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Transient analysis of non-Markovian retrial queueing system with priority services, second optional service, balking, server's interruptions and a stand by server

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Abstract

This paper deals with the transient analysis of batch arrival retrial queueing system with general retrial times, priority service, balking, second optional service, Bernoulli vacation, extended vacation, breakdown, delayed repair and stand by server. Here we assume that customers arrive in batches according to compound Poisson processes, where the blocked customers either with probability p join priority queue or with complementary probability q leave the service area and enter the retrial group (called orbit) are consider as low priority customers. The service rule for both priority customers follows FCFS discipline and only the customer at the head of the queue or head in orbit is allowed to retry for service. The low priority customers may balk the queue(orbit) with probability qb or may join the orbit with probability $q(1-b)$. First service is the essential for each customer and second service is the optional, after completion of the essential service the customer have a option for second service with probability r_1 or leave the system with probability $(1-r_1)$. The main server has four interruptions, there are after completion of each service the server has a option to go vacation with probaility θ or continue for next service with probability $(1-\theta)$, after completion of a vacation the server has option for extended vacation with probability r or continue to serve the next customer with probability $(1-r)$. While the server is working with the customer, it may breakdown at any instant and the server will be down for short interval of time. During these interruptions of the main server the stand-by-server serve the customers Further concept of delay time to repair is also introduced.

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The retrial time, service time for both priority and non-priority customers, vacation time, extended vacation time, delay time and repair time are all follows general(arbitrary) distribution, breakdown time and service time of stand by server follow exponential distribution. Finally, we derive the probability generating function of service time, vacation time, extended vacation time, delay time and repair time and some important performance measures of the model.

AMS subject classification: 60K25, 60K30, 90B22.
Keywords: Batch arrival, retrial times, transient solution, stand-by-server, Bernoulli vacation, extended vacation, unreliable server.

1. Introduction

The study on queuing models have become an indispensable area due to its wide applicability in real life situations, all the models considered have had the property that units proceed to service on a first-come, first-served basis. This is obviously not only the manner of service and there are many alternatives, such as last-come first-served, selection in random order and selection by priority. In order to offer different quality of service for different kinds of customers, we often control a queueing system by priority mechanism. This phenomenon is common in pratice. For example, in telecommunication transfer protocol, for guaranteeing different layers service for different customers, priority classes control may appear in header of IP package or in ATM cell. Priority control is also widely used in production practice, transportation management.

Retrial queues are characterized by the feature that arriving customers who find all servers busy and join the retrial group to try their luck again after a time period. Queues in which customers are allowed to conduct retrials have been extensively used to model many problems in telephone switching systems, telecommunication networks and computer systems for competing to gain service from a central processor unit.

Several authors discussed the single arrival and batch arrival retrial queueing systems with priority service. Ayyappan et al. (2010) have discussed a single server retrial queueing system with stand by server under pre-emptive priority service, Ayyappan et al. (2014) have discussed retrial queueing system with second optional service, random breakdown, set up time and Bernoulli vacation. Al-Jaraha and Madan (2003) have studied an $M/G/1$ queue with second optional service with general service time distribution, Atencia et al. (2005) have discussed a single-server retrial queue with general retrial times and Bernoulli schedule. Gautam Choudhury et al., (2012) have discussed a batch arrival retrial queue with general retrial times under Bernoulli vacation schedule for unreliable server and delaying repair, Jain et al. (2008) have discussed a bulk arrival retrial queue with unreliable server and priority subscribers, Jau-Chuan et al. (2009) have discussed a modified vacation policy for $M/G/1$ retrial queue with balking and feedback, Jinbiao et al. (2013) have discussed a single-server retrial G-queue with priority and unreliable server under Bernoulli vacation schedule. Khalaf (2014) have discussed a queueing system with four different main server's interruptions and a stand-by-server and Rajadurai et al. (2014) have discussed an analysis of an $M^{[X]}/(G_1, G_2)/1$ retrial queueing system with balking, optional re-service under modified vacation policy and service interruption.

This paper deals with the transient analysis of batch arrival retrial queueing system with general retrial times, priority service, balking, second optional service, Bernoulli vacation, extended vacation, breakdown, delayed repair and stand by server. Here we assume that customers arrive in batches according to compound Poisson processes, where the blocked customers either with probability p join priority queue or with complementary probability q leave the service area and enter the retrial group (called orbit) are consider as low priority customers. The service rule for both priority customers follows FCFS discipline and only the customer at the head of the queue or in orbit is allowed to retry for service. The arriving low priority customers may balk the queue(orbit) with probability qb or may join the orbit with probability $q(1-b)$. First service is the essential for each customer and second service is the optional, after completion of the essential service the customer have a option for second service with probability r_1 or leave the system with probability $(1-r_1)$. The main server has four interruptions, there are after completion of each service the server has a option to go vacation with probaility θ or continue for next service with probability $(1-\theta)$, after completion of a vacation the server has option for extended vacation with probability r or continue to serve the next customer with probability $(1-r)$. While the server is working with the customer, it may breakdown at any instant and the server will be down for short interval of time. During these interruptions of the main server the stand-by-server serve the customers Further concept of delay time to repair is also introduced.

The retrial time, service time for both priority and non-priority customers, vacation time, extended vacation time, delay time and repair time are all follows general(arbitrary) distribution, breakdown time and service time of stand by server follow exponential distribution. Finally, we derive the probability generating function of service time, vacation time, extended vacation time, delay time and repair time and some important performance measures of the model.

The rest of the paper is organized as follows: Mathematical description of our model in section (2). Practical apllications, equations governing of our model and the time dependent solution have been obtained in section (3) and (4). The corresponding steady state results have been derived explicitly in section (5). Average queue size and the average waiting time are computed in section (6) and (7). Some particular cases have been discussed in section (8). Numerical results and graphical illustrations in section (9). References are in section (10).

2. Mathematical description of our model

1. Customers arrive at the system in batches of variable size in a compound Poisson process. Let $\lambda c_i dt$
   $(i = 1, 2, 3, \ldots)$ be the first order probability that a batch of i customers arrives at

the system during a short interval of time $(t, t+dt)$, where $0 \leq c_i \leq 1$, $\sum_{i=1}^{\infty} c_i = 1$, and $\lambda > 0$, is the average arrival rate. If the server is idle upon an arrival, service of the arriving customers commences immediately. Otherwise, the arriving customers either with probability p joins the priority queue, where they waits to be served or with complementary probability q joins the retrial group (called orbit) as low priority customers.

2. The server must serve all the priority customers present in the system before taking up non-priority (retrial) customers for service. In other words, there is no priority customers present in the system at the time of starting service of a non-priority unit. Further, we assume that the server follows a non-preemptive priority rule, which means that if one or more priority customers arrive during the service time of a non-priority unit, the current service of a non-priority customer is not stopped and a priority customer will be taken up for service only after the current service of a non-priority customer is complete.

3. Assume that only the customer at the head of the orbit is allowed for access to the server. If the server is busy upon retrial, the customer join the orbit again. Such a process is repeated until the arriving customer finds the server idle and gets the requested service at the time of a retrial. Also upon arrival, if the customer finds the server busy or on interruptions, then the customers join the orbit with probability $q(1-b)$ or balk the orbit with probability qb. Successive inter retrial times of any customers follow a general(arbitrary) distribution function $A(x)$, density function $a(x)$ and the conditional completion rate for retrials is given by $\eta(x) = \dfrac{a(x)}{(1-A(x))}$

4. Each customer under priority and non-priority, by a single server provides essential service on first come - first served basis. As soon as customers completes the essential service they have an option for second service with probability r_1 or may leave the system with probability $(1-r_1)$. The service time for both essential and second optional service follows general(arbitrary) distributions with distribution functions $B_i(s)$ and the density functions $b_i(s)$, $i = 1, 2$.

5. Let $\mu_i(x)dx$ be the conditional probability of completion of essential and second optional service (for both priority and non-priority customers) during the interval $(x, x+dx]$, given that the elapsed service time is x, so that

$$\mu_i(x) = \frac{b_i(x)}{1 - B_i(x)},$$

and therefore,

$$b_i(s) = \mu_i(s) e^{-\int_0^s \mu_i(x)dx}.$$

6. We further assume that as soon as the completion of each service the server has the option to take a vacation of random length with probability θ, in which case the vacation starts immediately or else with probability $(1-\theta)$ he may decide to continue serving the next units present in the system, if any. After completion of a vacation the server has option for extended vacation with probability r or continue to serve the next customer with probability $(1-r)$.

7. The vacation time follows general (arbitrary) distribution with distribution function $V(s)$ and the density function $v(s)$. Let $\gamma(x)dx$ be the conditional probability of a completion of a vacation during the interval $(x, x+dx]$, given that the elapsed vacation time is x, so that
$$\gamma(x) = \frac{v(x)}{1-V(x)},$$
and therefore,
$$v(s) = \gamma(s)e^{-\int_0^s \gamma(x)dx}.$$

8. The extended vacation time follows general (arbitrary) distribution with distribution function $E(s)$ and the density function $e(s)$. Let $\theta(x)dx$ be the conditional probability of a completion of a extended vacation during the interval $(x, x+dx]$, given that the elapsed extended vacation time is x, so that
$$\theta(x) = \frac{e(x)}{1-E(x)},$$
and therefore,
$$e(s) = \theta(s)e^{-\int_0^s \theta(x)dx}.$$

9. On returning from vacation, extended vacation, the server instantly starts serving the customer at the head of the queue, if any. The server stays in the system for being available if there are no customers.

10. The system may break down at random and breakdowns are assumed to occur according to a Poisson stream with mean breakdown rate $\alpha > 0$. Whenever the system breaks down, its repairs do not start immediately and there is a delay time to start repair.

11. The delay time to start repair follows general (arbitrary) distribution with distribution function $D(s)$ and the density function $d(s)$. Let $\xi(x)dx$ be the conditional probability of a completion of a delay time during the interval $(x, x+dx]$, given that the elapsed delay time is x, so that
$$\xi(x) = \frac{d(x)}{1-D(x)},$$

and therefore,

$$d(s) = \xi(s)e^{-\int_0^s \xi(x)dx}.$$

12. The Repair time follows general (arbitrary) distribution with distribution function $R(s)$ and the density function $r(s)$. Let $\beta(x)dx$ be the conditional probability of a completion of a repair during the interval $(x, x+dx]$, given that the elapsed repair time is x, so that

$$\beta(x) = \frac{r(x)}{1 - R(x)},$$

and therefore,

$$r(s) = \beta(s)e^{-\int_0^s \beta(x)dx}.$$

13. When the main server is on vacation, extended vacation, waiting for repair to start or under repair in all these interruptions there is a stand by server serves the customer until the main server returns. The stand by server's service time is exponentially distributed with service rate is $\delta(> 0)$ and mean service time is $\frac{1}{\delta}$. When the main server rejoins the system after any interruption, the customer being served by the stand by server transferred to the main server to start a service over.

Now we obtain the probability generating function of the joint distribution of the status of the server and the number of customers in the queue by treating $P^0(t)$; $B^0(t)$; $D^0(t)$; $R^0(t)$; $V^0(t)$ and $E^0(t)$ are the elapsed retial time, elapsed service time, elapsed delay time, elapsed repair time, elapsed vacation time and elapsed extended vacation time of the server at time t, respectively as supplementary variables. Assuming that the system is empty initially. Let $N_1(t)$ and $N_2(t)$ be the number of customers in the priority queue and the orbit at time t, and $C(t)$ denote the server state at time t. At an arbitrary time t, the system can be described by means of the Markov process.
Probabilities at time t are as follows:
$C(t) = i(i = 0, 1, 2, 3, 4, 5, 6and7)$ define if the server is idle, the server is idle during retrial time, the server is busy with first essential service, the server is busy with second optional service, the server is on vacation, the server is on extended vacation, the server is on delay time and the server is on repair at time t respectively.
Introducing the supplementary variables $P^0(t)$; $B_i^0(t)(i = 1, 2,)$; $V^0(t)$, $E^0(t)$, $D^0(t)$ and $R^0(t)$ to obtain a bivariate Markov process $Z(t) = (N_1(t), N_2(t); X(t))$,

where

$$X(t) = \begin{cases} 0 \text{if} C(t) = 0, \\ P^0(t) \text{if} C(t) = 1, \\ B_1^0(t) \text{if} C(t) = 2, \\ B_2^0(t) \text{if} C(t) = 3, \\ V^0(t) \text{if} C(t) = 4, \\ E^0(t) \text{if} C(t) = 5, \\ D^0(t) \text{if} C(t) = 6, \\ R^0(t) \text{if} C(t) = 7. \end{cases}$$

Now we define the following limiting probabilities:
$P^0(t) = Pr\{N_1(t) = 0, N_2(t) = 0; X(t) = 0\}$;
$P^0_{0,n}(x;t)dx = Pr\{N_1(t) = 0, N_2(t) = n; X(t) = P^0(t); x < P^0(t) \le x + dx; x; t > 0; n \ge 1\}$;
$P^i_{m,n}(x;t)dx = Pr\{N_1(t) = m, N_2(t) = n; X(t) = B_i^0(t); x < B_i^0(t) \le x + dx; x; t > 0; m, n \ge 0\}$;
$V_{m,n}(x;t)dx = Pr\{N_1(t) = m, N_2(t) = n; X(t) = V^0(t); x < V^0(t) \le x + dx; x; t > 0; m, n \ge 0\}$;
$E_{m,n}(x;t)dx = Pr\{N_1(t) = m, N_2(t) = n; X(t) = E^0(t); x < E^0(t) \le x + dx; x; t > 0; m, n \ge 0\}$;
$D_{m,n}(x;t)dx = Pr\{N_1(t) = m, N_2(t) = n; X(t) = D^0(t); x < D^0(t) \le x + dx; x; t > 0;$
$m, n \ge 0\}$;
$R_{m,n}(x;t)dx = Pr\{N_1(t) = m, N_2(t) = n; X(t) = R^0(t); x < R^0(t) \le x + dx; x; t > 0; m, n \ge 0\}$;
and for fixed values of x and $m, n \ge 0$.
Further, it is assumed that
$P^0(0) = 0$, $P^0(\infty) = 1$, $B_i(0) = 0$, $B_i(\infty) = 1 (i = 1, 2,)$, $V(0) = 0$, $V(\infty) = 1$, $E(0) = 0$, $E(\infty) = 1$, $D(0) = 0$, $D(\infty) = 1$, $R(0) = 0$, $R(\infty) = 1$ and that $P^0(x)$, $B_i(x)(i = 1, 2,)$, $V(x)$ and $E(x)$ are continues at $x = 0$ and $D(y)$ and $R(y)$ are continues at $y = 0$.
Various stochastic processes involved in the system are assumed to be independent of each other.

3. Practical Applications

This section gives several examples of real systems which can be modelled as a retrial queue with two types of customers: Type I(priority) and Type II(non priority) customers arrive at the system according to compound Poisson processes with rates λp and λq, respectively.

1. Telephone in a bank
   We consider a man who working in a bank, he must attend the waiting line and a

telephone, although such a man gives attention to the telephone only when there are no customers in the waiting line. An arriving customers who finds this man idle enters the bank in order to be served one by one, by FCFS discipline; otherwise, they either with probability p goes in the bank to wait for his service in a queue or with complementary probability q decides to telephone later until they gets their service. This system can be modelled as an $M^{X_1}, M^{X_2}/G/1$ retrial queue where the customers waiting in the bank are consider as priority customers and the customers making calls are consider as non-priority customers.

2. Call Centers
We consider a call center which can attend to incoming and outgoing calls. If the server is free a calls (incoming or outgoing) occupies it immediately. Otherwise, if the server is busy, the incoming calls queued in an infinite buffer and the outgoing calls are blocked and must repeat their attempts later in order to get the service. Of course, as soon as the server is free, an incoming calls (if any) are served immediately. If both the buffer and the server are idle the outgoing calls only have opportunity to occupy the server. Thus the incoming calls have non-preemptive priority over the outgoing calls. Therefore, this type of call centers can be modelled as our retrial queue where a incoming calls are Type I customers and outgoing calls are Type II customers.

4. Equations Governing the System

The Kolmogorov forward equations to govern the model

$$\frac{d}{dt}P_0(t) = -\lambda P_0(t) + (1-r_1)(1-\theta)\int_0^\infty P_{0,0}^{(1)}(x,t)\mu_1(x)dx + (1-\theta)\int_0^\infty P_{0,0}^{(2)}(x,t)\mu_2(x)dx + (1-r)\int_0^\infty V_{0,0}(x,t)\gamma(x)dx + \int_0^\infty E_{0,0}(x,t)\theta(x)dx, \tag{1}$$

$$\frac{\partial}{\partial t}P_{0,n}^{(0)}(x,t) + \frac{\partial}{\partial x}P_{0,n}^{(0)}(x,t) = -(\lambda + a(x))P_{0,n}^{(0)}(x,t), n \geq 1, \tag{2}$$

$$\frac{\partial}{\partial t}P_{m,n}^{(1)}(x,t) + \frac{\partial}{\partial x}P_{m,n}^{(1)}(x,t) = -(\lambda + \mu_1(x) + \alpha)P_{m,n}^{(1)}(x,t) + \int_0^\infty R_{m,n}(x,y,t)\beta(y)dy + (1-\delta_{m0})\lambda p\sum_{i=1}^{m} C_i P_{m-i,n}^{(1)}(x,t) + (1-\delta_{0n})\lambda qb\sum_{i=1}^{n} C_i P_{m,n-i}^{(1)}(x,t) + \lambda q(1-b)P_{m,n}^{(1)}(x,t), \quad m,n \geq 0, \tag{3}$$

$$\frac{\partial}{\partial t}P^{(2)}_{m,n}(x,t)+\frac{\partial}{\partial x}P^{(2)}_{m,n}(x,t)=-(\lambda+\mu_2(x)+\alpha)P^{(2)}_{m,n}(x,t)$$
$$+\int_0^{\infty}R_{m,n}(x,y,t)\beta(y)dy+(1-\delta_{m0})\lambda p\sum_{i=1}^{m}C_iP^{(2)}_{m-i,n}(x,t)$$
$$+(1-\delta_{0n})\lambda qb\sum_{i=1}^{n}C_iP^{(2)}_{m,n-i}(x,t)+\lambda q(1-b)P^{(2)}_{m,n}(x,t),$$
$$m,n\geq 0, \tag{4}$$

$$\frac{\partial}{\partial t}V_{m,n}(x,t)+\frac{\partial}{\partial x}V_{m,n}(x,t)=-(\lambda+\gamma(x)+\delta)V_{m,n}(x,t)+p\delta V_{m+1,n}(x,t)$$
$$+\delta_{m0}q\delta V_{m,n+1}(x,t)+(1-\delta_{m0})\lambda p\sum_{i=1}^{m}C_iV_{m-i,n}(x,t)+(1-\delta_{0n})$$
$$\lambda qb\sum_{i=1}^{n}C_iV_{m,n-i}(x,t)+\lambda q(1-b)V_{m,n}(x,t),\ m,n\geq 0, \tag{5}$$

$$\frac{\partial}{\partial t}E_{m,n}(x,t)+\frac{\partial}{\partial x}E_{m,n}(x,t)=-(\lambda+\theta(x)+\delta)E_{m,n}(x,t)+p\delta E_{m+1,n}(x,t)$$
$$+\delta_{m0}q\delta E_{m,n+1}(x,t)+(1-\delta_{m0})\lambda p\sum_{i=1}^{m}C_iE_{m-i,n}(x,t)+(1-\delta_{0n})$$
$$\lambda qb\sum_{i=1}^{n}C_iE_{m,n-i}(x,t)+\lambda q(1-b)E_{m,n}(x,t),\ m,n\geq 0, \tag{6}$$

$$\frac{\partial}{\partial t}D_{m,n}(x,y,t)+\frac{\partial}{\partial x}D_{m,n}(x,y,t)=-(\lambda+\xi(y)+\delta)D_{m,n}(x,y,t)$$
$$+p\delta D_{m+1,n}(x,y,t)+\delta_{m0}q\delta D_{m,n+1}(x,y,t)+(1-\delta_{m0})\lambda p$$
$$\sum_{i=1}^{m}C_iD_{m-i,n}(x,y,t)+(1-\delta_{0n})\lambda qb\sum_{i=1}^{n}C_iD_{m,n-i}(x,y,t)$$
$$+\lambda q(1-b)D_{m,n}(x,y,t),\ m,n\geq 0, \tag{7}$$

$$\frac{\partial}{\partial t}R_{m,n}(x,y,t)+\frac{\partial}{\partial x}R_{m,n}(x,y,t)=-(\lambda+\beta(y)+\delta)R_{m,n}(x,y,t)$$
$$+p\delta R_{m+1,n}(x,y,t)+\delta_{m0}q\delta R_{m,n+1}(x,y,t)+(1-\delta_{m0})\lambda p$$
$$\sum_{i=1}^{m}C_iR_{m-i,n}(x,y,t)+(1-\delta_{0n})\lambda qb\sum_{i=1}^{n}C_iR_{m,n-i}(x,y,t)$$
$$+\lambda q(1-b)R_{m,n}(x,y,t),\ m,n\geq 0. \tag{8}$$

The above set of equations are to be solved under the following boundary conditions at x=0.

$$P_{0,n}^{(0)}(0,t) = (1-\theta)(1-r_1)\int_0^\infty P_{0,n}^{(1)}(x,t)\mu_1(x)dx$$
$$+(1-\theta)\int_0^\infty P_{0,n}^{(2)}(x,t)\mu_2(x)dx + (1-r)\int_0^\infty V_{0,n}(x,t)\gamma(x)dx$$
$$+\int_0^\infty E_{0,n}(x,t)\theta(x)dx,\ n \geq 1, \tag{9}$$

$$P_{m,n}^{(1)}(0,t) = \delta_{0n}\lambda p\sum_{i=1}^{m} C_{i+1}P_0(t) + \delta_{m0}\lambda q\sum_{i=1}^{n} C_{i+1}P_0(t)$$
$$+\delta_{m0}(1-\delta_{0n})\sum_{i=1}^{n}\lambda C_i\int_0^\infty P_{0,n+1-i}^{(0)}(x,t)dx$$
$$+\delta_{m0}\int_0^\infty P_{0,n+1}^{(0)}(x,t)a(x)dx + (1-r_1)(1-\theta)$$
$$\int_0^\infty P_{m+1,n}^{(1)}(x,t)\mu_1(x)dx + (1-\theta)\int_0^\infty P_{m+1,n}^{(2)}(x,t)\mu_2(x)dx$$
$$+(1-r)\int_0^\infty V_{m+1,n}(x,t)\gamma(x)dx + \int_0^\infty E_{m+1,n}(x,t)\theta(x)dx,$$
$$m,n \geq 0, \tag{10}$$

where δ_{mn} is the Kronecker delta

$$P_{m,n}^{2}(0,t) = r_1\int_0^\infty P_{m,n}^{(1)}(x,t)\mu_1(x)dx,\ m,n \geq 0, \tag{11}$$

$$V_{m,n}(0,t) = \theta(1-r_1)\int_0^\infty P_{m,n}^{(1)}(x,t)\mu_1(x)dx$$
$$+\theta\int_0^\infty P_{m,n}^{(2)}(x,t)\mu_2(x)dx,\ m,n \geq 0, \tag{12}$$

$$E_{m,n}(0,t) = r\int_0^\infty V_{m,n}(x,t)\gamma(x)dx,\ m,n \geq 0 \tag{13}$$

and at $y = 0$ and fixed values of x

$$D_{m,n}(x,0,t) = \alpha P_{m,n}^{(1)}(x,t) + \alpha P_{m,n}^{(2)}(x,t),\ m,n \geq 0, \tag{14}$$

$$R_{m,n}(x,0,t) = \int_0^\infty D_{m,n}(x,y,t)\xi(y)dy,\ m,n \geq 0 \tag{15}$$

and the normalization condition is

$$P_0+\sum_{n=1}^{\infty}\int_0^{\infty}P_{0,n}^{(0)}(x,t)dx+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}[\int_0^{\infty}P_{m,n}^{(1)}(x,t)dx+\int_0^{\infty}P_{m,n}^{(2)}(x,t)dx$$
$$+\int_0^{\infty}V_{m,n}(x,t)dx+\int_0^{\infty}V_{m,n}(x,t)dx+\int_0^{\infty}E_{m,n}(x,t)dx$$
$$+\int_0^{\infty}\int_0^{\infty}D_{m,n}(x,y,t)dydx+\int_0^{\infty}\int_0^{\infty}R_{m,n}(x,y,t)dydx]=1. \tag{16}$$

We assume that initially there are no customers in the system and the server is idle. So the initial conditions are

$$P_{m,n}^{(1)}(0)=P_{m,n}^{(2)}(0)=V_{m,n}(0)=E_{m,n}(0)=D_{m,n}(0)=R_{m,n}(0)=0$$
$$\text{and} P_0(0)=1. \tag{17}$$

Next, we define the following probability generating functions:

$$P_0(0,z_2)=\sum_{m=1}^{\infty}z_2^nP_{0,n}(0,t),\ A(z_1,z_2)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}z_1^mz_2^nA_{m,n}(x,t) \tag{18}$$

where $A=P^{(1)}, P^{(2)}, V, E, D, R$ and which are convergent inside the circle given by $|z_1|\leq 1, |z_2|\leq 1$. Taking Laplace transform of equations (1) to (16) and then solve it, we get

$$\overline{P}_0^{(0)}(x,z_2,s)=\overline{P}_0^{(0)}(0,z_2,s)\big[1-M(x)\big]e^{-(s+\lambda)x}, \tag{19}$$

$$\overline{P}^{(1)}(x,z_1,z_2,s)=\overline{P}^{(1)}(0,z_1,z_2,s)\big[1-B_1(x)]e^{-(\phi[z,s])x}, \tag{20}$$

$$\overline{P}^{(2)}(x,z_1,z_2,s)=\overline{P}^{(2)}(0,z_1,z_2,s)\big[1-B_2(x)]e^{-(\phi[z,s])x}, \tag{21}$$

$$\overline{V}(x,z_1,z_2,s)=\overline{V}(0,z_1,z_2,s)\big[1-V(x)\big]e^{-(b[z,s])x}, \tag{22}$$

$$\overline{E}(x,z_1,z_2,s)=\overline{E}(0,z_1,z_2,s)\big[1-E(x)\big]e^{-(b[z,s])x}, \tag{23}$$

$$\overline{D}(x,y,z_1,z_2,s)=\overline{D}(0,z_1,z_2,s)\big[1-D(y)\big]e^{-(b[z,s])y}, \tag{24}$$

$$\overline{R}(x,y,z_1,z_2,s)=\overline{R}(0,z_1,z_2,s)\big[1-R(y)\big]e^{-(b[z,s])y}, \tag{25}$$

where

$$
\begin{aligned}
a[z,s] &= s+\lambda-\lambda pC(z_1)-\lambda q+\lambda qb(1-C(z_2)),\\
b[z,s] &= s+\lambda-\lambda pC(z_1)-\lambda q+\lambda qb(1-C(z_2))+\delta-\delta\frac{p}{z_1}-\delta\frac{q}{z_2},\\
\phi[z,s] &= a[z,s]+\alpha(1-\overline{D}(b[z,s])\overline{R}(b[z,s]))\text{and}
\end{aligned}
$$

$$
\begin{aligned}
\overline{P}_0^{(0)}(0,z_2,s) = \overline{P}_0^{(1)}(0,z_2,s)\{&(1-r_1)\overline{B}_1(\phi_1[z,s])[1-\theta+\theta\overline{V}(b_1[z,s])\\
&\{1-r+r\overline{E}(b_1[z,s])\}]+r_1\overline{B}_1(\phi_1[z,s])\overline{B}_2(\phi_1[z,s])[1-\theta\\
&+\theta\overline{V}(b_1[z,s])\{1-r+r\overline{E}(b_1[z,s])\}]\}+1-(s+\lambda)\overline{P}_0(s), \qquad (26)
\end{aligned}
$$

$$
\begin{aligned}
&\overline{P}^{(1)}(0,z_1,z_2,s)\{z_1-\overline{B}_1(\phi[z,s])[1-\theta+\theta\overline{V}(b[z,s])\{1-r\\
&+r\overline{E}(b[z,s])\}][1-r_1+r_1\overline{B}_2(\phi[z,s])]\} = z_1\lambda\overline{P}_0(s)\left[p\frac{C(z_1)}{z_1}+q\frac{C(z_2)}{z_2}\right]\\
&+z_1\overline{P}_0^{(0)}(0,z_2,s)\left\{\lambda\frac{C(z_2)}{z_2}\left[\frac{1-\overline{M}(s+\lambda)}{(s+\lambda)}\right]+\frac{1}{z_2}\overline{M}(s+\lambda)\right\}\\
&-\overline{P}_0^{(1)}(0,z_2,s)\{\overline{B}_1(\phi_1[z,s])[1-\theta+\theta\overline{V}(b_1[z,s])\{1-r\\
&+r\overline{E}(b_1[z,s])\}][1-r_1+r_1\overline{B}_2(\phi_1[z,s])]\}. \qquad (27)
\end{aligned}
$$

Letting $z_1 = g(z_2)$ in (27) we get

$$
\begin{aligned}
&\overline{P}_0^{(1)}(0,z_2,s)\{\overline{B}_1(\phi_1[z,s])[1-\theta+\theta\overline{V}(b_1[z,s])\{1-r+r\overline{E}(b_1[z,s])\}]\\
&[1-r_1+r_1\overline{B}_2(\phi_1[z,s])]\} = g(z_2)\lambda\overline{P}_0(s)\left[p\frac{C(g(z_2))}{g(z_2)}+q\frac{C(z_2)}{z_2}\right]\\
&+g(z_2)\overline{P}_0^{(0)}(0,z_2,s)\left\{\lambda\frac{C(z_2)}{z_2}\left[\frac{1-\overline{M}(s+\lambda)}{(s+\lambda)}\right]+\frac{1}{z_2}\overline{M}(s+\lambda)\right\}, \qquad (28)
\end{aligned}
$$

where

$$
\begin{aligned}
a_1[z,s] &= s+\lambda-\lambda q+\lambda qb(1-C(z_2))\text{and}b_1[z,s]=s+\lambda-\lambda q+\lambda qb\\
&(1-C(z_2))+\delta-\delta\frac{q}{z_2},\ \phi_1[z,s]=a_1[z,s]+\alpha(1-\overline{D}(b_1[z,s])\overline{R}(b_1[z,s])),
\end{aligned}
$$

substitute (28) in (26) and (27) we get

$$
\begin{aligned}
&\overline{P}_0^{(0)}(0,z_2,s)\\
&=\frac{z_2-z_2(s+\lambda)\overline{P}_0(s)+z_2\lambda\overline{P}_0(s)pC(g(z_2)+\lambda qC(z_2)g(z_2)\overline{P}_0(s)}{z_2-g(z_2)\{C(z_2)[1-\overline{M}(s+\lambda)]+\overline{M}(s+\lambda)\}}, \qquad (29)
\end{aligned}
$$

$$\overline{P}^{(1)}(0, z_1, z_2, s) = \frac{\begin{array}{l} z_2\lambda\overline{P}_0(s)pC(z_1) + z_1\lambda\overline{P}_0(s)qC(z_2) - z_2\lambda\overline{P}_0(s)pC(g(z_2)) \\ -\ g(z_2)\lambda\overline{P}_0(s)qC(z_2) + \overline{P}_0^{(0)}(0, z_2, s)\left\{\lambda C(z_2\left[\frac{1-\overline{M}(s+\lambda)}{(s+\lambda)}\right]\right. \\ \left. +\ \overline{M}(s+\lambda)\}\{z_1 - g(z_2)\right\} \end{array}}{\begin{array}{l} z_2\{z_1 - \overline{B}_1(\phi[z,s])[1-\theta+\theta\overline{V}(b[z,s])\{1-r+r\overline{E}(b[z,s])\}] \\ \times\ [1-r_1+r_1\overline{B}_2(\phi[z,s])]\} \end{array}}, \tag{30}$$

where $\overline{P}_0^{(0)}(0, z_2, s)$ is given by (29)

$$\overline{P}^{(2)}(0, z_1, z_2, s) = r_1\overline{P}^{(1)}(0, z_1, z_2, s)\overline{B}_1(\phi[z, s]), \tag{31}$$

$$\overline{V}(0, z_1, z_2, s) = \theta\overline{P}^{(1)}(0, z_1, z_2, s)\overline{B}_1(\phi[z, s])[1 - r_1 + r_1\overline{B}_2(\phi[z, s])], \tag{32}$$

$$\begin{aligned}\overline{E}(0, z_1, z_2, s) = r\theta\overline{P}^{(1)}(0, z_1, z_2, s)\overline{B}_1(\phi[z, s])\overline{V}(b[z, s]) \\ [1 - r_1 + r_1\overline{B}_2(\phi[z, s])],\end{aligned} \tag{33}$$

$$\overline{D}(0, z_1, z_2, s) = \alpha[\overline{P}^{(1)}(z_1, z_2, s) + \overline{P}^{(2)}(z_1, z_2, s)], \tag{34}$$

$$\overline{R}(0, z_1, z_2, s) = \alpha[\overline{P}^{(1)}(z_1, z_2, s) + \overline{P}^{(2)}(z_1, z_2, s)]\overline{D}(b[z, s]). \tag{35}$$

Theorem 4.1. The inequality $\rho_1 - \rho_2 < 1$ is a necessary and sufficient condition for the system to be stable, under this condition the marginal PGF of the server's state, queue size and orbit size, distribution are given by

$$\overline{P}_0^{(0)}(z_2, s) = \overline{P}_0^{(0)}(0, z_2, s)\left[\frac{1 - \overline{M}(s + \lambda)}{(s + \lambda)}\right], \tag{36}$$

$$\overline{P}^{(1)}(z_1, z_2, s) = \overline{P}^{(1)}(0, z_1, z_2, s)\left[\frac{1 - \overline{B}_1(\phi[z, s])}{\phi[z, s]}\right], \tag{37}$$

$$\overline{P}^{(2)}(z_1, z_2, s) = r_1\overline{P}^{(1)}(0, z_1, z_2, s)\overline{B}_1(\phi[z, s])\left[\frac{1 - \overline{B}_2(\phi[z, s])}{\phi[z, s]}\right], \tag{38}$$

$$\overline{V}(z_1, z_2, s) = \theta\overline{P}^{(1)}(0, z_1, z_2, s)\overline{B}_1(\phi[z, s])[1 - r_1 + r_1\overline{B}_2(\phi[z, s])] \left[\frac{1 - \overline{V}(b[z, s])}{(b[z, s])}\right], \tag{39}$$

$$\overline{E}(z_1, z_2, s) = r\theta\overline{P}^{(1)}(0, z_1, z_2, s)\overline{B}_1(\phi[z, s])\overline{V}(b[z, s])[1 - r_1 + r_1\overline{B}_2(\phi[z, s])] \left[\frac{1 - \overline{E}(b[z, s])}{(b[z, s])])}\right], \tag{40}$$

$$\overline{D}(z_1, z_2, s) = \alpha[\overline{P}^{(1)}(z_1, z_2, s) + \overline{P}^{(2)}(z_1, z_2, s)] \left[\frac{1 - \overline{D}(b[z, s])}{(b[z, s])}\right], \tag{41}$$

$$\overline{R}(z_1, z_2, s) = \alpha[\overline{P}^{(1)}(z_1, z_2, s) + \overline{P}^{(2)}(z_1, z_2, s)]\overline{D}(b[z, s]) \left[\frac{1 - \overline{R}(b[z, s])}{(b[z, s])}\right]. \tag{42}$$

Proof. Integrating (19) to (23) with respect to x and using the well known result of renewal theory

$$\int_0^\infty \big[1 - H(x)\big]e^{-sx}dx = \frac{1 - \overline{h(s)}}{s}, \tag{43}$$

where $\overline{h(s)}$ is the LST of the distribution function of a random variable $H(x)$, we get the formulae (36) to (40), respectively. Similarly, integrating (24) and (25) with respect to y, we get

$$D(x, z) = \int_0^\infty D(x, y, z)dy = \frac{\alpha(\overline{P}^{(1)}(x, z) + \overline{P}^{(2)}(x, z))[1 - \overline{D}[b(z)]]}{b(z)}, \tag{44}$$

$$R(x, z) = \int_0^\infty R(x, y, z)dy = \frac{\alpha(\overline{P}^{(1)}(x, z) + \overline{P}^{(2)}(x, z))\overline{D}[b(z)][1 - \overline{R}[b(z)]]}{b(z)}. \tag{45}$$

Further integrating (44) and (45) with respect to x, we can get the formula (41) and (42), respectively. Thus we can obtain the complete solution for the probability generating function for the following states $\overline{P}_0^{(0)}(z_2, s)$, $\overline{P}^{(1)}(z_1, z_2, s)$, $\overline{P}^{(2)}(z_1, z_2, s)$, $\overline{V}(z_1, z_2, s)$, $\overline{E}(z_1, z_2, s)$, $\overline{E}(z_1, z_2, s)$, $\overline{D}(z_1, z_2, s)$ and $\overline{R}(z_1, z_2, s)$. ■

5. Steady state Analysis: Limiting Behaviour

In this section, we derive the steady state probability distribution for our queueing model. By applying the well-known Tauberian property,

$$\lim_{s\to 0} s\overline{f}(s) = \lim_{t\to\infty} f(t).$$

to the equations (36) to (42). In order to determine P_0 we use the normalizing condition from (16)

$$P_0 + P_0^0(1) + P^{(1)}(1,1) + P^{(2)}(1,1) + V(1,1) + E(1,1) + D(1,1) + R(1,1) = 1.$$

The steady state probability for priority and non-priority customers with second optional service, balking, Bernoulli and extended vacation for an unreliable server are

$$P_0^{(0)}(z_2) = P_0^{(0)}(0, z_2)\left[\frac{1-\overline{M}(\lambda)}{(\lambda)}\right], \tag{46}$$

$$P^{(1)}(z_1, z_2) = P^{(1)}(0, z_1, z_2)\left[\frac{1-\overline{B}_1(\phi[z])}{\phi[z]}\right], \tag{47}$$

$$P^{(2)}(z_1, z_2) = r_1 P^{(1)}(0, z_1, z_2)\overline{B}_1(\phi[z])\left[\frac{1-\overline{B}_2(\phi[z])}{\phi[z]}\right], \tag{48}$$

$$V(z_1, z_2) = \theta P^{(1)}(0, z_1, z_2)\overline{B}_1(\phi[z])[1 - r_1 + r_1\overline{B}_2(\phi[z])]\left[\frac{1-\overline{V}(b[z])}{(b[z])}\right], \tag{49}$$

$$\begin{aligned} E(z_1, z_2) &= r\theta P^{(1)}(0, z_1, z_2)\overline{B}_1(\phi[z])\overline{V}(b[z]) \\ &[1 - r_1 + r_1\overline{B}_2(\phi[z])]\left[\frac{1-\overline{E}(b[z])}{(b[z])])}\right], \end{aligned} \tag{50}$$

$$D(z_1, z_2) = \alpha[P^{(1)}(z_1, z_2) + P^{(2)}(z_1, z_2)]\left[\frac{1-\overline{D}(b[z])}{(b[z])}\right], \tag{51}$$

$$R(z_1, z_2) = \alpha[P^{(1)}(z_1, z_2) + P^{(2)}(z_1, z_2)]\overline{D}(b[z])\left[\frac{1-\overline{R}(b[z])}{(b[z])}\right], \tag{52}$$

where

$$P_0^{(0)}(0,z_2)=\frac{-z_2\lambda P_0+z_2\lambda P_0pC[g(z_2)]+\lambda qC(z_2)g(z_2)P_0}{z_2-g(z_2)\{C(z_2)[1-\overline{M}(\lambda)]+\overline{M}(\lambda)\}}, \tag{53}$$

$$P^{(1)}(0,z_1,z_2)=\frac{\begin{array}{l}z_2\lambda P_0pC(z_1)+z_1\lambda P_0qC(z_2)-z_2\lambda P_0pC[g(z_2)]\\ -\ g(z_2)\lambda P_0qC(z_2)+P_0^{(0)}(0,z_2)\{C(z_2)\\ +\ \overline{M}(\lambda)[1-C(z_2)]\}\{z_1-g(z_2)\}\end{array}}{z_2\{z_1-f(z)\}}, \tag{54}$$

$f(z)=\overline{B}_1(\phi[z])[1-\theta+\theta\overline{V}(b[z])\{1-r+r\overline{E}(b[z])\}][1-r_1+r_1\overline{B}_2(\phi[z])]$.
Let $P_q(z_1,z_2)$ be the probability generating function of the queue size irrespective of the state of the system

$$P_q(z_1,z_2)=P_0^{(0)}(z_2)+P^{(1)}(z_1,z_2)+P^{(2)}(z_1,z_2)+V(z_1,z_2)+E(z_1,z_2)+D(z_1,z_2)+R(z_1,z_2), \tag{55}$$

$$P_q(z_1,z_2)=\frac{\begin{array}{l}P_0^{(0)}(0,z_2)P_0\{[\frac{1-\overline{M}(\lambda)}{\lambda}]D(z)+\{C(z_2)+\overline{M}(\lambda)[1-C(z_2)]\}\\ \times\ \{z_1-g(z_2)\}N_r(z)\}+\{z_2\lambda P_0pC(z_1)+z_1\lambda P_0qC(z_2)\\ -\ z_2\lambda P_0pC(g(z_2))-g(z_2)\lambda P_0qC(z_2)\}N_r(z)\end{array}}{D(z)}, \tag{56}$$

where

$$N_r(z)=\{[1-\overline{B}_1(\phi[z])]+r_1\overline{B}_1(\phi[z])[1-\overline{B}_2(\phi[z])]\}\{b(z)+\alpha[1-\overline{D}(b(z))\overline{R}(b(z))]\}+\theta\phi(z)\overline{B}_1(\phi[z])[1-r_1+r_1\overline{B}_2(\phi[z])]\{[1-\overline{V}(b(z))]+r\overline{V}(b(z))[1-\overline{E}(b(z))]\},$$

$$D(z)=z_2\{z_1-f(z)\}\phi[z]b(z),$$

$$P_0=\frac{1-[\rho_1-\rho_2]}{D_r}. \tag{57}$$

$$D_r=[1-[\rho_1-\rho_2]]+P_0^{(0)}(0,1)\{[1-\overline{M}(\lambda)][1-[\rho_1-\rho_2]]+[1-g'(1)][(E(B_1)+r_1E(B_2))(1+\alpha[E(D)+E(R)])+\theta(E(V)+rE(E_v))]\}+\{\lambda[1+E(I)]-\lambda(p[1+E(I)g'(1)]+q[g'(1)+E(I)])\}\{(E(B_1)+r_1E(B_2))(1+\alpha[E(D)+E(R)])+\theta(E(V)+rE(E_v))\},$$

$$P_0^{(0)}(0,1) = \frac{q[g'(1)-1] + E(I)[pg'(1)+q]}{1-g'(1)-E(I)[1-\overline{M}(\lambda)]},$$

$$\rho_1 = E(I)[\lambda p + \lambda qb]\{(E(B_1)+r_1E(B_2))[1+\alpha(E(D)+E(R))] \\ +\theta(E(V)+rE(E_v))\},$$

$$\rho_2 = \delta\{\alpha(E(D)+E(R))(E(B_1)+r_1E(B_2)) + \theta(E(V)+rE(E_v))\}.$$

Equation (57) gives the probability that the server is idle. Substituting equation (57) in equation (56), we have completely and explicitly determined $P_q(z_1, z_2)$, the probability generating function of the queue size.

Theorem 5.1. If the system is in steady state condition, then we have

(i) The system is free with probability

$$P_0 = \frac{1-[\rho_1-\rho_2]}{D_r},$$

(ii) The system is occupied with probability

$$P_0^{(0)}(1) + P^{(1)}(1,1) + P^{(2)}(1,1) = P_0^{(0)}(0,1)\frac{[1-\overline{M}(\lambda)]}{\lambda} \\ +\frac{[E(B_1)+r_1E(B_2)]\{\lambda P_0[1+E(I)] - \lambda P_0(p[1+E(I)g'(1)]}{[1-\rho_1+\rho_2]} \\ +\frac{q[g'(1)+E(I)]) + P_0^{(0)}(0,1)[1-g'(1)]\}}{[1-\rho_1+\rho_2]},$$

(iii) The server is idle with probability

$$P_0 + P_0^{(0)}(1) = 1 \\ -\frac{\{[E(B_1)+r_1E(B_2)][1+\alpha(E(D)+E(R))] + \theta[E(V)+rE(E_v)]\}}{[1-\rho_1+\rho_2]} \\ \times\{\lambda P_0[1+E(I)] - \lambda P_0(p[1+E(I)g'(1)] + q[g'(1)+E(I)]) \\ +P_0^{(0)}(0,1)[1-g'(1)]\},$$

(iv) The server is busy with probability

$$P^{(1)}(1,1) + P^{(2)}(1,1) \\ = \frac{[E(B_1)+r_1E(B_2)]\{\lambda P_0[1+E(I)] - \lambda P_0(p[1+E(I)g'(1)]}{[1-\rho_1+\rho_2]} \\ +\frac{q[g'(1)+E(I)]) + P_0^{(0)}(0,1)[1-g'(1)]\}}{[1-\rho_1+\rho_2]},$$

(v) The server is waiting for repair with probability

$$D(1,1) = \frac{\alpha E(D)[E(B_1) + r_1 E(B_2)]\{\lambda P_0[1 + E(I)] - \lambda P_0(p[1 + E(I)g'(1)]}{[1 - \rho_1 + \rho_2]} + \frac{q[g'(1) + E(I)]) + P_0^{(0)}(0,1)[1 - g'(1)]\}}{[1 - \rho_1 + \rho_2]},$$

(vi) The server is under repair with probability

$$R(1,1) = \frac{\alpha E(R)[E(B_1) + r_1 E(B_2)]\{\lambda P_0[1 + E(I)] - \lambda P_0(p[1 + E(I)g'(1)]}{[1 - \rho_1 + \rho_2]} + \frac{q[g'(1) + E(I)]) + P_0^{(0)}(0,1)[1 - g'(1)]\}}{[1 - \rho_1 + \rho_2]},$$

(vii) The server is on vacation with probability

$$V(1,1) = \frac{\theta E(V)\{\lambda P_0[1 + E(I)] - \lambda P_0(p[1 + E(I)g'(1)]}{[1 - \rho_1 + \rho_2]} + \frac{q[g'(1) + E(I)]) + P_0^{(0)}(0,1)[1 - g'(1)]\}}{[1 - \rho_1 + \rho_2]},$$

(viii) The server is on extended vacation with probability

$$E(1,1) = \frac{\theta r E(E_v)\{\lambda P_0[1 + E(I)] - \lambda P_0(p[1 + E(I)g'(1)] + q[g'(1) + E(I)])}{[1 - \rho_1 + \rho_2]} + \frac{P_0^{(0)}(0,1)[1 - g'(1)]\}}{[1 - \rho_1 + \rho_2]}.$$

Proof. Note that

$$P_0^{(0)}(1) + P^{(1)}(1,1) + P^{(2)}(1,1) = \lim_{\substack{z_1 \to 1 \\ z_2 \to 1}} [P_0^{(0)}(z_2) + P^{(1)}(z_1, z_2) + P^{(2)}(z_1, z_2)],$$

$$P_0 + P_0^{(0)}(1) = P_0 + \lim_{z_2 \to 1} P_0^{(0)}(z_2),$$

$$P^{(1)}(1,1) + P^{(2)}(1,1) = \lim_{\substack{z_1 \to 1 \\ z_2 \to 1}} [P^{(1)}(z_1, z_2) + P^{(2)}(z_1, z_2)],$$

$$D(1,1) = \lim_{\substack{z_1 \to 1 \\ z_2 \to 1}} D(z_1, z_2),\ R(1,1) = \lim_{\substack{z_1 \to 1 \\ z_2 \to 1}} R(z_1, z_2),$$

$$V(1,1)=\lim_{\substack{z_1\to 1\\ z_2\to 1}} V(z_1,z_2)\text{and}$$

$$E(1,1)=\lim_{\substack{z_1\to 1\\ z_2\to 1}} E(z_1,z_2)$$

by direct calculation we get the above formulae. ■

Theorem 5.2. The availability of the server and failure of the server under the steady state condition are given by

$$\begin{aligned}A_v = 1-&\frac{\{\alpha[E(B_1)+r_1E(B_2)][E(D)+E(R)]+\theta[E(V)+rE(E_v)]\}}{[1-\rho_1+\rho_2]}\\ &\times\{\lambda P_0[1+E(I)]-\lambda P_0(p[1+E(I)g'(1)]+q[g'(1)+E(I)])\\ &+P_0^{(0)}(0,1)[1-g'(1)]\}\end{aligned}$$

and

$$M_f=\frac{\alpha[E(B_1)+r_1E(B_2)]\{\lambda P_0[1+E(I)]-\lambda P_0(p[1+E(I)g'(1)]}{[1-\rho_1+\rho_2]}$$

$$+\frac{q[g'(1)+E(I)])+P_0^{(0)}(0,1)[1-g'(1)]\}}{[1-\rho_1+\rho_2]}.$$

Proof. By considering the following equations we get the results

$$A_v=P_0+\lim_{\substack{z_1\to 1\\ z_2\to 1}}[P^{(1)}(z_1,z_2)+P^{(2)}(z_1,z_2)]$$

and

$$M_f=\alpha[P^{(1)}(1,1)+P^{(2)}(1,1)].$$

■

6. The Average Queue Length

The mean number of customers in priority queue under the steady state is

$$L_{q_1}=\frac{d}{dz_1}P_{q_1}(z_1,1)|_{z_1=1}$$

and the mean number of customers in the orbit under the steady state is

$$L_{q_2}=\frac{d}{dz_2}P_{q_2}(1,z_2)|_{z_2=1},$$

then

$$L_{q_1} = \frac{D_1'''(1)[N_1''''(1) + N_2''''(1)] - D_1''''(1)[N_1'''(1) + N_2'''(1)]}{4(D_1'''(1))^2}, \tag{58}$$

$$L_{q_2} = \frac{D_2'''(1)[N_3''''(1) + N_4''''(1)] - D_2''''(1)[N_3'''(1) + N_4'''(1)]}{4(D_2'''(1))^2} \tag{59}$$

where

$$N_1'''(1) = P_0^{(0)}(0,1)\{[\frac{1-\overline{M}(\lambda)}{\lambda}]D_1'''(1) + 3N_{r1}''(1)\},$$

$$N_1''''(1) = P_0^{(0)}(0,1)\{[\frac{1-\overline{M}(\lambda)}{\lambda}]D_1''''(1) + 4N_{r1}'''(1)\},$$

$$N_2'''(1) = 3N_{r1}''(1)\lambda P_0[pE(I) + q],$$

$$N_2''''(1) = 4N_{r1}'''(1)\lambda P_0[pE(I) + q] + 6N_{r1}''(1)\lambda P_0 pE(I[I-1]),$$

$$D_1'''(1) = 6\{\lambda pE(I) - \delta p\}\{\lambda pE(I)[1 + \alpha[E(D) + E(R)]] - \alpha\delta p[E(D) + E(R)]\}\{1 - [\rho_1 - \rho_2]\},$$

$$\begin{aligned} D_1''''(1) = {} & -12\{\lambda pE(I) - \delta p\}\{\lambda pE(I)[1 + \alpha[E(D) + E(R)]] - \alpha\delta p[E(D) \\ & + E(R)]\}\{\{\lambda pE(I)[1 + \alpha[E(D) + E(R)]] - \alpha\delta p[E(D) + E(R)]\}^2[E(B_1^2) \\ & + 2r_1E(B_1)E(B_2) + r_1E(B_2^2)] + \{\lambda pE(I) + \alpha[(\lambda pE(I) - \delta p)^2[E(D^2) + 2 \\ & E(D)E(R) + E(R^2)] + [\lambda pE(I[I-1]) + 2\delta p][E(D) + E(R)]]\}(E(B_1) \\ & + r_1E(B_2)) + \theta(\lambda pE(I) - \delta p)^2[E(V^2) + 2rE(V)E(E_v) + rE(E_v^2)] \\ & + \theta[\lambda pE(I[I-1]) + 2\delta p][E(V) + rE(E_v)] + 2\theta(\lambda pE(I) - \delta p)\{\lambda pE(I) \\ & [1 + \alpha[E(D) + E(R)]] - \alpha\delta p[E(D) + E(R)]\}[E(V) + rE(E_v)](E(B_1) \\ & + r_1E(B_2))\} + 12\{\{\lambda pE(I) - \delta p\}\{\lambda pE(I[I-1]) + \alpha[(\lambda pE(I) - \delta p)^2 \\ & [E(D^2) + 2E(D)E(R) + E(R^2)] + [\lambda pE(I[I-1]) + 2\delta p][E(D) + E(R)]]\} \\ & + \{\lambda pE(I)[1 + \alpha[E(D) + E(R)]] - \alpha\delta p[E(D) + E(R)]\}[\lambda pE(I[I-1]) \\ & + 2\delta p]\}\{1 - [\rho_1 - \rho_2]\}, \end{aligned}$$

$$\begin{aligned} N_{r1}''(1) = {} & 2\{\lambda pE(I) - \delta p\}\{\lambda pE(I)[1 + \alpha[E(D) + E(R)]] - \alpha\delta p[E(D) \\ & + E(R)]\}\{(E(B_1) + r_1E(B_2))[1 + \alpha[E(D) + E(R)]] \\ & + \theta[E(V) + rE(E_v)]\}, \end{aligned}$$

$$N_{r1}'''(1) = 3\{\lambda pE(I) - \delta p\}\big[\{\lambda pE(I)[1+\alpha[E(D)+E(R)]] - \alpha\delta p[E(D) +E(R)]\}^2[E(B_1^2) + 2r_1E(B_1)E(B_2) + r_1E(B_2^2)] + \{\lambda pE(I[I-1]) +\alpha[(\lambda pE(I)-\delta p)^2[E(D^2)+2E(D)E(R)+E(R^2)] + [\lambda pE(I[I-1]) +2\delta p][E(D)+E(R)]]\}(E(B_1)+r_1E(B_2))[1+\alpha[E(D)+E(R)]]\big] + 3 \{\lambda pE(I)[1+\alpha[E(D)+E(R)]] - \alpha\delta p[E(D)+E(R)]\}(E(B_1)+r_1E(B_2)) [\{\lambda pE(I[I-1]) + 2\delta p\} + \alpha[(\lambda pE(I)-\delta p)^2[E(D^2)+2E(D)E(R)+E(R^2)] +[\lambda pE(I[I-1]) + 2\delta p][E(D)+E(R)]]] + 3\theta\{\lambda pE(I)-\delta p\}\{\lambda pE(I[I-1]) +\alpha[(\lambda pE(I)-\delta p)^2[E(D^2)+2E(D)E(R)+E(R^2)] + [\lambda pE(I[I-1]) +2\delta p][E(D)+E(R)]]\}[E(V)+rE(E_v)] + 6\theta\{\lambda pE(I)-\delta p\}\{\lambda pE(I) [1+\alpha[E(D)+E(R)]] - \alpha\delta p[E(D)+E(R)]\}^2[E(V)+rE(E_v)](E(B_1) +r_1E(B_2)) + 3\theta\{\lambda pE(I)[1+\alpha[E(D)+E(R)]] - \alpha\delta p[E(D)+E(R)]\} \{(\lambda pE(I)-\delta p)^2[E(V^2)+2rE(V)E(E_v)+rE(E_v^2)] + [\lambda pE(I[I-1]) +2\delta p][E(V)+rE(E_v)]\},$$

$$N_3'''(1) = P_0^{(0)}(0,1)\{[\frac{1-\overline{M}(\lambda)}{\lambda}]D_2'''(1) - 3g'(1)N_{r2}''(1)\},$$

$$N_3''''(1) = 4P_0^{(0)'}(0,1)\{[\frac{1-\overline{M}(\lambda)}{\lambda}]D_2'''(1) - 3g'(1)N_{r2}''(1)\} + P_0^{(0)}(0,1) \{[\frac{1-\overline{M}(\lambda)}{\lambda}]D_2''''(1) - 12E(I)[1-\overline{M}(\lambda)]g'(1)N_{r2}''(1) -6g''(1)N_{r2}''(1) - 4g'(1)N_{r2}'''(1)\},$$

$$N_4'''(1) = 3N_{r2}''(1)\{\lambda P_0[p+qE(I)] - \lambda P_0[p+pE(I)g'(1)+qg'(1)+qE(I)]\},$$

$$N_4''''(1) = 4N_{r2}'''(1)\lambda P_0[p+qE(I)] - \lambda P_0[p+pE(I)g'(1)+qg'(1)+qE(I)] +6N_2''(1)\{\lambda P_0qE(I[I-1]) - \lambda P_0[2pE(I)g'(1) + p(E(I[I-1])[g'(1)]^2 +E(I)g''(1)) + g''(1)q + 2qg'(1)E(I) + qE(I[I-1])]\},$$

$$D_2'''(1) = -6\{\lambda qbE(I) - \delta q\}\{\lambda qbE(I)[1+\alpha[E(D)+E(R)]] - \alpha\delta q[E(D) +E(R)]\}\{\rho_1 - \rho_2\},$$

$$
\begin{aligned}
D_2''''(1) = {} & -12\{\lambda qbE(I)-\delta q\}\{\lambda qbE(I)[1+\alpha[E(D)+E(R)]]-\alpha\delta q[E(D)\\
&+E(R)]\}\{2[\rho_1-\rho_2]+\{\lambda qbE(I)[1+\alpha[E(D)+E(R)]]-\alpha\delta q[E(D)\\
&+E(R)]\}^2[E(B_1^2)+2r_1E(B_1)E(B_2)+r_1E(B_2^2)]+\{\lambda qbE(I)+\alpha\\
&[(\lambda qbE(I)-\delta q)^2[E(D^2)+2E(D)E(R)+E(R^2)]+[\lambda qbE(I[I-1])\\
&+2\delta q][E(D)+E(R)]]\}(E(B_1)+r_1E(B_2))+\theta(\lambda qbE(I)-\delta q)^2[E(V^2)\\
&+2rE(V)E(E_v)+rE(E_v^2)]+\theta[\lambda qbE(I[I-1])+2\delta q][E(V)+rE(E_v)]\\
&+2\theta(\lambda qbE(I)-\delta q)\{\lambda qbE(I)[1+\alpha[E(D)+E(R)]]-\alpha\delta q[E(D)+E(R)]\}\\
&[E(V)+rE(E_v)](E(B_1)+r_1E(B_2))\}-12\{\{\lambda qbE(I)-\delta q\}\{\lambda qbE(I[I-1])\\
&+\alpha[(\lambda qbE(I)-\delta q)^2[E(D^2)+2E(D)E(R)+E(R^2)]+[\lambda qbE(I[I-1])\\
&+2\delta q][E(D)+E(R)]]\}+\{\lambda qbE(I)[1+\alpha[E(D)+E(R)]]-\alpha\delta q[E(D)\\
&+E(R)]\}[\lambda qbE(I[I-1])+2\delta q]\}\{[\rho_1-\rho_2]\},
\end{aligned}
$$

$$
\begin{aligned}
N_{r2}''(1) = {} & 2\{\lambda qbE(I)-\delta q\}\{\lambda qbE(I)[1+\alpha[E(D)+E(R)]]-\alpha\delta q[E(D)\\
&+E(R)]\}\{(E(B_1)+r_1E(B_2))[1+\alpha[E(D)+E(R)]]\\
&+\theta[E(V)+rE(E_v)]\},
\end{aligned}
$$

$$
\begin{aligned}
N_{r2}'''(1) = {} & 3\{\lambda qbE(I)-\delta q\}\big[\{\lambda qbE(I)[1+\alpha[E(D)+E(R)]]-\alpha\delta q[E(D)\\
&+E(R)]\}^2[E(B_1^2)+2r_1E(B_1)E(B_2)+r_1E(B_2^2)]+\{\lambda qbE(I[I-1])\\
&+\alpha[(\lambda qbE(I)-\delta q)^2[E(D^2)+2E(D)E(R)+E(R^2)]+[\lambda qbE(I[I-1])\\
&+2\delta q][E(D)+E(R)]]\}(E(B_1)+r_1E(B_2))[1+\alpha[E(D)+E(R)]]\big]+3\\
&\{\lambda qbE(I)[1+\alpha[E(D)+E(R)]]-\alpha\delta q[E(D)+E(R)]\}(E(B_1)+r_1\\
&E(B_2))[\{\lambda qbE(I[I-1])+2\delta q\}+\alpha[(\lambda qbE(I)-\delta q)^2[E(D^2)+2E(D)\\
&E(R)+E(R^2)]+[\lambda qbE(I[I-1])+2\delta q][E(D)+E(R)]]]+3\theta\{\lambda qbE(I)\\
&-\delta q\}\{\lambda qbE(I[I-1])+\alpha[(\lambda qE(I)-\delta q)^2[E(D^2)+2E(D)E(R)+E(R^2)]\\
&+[\lambda qbE(I[I-1])+2\delta q][E(D)+E(R)]]\}[E(V)+rE(E_v)]+6\theta\{\lambda qbE(I)\\
&-\delta q\}\{\lambda qbE(I)[1+\alpha[E(D)+E(R)]]-\alpha\delta q[E(D)+E(R)]\}^2[E(V)+r\\
&E(E_v)](E(B_1)+r_1E(B_2))+3\theta\{\lambda qbE(I)[1+\alpha[E(D)+E(R)]]-\alpha\delta q\\
&[E(D)+E(R)]\}\{(\lambda qbE(I)-\delta q)^2[E(V^2)+2rE(V)E(E_v)+rE(E_v^2)]\\
&+[\lambda qbE(I[I-1])+2\delta q][E(V)+rE(E_v)]\}.
\end{aligned}
$$

7. The Average Waiting Time in the Queue

Average waiting time of a customers in the priority queue is

$$W_{q_1}=\frac{L_{q_1}}{\lambda p}. \tag{60}$$

Average waiting time of a customers in the orbit is

$$W_{q_2} = \frac{L_{q_2}}{\lambda q b}, \tag{61}$$

where L_{q_1} and L_{q_2} have been found in equations (58) and (59).

8. Particular Case

Case: I

If we let no priority arrival, no balking, no stand by server, no second optional service and no extended vacation, i.e. $q = 1, z_2 = z, p = 0, b = 1, \delta = 0, r_1 = 0, r = 0$ then our model reduces to $M^{[X]}/G/1$ retrial queueing system with general retrial times under Bernoulli vacation for unreliable server and delaying repair.

$$P^{(0)}(z) = \frac{P_0\{z - C(z)\overline{B}_1(\phi_1[z])[1-\theta+\theta\overline{V}(b_1[z])]\}[1-\overline{M}(\lambda)]}{\overline{B}_1(\phi_1[z])[1-\theta+\theta\overline{V}(b_1[z])]\{C(z)[1-\overline{M}(\lambda)]+\overline{M}(\lambda)\} - z},$$

$$P^{(1)}(z) = \frac{\lambda P_0\overline{M}(\lambda)[1-C(z)]\{1-\overline{B}_1(\phi_1[z]\}}{[\overline{B}_1(\phi_1[z])[1-\theta+\theta\overline{V}(b_1[z])]\{C(z)[1-\overline{M}(\lambda)]+\overline{M}(\lambda)\} - z]\phi_1[z]},$$

$$V(z) = \frac{\theta\lambda P_0\overline{M}(\lambda)[1-C(z)]\overline{B}_1(\phi_1[z]\{1-\overline{V}(b_1(z))\}}{[\overline{B}_1(\phi_1[z])[1-\theta+\theta\overline{V}(b_1[z])]\{C(z)[1-\overline{M}(\lambda)]+\overline{M}(\lambda)\} - z]b_1(z)},$$

$$D(z) = \frac{\alpha P_0\overline{M}(\lambda)\{1-\overline{B}_1(\phi_1[z])\}\{1-\overline{D}(b_1(z))\}}{[\overline{B}_1(\phi_1[z])[1-\theta+\theta\overline{V}(b_1[z])]\{C(z)[1-\overline{M}(\lambda)]+\overline{M}(\lambda)\} - z]\phi_1[z]},$$

$$R(z) = \frac{\alpha P_0\overline{M}(\lambda)\{1-\overline{B}_1(\phi_1[z])\}\overline{D}(\phi_1[z])\{1-\overline{R}(b_1(z))\}}{[\overline{B}_1(\phi_1[z])[1-\theta+\theta\overline{V}(b_1[z])]\{C(z)[1-\overline{M}(\lambda)]+\overline{M}(\lambda)\} - z]\phi_1[z]}.$$

This result is coincide with Gautam Choudhury, Jau-Chaun Ke. (5)

Case: II

If we let no priority arrival, no balking, no stand by server, no extended vacation, no breakdown and $\overline{M}(\lambda) \to 1$ i.e. $q = 1, z_2 = z, p = 0, b = 1, \delta = 0, r = 0, \alpha = 0$ then our model reduces to $M^{[X]}/G/1$ queueing system with second optional service and Bernoulli vacation.

$$P^{(1)}(z) = \frac{P_0\{1-\overline{B}_1(a_1[z]\}}{[\overline{B}_1(a_1[z])[1-\theta+\theta\overline{V}(a_1[z])] - z]},$$

$$P^{(2)}(z) = \frac{r_1 P_0\overline{B}_1(a_1[z])\{1-\overline{B}_2(a_1[z])\}}{[\overline{B}_1(a_1[z])[1-\theta+\theta\overline{V}(a_1[z])] - z]},$$

$$V(z) = \frac{\theta P_0\overline{B}_1(a_1[z])[1-r_1+r_1\overline{B}_2(a_1[z])]\{1-\overline{V}(a_1(z))\}}{[\overline{B}_1(a_1[z])[1-\theta+\theta\overline{V}(a_1[z])] - z]}.$$

In this case if $C(z) = z, \theta = 0$ then this model is coincide with Al-Jaraha and Madan (1).

Case: III

If we let no priority arrival, no balking, no stand by server, no second optional service and no extended vacation, no breakdown and no vacation i.e.$q = 1, z_2 = z, p = 0, b = 1, \delta = 0, r_1 = 0, r = 0,$
$\alpha = 0$ and $\theta = 0$ then our model reduces to $M^{[X]}/G/1$ retrial queueing system.

$$P^{(0)}(z) = \frac{P_0\{z - C(z)\overline{B}_1(a_1[z])\}[1 - \overline{M}(\lambda)]}{\overline{B}_1(a_1[z])\{C(z)[1 - \overline{M}(\lambda)] + \overline{M}(\lambda)\} - z},$$
$$P^{(1)}(z) = \frac{P_0\overline{M}(\lambda)\{1 - \overline{B}_1(a_1[z]\}}{[\overline{B}_1(a_1[z])\{C(z)[1 - \overline{M}(\lambda)] + \overline{M}(\lambda)\} - z]}.$$

In this case if $C(z) = z$, then this model is coincide with Comez corral (6).

9. Numerical Analysis

The above queueing model is analysed numerically with the following assumptions.

1. Service time distribution for essential service follows exponential distribution the mean service rate is $\mu_1 = 10$.
2. service time for the second optional service follows exponential distribution with $r_1 = 0.5$ and $\mu_2 = 8$.
3. The service time of stand by server follows exponential distribution with parameter $\delta = 7$.
4. Retrial time follows exponential distribution with parameter $\nu = 5$ and $\overline{M}(\lambda) = [\frac{\nu}{\nu + \lambda}]$.
5. If the arriving customers join the priority queue with probability $p = 0.8$
6. If the arriving customers join the non-priority queue(orbit) with probability $q = 0.2$
7. $g'(1) = [\frac{\lambda q b}{\mu_1 - \lambda p}], g''(1) = [\frac{2\lambda q b \mu_1}{(\mu_1 - \lambda p)^2}]$.
8. Vacation time follows exponential distribution with $\theta = 7$ and $\gamma = 3$.
9. Extended vacation time follows exponential distribution with $r = 0.3$ and $\theta_v = 6$.
10. Random Breakdown follows exponential distribution with parameter $\alpha = 2$.
11. Delay time to start repair follows exponential distribution with $\xi = 2$.

12. Repair time follows exponential distribution with $\beta = 6$.

13. Arriving low priority customers may balk the orbit with probability $b = 0.3$.

14. Arrivals are single and the average arrival rate ranging from $\lambda = 0.1$ to 1.0.

Table 1: Effect of λ on various queue characteristics

λ	P_0	L_{q1}	L_{q2}	W_{q1}	W_{q2}
0.1	0.9520	0.0035	0.0598	0.0440	9.9740
0.2	0.9383	0.0066	0.0603	0.0411	5.0220
0.3	0.9246	0.0100	0.0610	0.0416	3.3883
0.4	0.9110	0.0138	0.0620	0.0430	2.5836
0.5	0.8975	0.0179	0.0633	0.0448	2.1102
0.6	0.8841	0.0225	0.0649	0.0468	1.8021
0.7	0.8707	0.0274	0.0667	0.0489	1.5881
0.8	0.8573	0.0327	0.0688	0.0512	1.4327
0.9	0.8441	0.0385	0.0711	0.0535	1.3161
1.0	0.8309	0.0448	0.0736	0.0560	1.2266

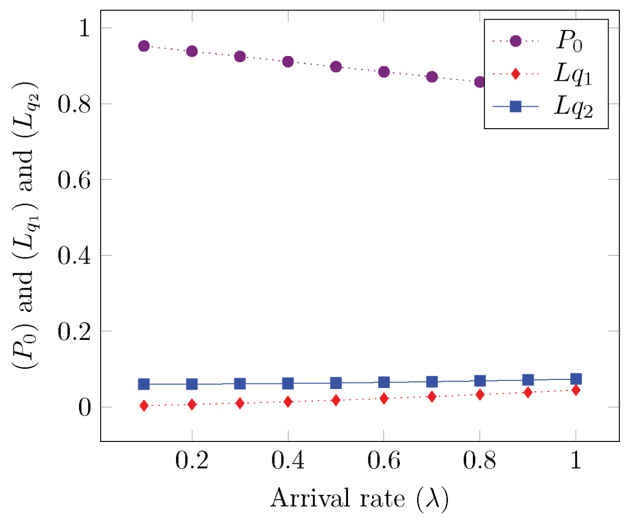

Figure 1: Average queue lengths of priority and non-priority customers verses arrival rate λ

In this case increasing the arrival rate the idle time decreases as well as the queue size and orbit size increases in table 1 and graph 1.

15. In table 2, increasing the essential service rate $\mu_1 = 1, 2, 3$. Then idle time increases as well as decreases the queue size and orbit size it also shown in graph 2.

Results are presented for the values of λ and μ_1 in the following tables with their corresponding graphical representations.

Table 2: Effect of μ_1 on various queue characteristics

μ_1	P_0	L_{q_1}	L_{q_2}	W_{q_1}	W_{q_2}
1	0.0852	3.5371	0.0904	0.7369	0.2511
2	0.1460	2.3353	0.0586	0.4865	0.1627
3	0.1673	1.7090	0.0175	0.3561	0.0487

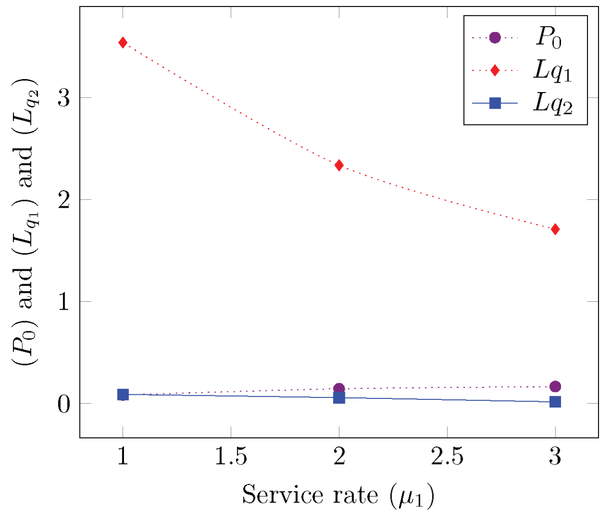

Figure 2: Average queue lengths of priority and non-priority customers verses essential service rate μ_1

10. Conclusion

In this paper we discussed the transient analysis of batch arrival retrial queueing system with general retrial times, priority service, balking, second optional service, Bernoulli vacation, extended vacation, breakdown, delayed repair and stand by server. The server provides two types of service namely priority and non-priority under non-preemptive priority rule. We derived the probability generating functions of the number of customers in the priority queue and non-priority customers in the orbit are found by using the supplementary variable technique. Average queue size, the average waiting time for the priority and non-priority customers and numerical results are also obtained.

References

[1] Al-Jaraha, J. and Madan, K., An $M/G/1$ queue with second optional service with general service time distribution: *International Journal of Information and Management Sciences,* Vol. 14, No. 2, (2003), pp. 47–56.

[2] Ayyappan, G. and A. Muthu Ganapathi Subramanian, Single server Retrial queueing system with stand by server under pre-emptive priority service: *International Journal of Computational and Applied Mathematics*, Volume 5, No 4, (2010), pp. 487–499.

[3] Ayyappan, G. and S. Shyamala, $M^{[X]}/G/1$ Retrial Queueing system with Second Optional Service, Random breakdown, Set up time and Bernoulli vacation: *Int. J. Open Problems Compt. Math.*, Vol. 7, No. 1, (2014).

[4] Atencia, I., Moreno, P., A single-server retrial queue with general retrial times and Bernoulli schedule: *Applied Mathematics and Computation*, 162, (2005), 855–880.

[5] Gautam Choudhury, Jau-Chuan Ke, A batch arrival retrial queue with general retrial times under Bernoulli vacation schedule for unreliable server and delaying repair: *Applied Mathematical Modelling*, 36, (2012), 255–269.

[6] Gomez Corral, A., Stochastic analysis of a single server retrial queue with general retrial timesP: *Nav. Logis.*, vol. 46, (1999), pp. 561–581.

[7] Jain, M. and Charu Bhargava, Bulk Arrival Retrial Queue with Unreliable Server and Priority Subscribers: *International Journal of Operations Research*, Vol. 5, No. 4, (2008), 242–259.

[8] Jau-Chuan Ke, Fu-Min Chang, Modified vacation policy for $M/G/1$ retrial queue with balking and feedback: *Computers and Industrial Engineering*, 57, (2009), 433–443.

[9] Jinbiao Wu, Zhaotong Lian, A single-server retrial G-queue with priority and unreliable server under Bernoulli vacation schedule: *Computers & Industrial Engineering*, 64, (2013), 84–93.

[10] Khalaf, R.F., Queueing system with four different main server's interruptions and a stand-by-server: *International Journal of Statics and Probability*, Vol. 3, No. 1 (2014).

[11] Rajadurai, P., Saravanarajan, M.C., Chandrasekaran, V.M., Analysis of an $M^{[X]}/(G_1, G_2)/1$ retrial queueing system with balking, optional re-service under modified vacation policy and service interruption: *Ain Shams Engineering Journal*, 5, (2014), 935–950.