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Oscillation of a Functional Differential Equations of Second Order

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Abstract

In this work, we establish the oscillation criteria for second order nonlinear neutral delay differential equations of the form:

$$(r(t)(x(t) + p(t)x(\tau(t)))')' + q(t)G(x(\sigma(t))) + v(t)H(x(\eta(t))) = 0$$

under the assumption that

$$\int_0^\infty \frac{ds}{r(s)} = \infty,$$

$$\int_0^\infty \frac{ds}{r(s)} < \infty,$$

various ranges of $p(t)$.

AMS subject classification: 34K, 34C10.
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1. Introduction

Consider the nonlinear neutral delay differential equations of the form:

$$(r(t)(x(t) + p(t)x(\tau(t)))')' + q(t)G(x(\sigma(t))) + v(t)H(x(\eta(t))) = 0, \tag{1}$$

where $r,\ q,\ v,\ \tau,\ \sigma, \eta \in C(\mathbb{R}_+, \mathbb{R}_+),\ p \in C(\mathbb{R}_+, \mathbb{R}),\ G \in C(\mathbb{R}, \mathbb{R})$ such that $xG(x) > 0, xH(x) > 0$ for $x \neq 0$ and $\tau(t) \leq t,\ \sigma(t) \leq t,\ \eta(t) \leq t$ with $\lim_{t\to\infty} \tau(t) = \infty = \lim_{t\to\infty} \sigma(t) = \infty = \lim_{t\to\infty} \eta(t)$ and $G,\ H \in C(\mathbb{R}, \mathbb{R})$ satisfying the property $xG(x) > 0,\ \ uH(u) > 0$ for $x, u \neq 0$.

In this work, our objective is to establish the sufficient condition results for oscillation of all solution of (1) without the comparison results under the assumption

$$(A_0)\ \int_0^\infty \frac{ds}{r(s)} = \infty,$$

$$(A_1)\ \int_0^\infty \frac{ds}{r(s)} < \infty,$$

for all ranges of $p(t)$ with $|p(t)| < \infty$.

Baculikova et al. [5], have studied the oscillatory behavior of solutions of

$$(r(t)(x(t) + p(t)x(t - \tau(t)))')' + q(t)(x(\sigma(t)) = 0,$$

and they have established the sufficient conditions for oscillation by means of comparison results.

Baculikova et al. [7], have studied the oscillatory behavior of solutions of

$$(r(t)(x(t) + p(t)x(\tau(t)))')' + q(t)(x(\sigma(t))) + v(t)(x(\eta(t))) = 0,$$

they have established the sufficient conditions for oscillation by means comparison results.

Baculikova et al. [6], have studied the oscillation the oscillatory behavior of solutions of

$$(a(t)[z'(t)]^\gamma + q(t)x^\beta(\sigma(t))) = 0,$$

where $z(t) = x(t) + p(t)x(\tau(t))$.

(a) $\gamma,\ \beta$ are the ratio of two odd positive integers,

(b) $\sigma(t) \leq t,\ \sigma(t)$ nondreasing,

(c) $a,\ p\ q$, are positive $o \leq p(t) \leq p_0 < \infty,\ \lim_{t\to\infty} \tau(t) = \infty = \lim_{t\to\infty} \sigma(t),\ \tau'(t) \geq \tau_0 > 0$ and $\tau o \sigma = \sigma o \tau$.

They have established the sufficient conditions for oscillation by means comparison results.

In this work, sufficient conditions are obtained for oscillation of all solutions of (1) without using the comparison results.

This work would be interesting, than the works [1], [3], [8], [16] and [17], and the references cited therein.

Neutral delay differential equations find numerous applications in electric network. For example, they are frequently used for the study of distributed networks containing lossless transmission lines which arises in high speed computers where the lossless transmission lines are used to interconnect switching circuits(see for example [13]). The problem of obtaining sufficient conditions to ensure the second order differential equations which are special cases of (1) is oscillatory has received a great attention.

Since the second order equations have the applied applications there is the permanent interest in obtaining new sufficient conditions for the oscillation or nonoscillation of the solution of varietal type of the second order equations.

By a solution of (1), we mean a continuously differentiable function $x(t)$ which is defined for $t \geq T^* = \min\{\tau(t_0),\ \sigma(t_0),\ \eta(t_0)\}$ such that $x(t)$ satisfies (1) for all $t \geq t_0$. In the sequel, it will always be assumed that the solutions of (1) exist on some half line $[t_1, \infty)$, $t_1 \geq t_0$. A solution of (1) is said to be oscillatory, if it has arbitrarily large zeros; otherwise, it is called nonoscillatory. Equation (1) is called oscillatory, if all its solutions are oscillatory.

Oscillation Criteria for (1).

In this section we establish the oscillation criteria for (1). We need the following hypothesis for our use in the sequel:

(A_2) there exists $\lambda > 0$ such that $G(u) + G(v) \geq \lambda G(u + v)$ and there exists $\mu > 0$ such that $H(u) + H(v) \geq \mu H(u + v)$

(A_3) $G(uv) \leq G(u)G(v)$ and $H(uv) \leq H(u)H(v)$ for $u, v \in \mathbb{R}$, and $G(p) \geq H(p)$ and $u, v > 0$

(A_4) $G(-u) = -G(u), H(-u) = -H(u),\ u \in \mathbb{R}$,

(A_5) $\displaystyle\int_0^\infty [Q(t) + V(t)]\, dt = \infty,\ Q(t) = \min\{q(t), q(\tau(t))\}$, $V(t) = \min\{v(t), v(\tau(t))\}$;

(A_6) $\displaystyle\int_0^\infty [q(t) + v(t)]\, dt = \infty$;

(A_7) $\tau^n(t) = \tau(\tau^{n-1}(t)),\ \lim_{n\to\infty} \tau^n(t) < \infty$;

(A_8) $\displaystyle\int_{t_0}^\infty [Q(t)G(R(\sigma(t))) + V(t)H(R(\eta(t)))]\, dt = \infty,\ R(t) = \int_t^\infty \frac{ds}{r(s)}$;

(A$_9$) $\displaystyle\int_0^\infty \frac{1}{r(t)} \int_T^t [Q(s)G(R(\sigma(s))) + V(s)H(R(\eta(s)))]\, ds dt = \infty$, for every $T > 0$;

(A$_{10}$) $\displaystyle\int_0^\infty [q(t)G(R(\sigma(t))) + v(t)H(R(\eta(t)))]\, dt = \infty,\ R(t) = \int_t^\infty \frac{ds}{r(s)}$;

(A$_{11}$) $\displaystyle\int_0^\infty \frac{1}{r(t)} \int_T^t [q(s)G(R(\sigma(s))) + v(s)H(R(\eta(s)))]\, ds dt = \infty$, for every $T > 0$;

2. Main Results

This section deals with the necessary and sufficient conditions for oscillation of all solutions of (1). Throughout our discussion, we use the notation

$$z(t) = x(t) + p(t)x(\tau(t)). \tag{2}$$

Lemma 2.1. [5] Assume that (A_0) hold. Let $x(t)$ be a positive solution of (1) defined on $[t_0, \infty)$ such that $z(t) > 0$, then $z(t)$ satisfies $z(t) > 0$, $r(t)z'(t) > 0$, $(r(t)z'(t))' < 0$ eventually.

Lemma 2.2. Assume that (A_1) hold. Let $x(t)$ be any continuous function defined on $[t_0, \infty)$ such that $(r(t)x'(t))' \leq 0$ for $t \geq t_0$. If $x'(t) < 0$ for $t \geq t_0$, then $x(t) \geq -R(t)r(t)x'(t)$.

Proof. For $s \geq t$, $r(s)x'(s) \leq r(t)x'(t)$ implies that $x'(s) \leq \dfrac{r(t)x'(t)}{r(s)}$ which on integration from t to s, we get

$$\int_t^s x'(\theta)d\theta \leq r(t)x'(t) \int_t^s \frac{d\theta}{r(\theta)},$$

that is,

$$x(t) + r(t)x'(t)R(t) \geq x(s) > 0,$$

implies that

$$x(t) \geq -R(t)r(t)x'(t),$$

This completes the proof of the lemma. ■

Theorem 2.3. Let $0 \leq p(t) < p < \infty$ and $\tau(\sigma(t)) = \sigma(\tau(t))$ and $\tau(\eta(t)) = \eta(\tau(t))$ for $t > 0$. Assume that (A_0), $(A_2) - (A_5)$ and (A_3), (A_4) hold. Then every solution of (1) oscillates.

Proof. On the contrary, without loss of generality we may assume that $x(t)$ be a nonoscillatory solution of (1) such that

$$x(t) > 0,\ x(\tau(t)) > 0,\ x(\sigma(t)) > 0,\ x(\eta(t)) > 0\ for\ t \geq t_1.$$

Defining $z(t)$ as in Lemma 2.1 and then taking the lemma into account, it follows that $z(t) \geq C$ for $t \geq t_2$. Using (1), it follows that

$$\begin{aligned}(r(t)z'(t))' + q(t)G(x(\sigma(t)) + v(t)H(x(\eta(t)) + G(p)(r(\tau(t))z'(\tau(t)))' \\ +G(p)q(\tau(t))G(x(\sigma(\tau(t))) + G(p)v(\tau(t))H(x(\eta(\tau(t))) = 0.\end{aligned}$$

Using (A_2) and (A_3) in the above equation, we obtain

$$(r(t)z'(t))' + G(p)(r(\tau(t))z'(\tau(t)))' + \lambda Q(t)G(z(\sigma(t))) + \mu V(t)H(z(\eta(t))) \leq 0, \quad (3)$$

due to $\sigma(\tau(t)) = \tau(\sigma(t))$ and $\eta(\tau(t)) = \tau(\eta(t))$. Consequently, there exists $t_3 > t_2$ such that

$$\lambda G(C)Q(t) + \mu H(C)V(t) \leq -(r(t)z'(t))' - G(p)(r(\tau(t))z'(\tau(t)))' \quad (4)$$

for $t \geq t_3$. Integrating (3) from t_3 to $+\infty$, we obtain a contradiction to (A_5). This completes the proof of the theorem. ■

Remark 2.4. Equation (1) includes a class of nonlinear neutral differential equations when $p(t) \geq 0$. It is learnt that G and H could be linear, sublinear or superlinear also.

Theorem 2.5. Let $-1 < -p \leq p(t) \leq 0$, $p > 0$. Assume that (A_0), (A_4) and (A_6) hold. Then every unbounded solution of (1) oscillates.

Proof. Let $x(t)$ be an unbounded nonoscillatory solution of (1). Proceeding as in the proof of Theorem 2.3, we can find a $t_1 > t_0$ such that $x(t) > 0, x(\tau(t)) > 0, x(\sigma(t)) > 0$ and $x(\eta(t)) > 0$ for $t \geq t_1$. Since $z(t)$ is monotonic, then either $z(t) > 0$ or $z(t) < 0$ for $t \geq t_1$. Clearly, Lemma 2.1 holds when $z(t) > 0$ for $t \geq t_1$. Using the fact that $z(t) \leq x(t)$ for $t \geq t_1$, it follows from (1) that

$$(r(t)z'(t))' + q(t)G(z(\sigma(t))) + v(t)H(z(\eta(t))) \leq 0.$$

The rest of this case follows from Theorem 2.3. Suppose that $z(t) < 0$ for $t \geq t_1$. Since $x(t)$ is unbounded, then there exists $\{\sigma_n\}$ such that $\sigma_n \to \infty$ and $x(\sigma_n) \to \infty$ as $n \to \infty$ and $x(\sigma_n) = \max\{x(s) : t_1 \leq s \leq \sigma_n\}$. Indeed,

$$\begin{aligned} z(\sigma_n) &= x(\sigma_n) + p(\sigma_n)x(\tau(\sigma_n)) \\ &\geq x(\sigma_n) - px(\tau(\sigma_n)) \\ &\geq x(\sigma_n) - px(\sigma_n) \\ &= (1-p)x(\sigma_n)(\because 1-p > 0) \end{aligned}$$

implies that $z(t) > 0$, which is absurd. This completes the proof of the theorem. ■

Remark 2.6. We may note that G and H could be linear, superlinear or sublinear in the Theorem 2.5. If $z(t) < 0$ for $t \geq t_2$, then $x(t) < x(\tau(t))$ implies that

$$x(t) \leq x(\tau(t)) \leq x(\tau^2(t)) \leq x(\tau^3(t)) \leq .... \leq x(\tau^n(t))...$$

holds. Consequently, $x(t)$ is bounded due to (A_7) and hence $z(t)$ is bounded. When $z'(t) < 0$ for $t \geq t_2$, it happens that $r(t)z'(t) \leq r(t_2)z'(t_2)$ which then implies that $\lim_{t\to\infty} z(t) = -\infty$. Therefore, this case doesn't arise in Theorem 2.5.

Theorem 2.7. Let $-1 < -p \leq p(t) \leq 0$, $p > 0$. Assume that (A_0), (A_4), (A_6) and (A_7)hold. Then every solution of (1) either oscillates or converges to zero.

Proof. The proof of theorem follows from Theorem 2.5 and Remark 2.6. In case $z(t) < 0$,

$$\begin{aligned}
0 &= \lim_{t\to\infty} z(t) \\
&= \limsup_{t\to\infty}(x(t) + p(t)x(\tau(t))) \\
&\geq \limsup_{t\to\infty}(x(t) - px(\tau(t))) \\
&\geq \limsup_{t\to\infty} x(t) + \liminf_{t\to\infty}(-px(\tau(t))) \\
&= (1-p)\limsup_{t\to\infty} x(t)
\end{aligned}$$

implies that $\limsup_{t\to\infty} x(t) = 0$. Hence, $\lim_{t\to\infty} x(t) = 0$. This completes the proof of the theorem. ■

Theorem 2.8. Let $-\infty < -p \leq p(t) \leq -1$, $p > 1$. Assume that (A_0), (A_4) and (A_6) hold. Then every bounded solution of (1) either oscillates or converges to zero.

Proof. The proof of the theorem follows from the proof of theorem 2.7. In case $z(t) < 0$, $r(t)z'(t) > 0$, $(r(t)z'(t))' < 0$, we assert that $\liminf_{t\to\infty} x(t) = 0$. Otherwise, let there exists $a > 0$ and $t_4 > t_3$ such that $x(\tau(t)) \geq a$. Integrating (1) from t_4 to t we get a contradiction to (A_6). Therefore, our assertion is true. Hence, there exists $\{\delta_n\}_{n=1}^{\infty} \subset [t_4, \infty)$ such that $\delta_n \to \infty$ as $n \to \infty$ and $\lim_{n\to\infty} x(\delta_n) = 0$. Let $\lim_{t\to\infty} z(t) = l$, $l \in (-\infty, 0]$. For $t \geq t_4$, we have

$$z(\tau^{-1}(t)) - z(t) = x(\tau^{-1}(t)) + [p(\tau^{-1}(t)) - 1]x(t) - p(t)x(\tau(t))$$

implies that

$$\lim_{t\to\infty}[x(\tau^{-1}(t)) + \{p(\tau^{-1}(t)) - 1\}x(t) - p(t)x(\tau(t))] = 0.$$

Equivalently,

$$\lim_{n\to\infty}[x(\tau^{-1}(\delta_n)) + \{p(\tau^{-1}(\delta_n)) - 1\}x(\delta_n) - p(\delta_n)x(\tau(\delta_n))] = 0$$

implies that

$$\lim_{n\to\infty}[x(\tau^{-1}(\delta_n)) - p(\delta_n)x(\tau(\delta_n))] = 0.$$

Using the fact that

$$x(\tau^{-1}(\delta_n)) - p(\delta_n)x(\tau(\delta_n)) \geq -p(\delta_n)x(\tau(\delta_n)),$$

then it follows that

$$\lim_{n\to\infty} sup[-p(\delta_n)x(\tau(\delta_n))] = 0,$$

that is, $\lim_{n\to\infty}[-p(\delta_n)x(\tau(\delta_n))] = 0$. Ultimately,

$$l = \lim_{n\to\infty} z(\delta_n) = \lim_{n\to\infty}[x(\delta_n) + p(\delta_n)x(\tau(\delta_n))] = 0.$$

As a result

$$\begin{aligned} 0 &= \lim_{t\to\infty} z(t) \\ &= \limsup_{t\to\infty}(x(t) + p(t)x(\tau(t))) \\ &\geq \limsup_{t\to\infty}(x(t) - px(\tau(t))) \\ &\geq \limsup_{t\to\infty} x(t) + \liminf_{t\to\infty}(-px(\tau(t))) \\ &= (1-p)\limsup_{t\to\infty} x(t) \end{aligned}$$

implies that $\limsup_{t\to\infty} x(t) = 0$ and thus $\lim_{t\to\infty} x(t) = 0$. ■

Theorem 2.9. Let $0 \leq p(t) < p < \infty$ and $\tau(\sigma(t)) = \sigma(\tau(t))$ and $\tau(\eta(t)) = \eta(\tau(t))$ for $t > 0$. Assume that $(A_1) - (A_3)$, (A_4), (A_8) and (A_9) hold. Then every solution of (1) oscillates.

Proof. Proceeding as in the proof of Theorem 2.3 we can find $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$, $x(\eta(t)) > 0$ for $t \geq t_1$. It follows that $z(t)$ and $r(t)z'(t)$ are of one sign on $[t_1, \infty)$. We consider two cases upon the sign of $r(t)z'(t)$, that is, $z'(t) > 0$ and $z'(t) < 0$ for $t \geq t_1$. Suppose the former holds. Then there exists $t_2 > t_1$ and a $C_1 > 0$ such that $z(t) \geq C_1$ for $t \geq t_2$. It is easy to verify that there exists, $C > 0$ and $t_3 > t_2$ such that $z(t) \geq CR(t)$ for $t \geq t_3$($\because \lim_{t\to\infty} R(t) = 0$ and $R(t)$ is bounded). Consequently, we can write the inequality (3) as

$$\lambda G(CR(\sigma(t)))Q(t) + \mu H(CR(\eta(t)))V(t) \leq -(r(t)z'(t))' - G(p)(r(\tau(t))z'(\tau(t)))' \tag{5}$$

for $t \geq t_3$. Integrating (5) from t_3 to ∞, we obtain a contradiction. Ultimately, the latter holds. From Lemma 2.2, we have that $z(t) \geq -R(t)r(t)z'(t)$ for $t \geq t_2 > t_1$. Since $r(t)z'(t)$ is nonincreasing, then we can find a constant $C > 0$ and $t_3 > t_2$ such that $r(t)z'(t) \leq -C$ and $z(t) \geq CR(t)$ for $t \geq t_3$. Integrating (5) from t_3 to t, we obtain that

$$\lambda G(C)\mu H(C) \int_{t_3}^{t} [Q(s)G(R(\sigma(s))) + V(s)H(R(\eta(s)))]\, ds \leq -(1+G(p))(r(t)z'(t))$$

due to nonincreasing $r(t)z'(t)$. Hence

$$\frac{\lambda\mu G(C)H(C)}{r(t)} \int_{t_3}^{t} [Q(s)G(R(\sigma(s))) + V(s)H(R(\eta(s)))]\, ds \leq -(1+G(p))z'(t).$$

Since $\lim_{t\to\infty} z(t)$ exists, then it follows the above inequality that

$$\lambda\mu G(C)H(C) \int_{t_3}^{\infty} \frac{1}{r(t)} \int_{t_3}^{t} [Q(s)G(R(\sigma(s))) + V(s)H(R(\eta(s)))]\, ds dt < \infty,$$

a contradiction to (A_9). Thus the proof of the theorem is complete. ■

Theorem 2.10. Let $-1 < -p \leq p(t) \leq 0$, $p > 0$. Assume that (A_1), (A_4), (A_{10}) and (A_{11}) hold. Then every unbounded solution of (1) oscillates.

Proof. Proceeding as in the proof of Theorem 2.5, we conclude that $z(t)$ is monotonic on $[t_1, \infty)$. Hence there exists $t_2 > t_1$ such that $z(t) > 0$ or $z(t) < 0$ for $t \geq t_2$. Consider that $z(t) > 0$ on $[t_2, \infty)$. Using the same type of reasoning as in the proof of Theorem 2.9, we can find $C > 0$ and $t_3 > t_2$ such that $z(t) \geq CR(t)$ for $t \geq t_3$. Since $z(t) \leq x(t)$, then it follows that $x(t) \geq CR(t)$ on $[t_3, \infty)$, if we assume that $z'(t) > 0$. Therefore(1) becomes

$$(r(t)z'(t))' + G(C)G(R(\sigma(t)))q(t) + H(C)(R(\eta(t)))v(t) \leq 0 \tag{6}$$

for $t \geq t_3$. Integrating (6) from t_3 to ∞, we obtain a contradiction to (A_{10}). Hence $z'(t) < 0$ for $t \geq t_2$. Rest of this case follows from the proof of Theorem 2.9.

Next, we suppose that $z(t) < 0$ on $[t_2, \infty)$. Let there exist $t_3 > t_2$ such that $z'(t) > 0$ or $z'(t) < 0$ for $t \geq t_3$. Rest of this case follows from the proof of Theorem 2.5. Thus the proof of the theorem is complete. ■

Theorem 2.11. Let $-1 < -p \leq p(t) \leq 0$, $p > 0$. Assume that (A_1), (A_4), (A_7), (A_{10}) and (A_{11}) hold. Then every solution of (1) either oscillates or converges to zero.

Proof. The proof of the theorem follows from the proof of Theorem 2.7 and 6. Due to (A_7), $x(t)$ is bounded and hence $z(t)$ is bounded. Therefore, $\lim_{t\to\infty} z(t)$ exists, when $z(t) < 0$. Hence, the theorem is proved. ■

Theorem 2.12. Let $-\infty < -p \le p(t) \le -1$, $p > 0$. Assume that (A_1), (A_4), (A_{10}) and (A_{11}) hold. Then every bounded solution of (1) either oscillates or converges to zero.

Proof. The proof of the theorem can be followed from the proof of Theorem 2.11 and 2.8. Hence, the details are omitted. ■

3. Examples

Example 3.1.

$$(4(x(t) + x(t-\pi))')' + e^t G(x(t-2\pi)) + e^t H(x(t-3\pi)) = 0, t \ge 2\pi, \qquad (7)$$

where $G(x) = H(x) = x^3$. All conditions of Theorem 2.3 are satisfied for (1). Hence, every solution of (1) oscillates. In particular, $x(t) = sint$ is one of such solution of (1).

Example 3.2.

$$(e^t(x(t) + x(t-\pi))')' + e^{3t} G(x(t-2\pi)) + e^{3t} H(x(t-3\pi)) = 0, t \ge 2\pi, \qquad (8)$$

where $G(x) = H(x) = x^3$. All conditions of Theorem 2.9 are satisfied for (1). Hence, every solution of (1) oscillates. In particular, $x(t) = sint$ is one of such solution of (1).

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