdot x + N \subseteq X \Rightarrow g^{-1} \cdot x \in \underline{N}(X) \Rightarrow x \in g \cdot \underline{N}(X)$
. Thus $\underline{N}(g \cdot X) = g \cdot \underline{N}(X)$

From (7) $\overline{N}(g \cdot X) = g \cdot X + N = g \cdot X + g \cdot N = g \cdot (X + N) = g \cdot \overline{N}(X)$.

Theorem 4.3 Let N_1 and N_2 be G-submodules of M and X and Y be non empty subsets of M. Then the following statements are true.

1) $\underline{N_1 \cap N_2}(X) \supseteq \underline{N_1}(X) \cup \underline{N_2}(X)$

2) $\overline{N_1 \cap N_2}(X) \subseteq \overline{N_1}(X) \cap \overline{N_2}(X)$

3) $\overline{N_1 + N_2}(X+Y) = \overline{N_1}(X) + \overline{N_2}(Y)$

4) $\underline{N_1 + N_2}(X+Y) \supseteq \underline{N_1}(X) + \underline{N_2}(Y)$

5) If $N_1 \subseteq N_2$ then $\overline{N_1}(X) \subseteq \overline{N_2}(X)$ and $\underline{N_2}(X) \subseteq \underline{N_1}(X)$

Proof.

(1) If $x \in \underline{N_1}(X) \cup \underline{N_2}(X)$ then $x \in \underline{N_1}(X)$ or $x \in \underline{N_2}(X)$ so that $(x+N_1) \subseteq X$ or $(x+N_2) \subseteq X$. Thus $x+(N_1 \cap N_2) \subseteq X$ which implies that $x \in \underline{N_1 \cap N_2}(X)$.

(2) By Theorem 4.2,
$\overline{N_1 \cap N_2}(X) = X + (N_1 \cap N_2) \subseteq (X+N_1) \cap (X+N_2) = \overline{N_1}(X) \cap \overline{N_2}(X).$

(3) By Theorem 4.2,
$\overline{N_1 + N_2}(X+Y) = (X+Y)+(N_1+N_2) = (X+N_1)+(Y+N_2) = \overline{N_1}(X) + \overline{N_2}(Y).$

(4) If $x \in \underline{N_1}(X) + \underline{N_2}(Y)$ then there exists $y \in \underline{N_1}(X), z \in \underline{N_2}(Y)$ such that $x = y+z$. Since $y \in \underline{N_1}(X)$ and $z \in \underline{N_2}(Y)$ we have $(y+N_1) \subseteq X$ and $(z+N_2) \subseteq Y$. Thus $x+(N_1+N_2) = (y+z)+(N_1+N_2) = (y+N_1)+(z+N_2) \subseteq X+Y$. So $x \in \underline{N_1+N_2}(X+Y)$. Therefore $\underline{N_1+N_2}(X+Y) \supseteq \underline{N_1}(X) + \underline{N_2}(Y).$

(5) If $N_1 \subseteq N_2$ then $\overline{N_1}(X) = X + N_1 \subseteq X + N_2 = \overline{N_2}(X)$. Thus $\overline{N_1}(X) \subseteq \overline{N_2}(X)$. Now let $x \in \underline{N_2}(X)$. Then $x+N_2 \subseteq X$. Since $N_1 \subseteq N_2$ we have $x+N_1 \subseteq X$ which implies that $x \in \underline{N_1}(X)$. Therefore $\underline{N_2}(X) \subseteq \underline{N_1}(X)$.

Definition 4.4 Let (M,N) be an approximation space, a non empty subset X of M is called a N*-lower rough* G*-submodule* if $\underline{N}(X)$ is a G-submodule of

M. X is called a N *-upper rough* G *-submodule* if $\overline{N}(X)$ is a G -submodule of M. X is called a N *-rough* G *-submodule* if both $\underline{N}(X)$ and $\overline{N}(X)$ are G -submodules of M.

Theorem 4.5 Let (M,N) be an approximation space.

1) If $\phi \neq X \subseteq N$ then X is a N -upper rough G -submodule and $\overline{N}(X) = N$.

2) If $\phi \neq X \subseteq M$ and X is a N -lower rough G -submodule of M then $N \subseteq X$.

Proof.

1) From (7) of theorem 4.2 we have $\overline{N}(X) = X + N$. Also $X + N \subseteq N[\because N$ is a G -submodule of M]. $\therefore \overline{N}(X) \subseteq N$

On the other hand, Let
$n \in N \Rightarrow (n+N) \cap X = N \cap X = X \neq \phi[\because X \subseteq N] \Rightarrow n \in \overline{N}(X)$. Thus we have $N \subseteq \overline{N}(X)$. $\therefore \overline{N}(X) = N$

2) Since X is a N -lower rough G -submodule of M we have $\underline{N}(X)$ is a G -submodule of $M \Rightarrow 0 \in \underline{N}(X) \Rightarrow 0 + N \in X \Rightarrow N \subseteq X$.

Theorem 4.6 Let N_1 and N_2 be G -submodules of M. Then

1) $\overline{N_1 + N_2}(X) = \overline{N_1}(X) + \overline{N_2}(X)$

2) $\underline{N_1 + N_2}(X) = \underline{N_1}(X) + \underline{N_2}(X)$

Proof. The proof is similar to that of theorem 4.2 in [16].

Theorem 4.7 Let P and Q be two G -submodules of M. Then $\overline{P}(Q)$ is a G -submodule of M.

Proof. Let $p, q \in \overline{P}(Q), k \in K, g \in G \Rightarrow (p+P) \cap Q \neq \phi$ and $(q+P) \cap Q \neq \phi$. So there exists $x \in (p+P) \cap Q$ and $y \in (q+P) \cap Q$. Since Q is a G-submodule of M we get $x - y \in Q$ and $x - y \in (p+P) - (q+P) = p - q + P$. Hence $(p - q + P) \cap Q \neq \phi$ which implies that $p - q \in \overline{P}(Q)$.

Also we have $kx \in Q$ and $kx \in k(p+P) = kp + kP = kp + P$. So $(kp + P) \cap Q \neq \phi \Rightarrow kp \in \overline{P}(Q)$.

Also we have $g \cdot x \in Q$ and $g \cdot x \in g \cdot (p+P) = g \cdot p + g \cdot P = g \cdot p + P$. So $(g \cdot p + P) \cap Q \neq \phi \Rightarrow g \cdot p \in \overline{P}(Q)$.

$\therefore \overline{P}(Q)$ is a G-submodule of M.

Theorem 4.8 Let P and Q be two G-submodules of M. Then $\underline{P}(Q)$ is non empty then $\underline{P}(Q) = Q$.

Proof. We know that $\underline{P}(Q) \subseteq Q$. Now we will show that $Q \subseteq \underline{P}(Q)$. Since $\underline{P}(Q) \neq \phi$ there exists an $x \in \overline{P}(Q) \Rightarrow (x+P) \subseteq Q$. Since $0 \in P$ we have $x + 0 = x \in Q$ then $-x \in Q. \therefore P = -x + x + P \subseteq -x + Q = Q$. Let $q \in Q$. Since $P \subseteq Q$ and Q is a G-submodule we have $q + P \subseteq Q \Rightarrow q \in \underline{P}(Q). \therefore \underline{P}(Q) = Q$.

Corollary 4.9 Let P and Q be two G-submodules of M such that $P \subseteq Q$ then $(\underline{P}(Q), \overline{P}(Q))$ is a rough G-submodule of M.

5 CONCLUSION

In this paper we have introduced the notion of rough G-submodule in a G-module and proved some related results. We hope that this will be useful in the theory of rough sets and approximate reasoning. The theory of rough sets and fuzzy sets can be extended in a similar way to other algebraic structures.

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