Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 14, Number 10 (2018), pp. 1285–1309

http://www.ripublication.com/gjpam.htm

Nonlinear stability and asymptotics of solutions to perturbed dynamic problems

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Abstract

In this paper we study the behaviour of solutions to nonlinear partial differential equations describing a tumour growth problem. We present nonlinear energy stability results using an integral inequality technique and then analyze the asymptotic behaviour of the studied model solutions when the initial data and external force are perturbed.

AMS subject classification:
Keywords: Tumour, nonlinear stability, asymptotic behaviour, perturbed solutions.

1. Introdution

The need to adapt to forms of cancer [17] and the various phenomena involved in its growth have led to the establishment of numerous mathematical models [4] in recent decades. Even if simplified compared to biological reality, the interest of developing such models lies in their ability to gather a large amount of information accumulated by biologists. Indeed, it is important to understand that the mathematical complexity of a model is not a sufficient criterion to judge its relevance. Thus, the character of this phenomenon (cancer cells have a fluid type motion) motivated us to use relatively simple nonlinear partial differential equations that describe the disease. The objective of this work is to solve some problems which are of major importance in control theory, namely the problem of the stability of solutions of a system of partial nonlinear differential equations in a bounded domain $I \subset \mathbb{R}^3$, strictly lipchitzian [18] and having a compact border. The concept of stability plays an essential role in the study of the behaviour of dynamic systems and in the synthesis of control laws for these systems. Thus, the problem of stability of dynamic systems has been and remains a major concern in the work of automation engineers and engineers [15]. In the literature, there are several notions of stability, very often linked to the nature of the studied systems, their environments, specifications and desired performance. Among these notions of stability, the best known is Lyapunov's stability, established in 1892 by the Russian mathematician Lyapunov[16]. His contribution consists in a qualitative characterization of stability by a study of the trajectories of dynamic systems, using auxiliary functions now called Lyapunov functions. Thus, stability in the sense of Lyapunov is the most known and used in the literature. Nevertheless, several works present other notions of stability that allow to solve several practical cases of systems studies, namely if we move the system away from the equilibrium position, will he return to it? Or again: can a small perturbed, which moves the system slightly away from its stationary regime, have a much more significant influence over time? In recent years, the tumour growth process and its dynamics have been intensively studied [14]. Tumor progression is a complex process. To describe this process, mathematical models are needed. Several papers have been devoted to the development of mathematical models to describe the process [3, 6, 8]. Most of these models are based on the reaction-diffusion equations [19] and the law of mass conservation. The analysis and stability of such mathematical models has generated great interest, and many results have been established [7, 4, 15, 6]. However, all applications to partial differential equations (PDEs) must be stable. This is more important in medical cases, where failure to take into account the stability problems of numerical methods may lead to erroneous judgments by physicians which in turn may lead to the development of incorrect treatment methods. In this paper, we will focus on some particular notions of stability. In particular, nonlinear energetic stability and asymptotic stability are cited.

The structure of this paper is as follows: after this introduction, section 2 describes the tumour growth model. In section 3, using the energy method, we study the non-linear energy stability of the solution and the perturbed solution using an integral inequality technique. Finally in section 4 we study the asymptotic behaviour of the model solution. It is clear that any perturbed solution of the nonlinear partial differential equations converges asymptotically to the solution of the original system.

2. Model description

In all fields of biology, the use of mathematics as a tool for presentation and forecasting is becoming more and more important. It is clear that building a mathematical model of a biological phenomenon is not an end in itself. The use of such a model will in no way replace insufficient biological knowledge. It will make it possible to take some steps along the path of this knowledge and to formalize everything we know on this subject in a system under rigorous hypotheses. In this paper, we present a model based on reaction-diffusion nonlinear differential equations, describing the proliferative evolution of tumor cells across a given domain.

Let x denote the size of the tumor, and t the time parameter. Consider a time-dependent reference region $I_t = I \times (0, T)$, $T > 0$ occupied by the tumor where I is a bounded open set of $\mathbb{R}^3$ and either $\partial I_t = \partial I \times (0, T)$, its smooth enough boundary.

Note $\nu = \nu(x, t)$ and $\xi_e = \xi_e(x, t)$, vector functions designating respectively the proliferation rate of cancer cells, the density of external forces (healthy cells + nutrients) and the scalar function $\rho = \rho(x, t)$, the density of tumor cells. A general form of the system is as follows:

$$\begin{cases} \dfrac{\partial \rho \nu_i}{\partial t} + \displaystyle\sum_{j=1}^{3} \dfrac{\partial}{\partial x_j}(\rho \nu_i \otimes \nu_i) = \rho \xi_e + L_{\lambda,\mu}(\nu), \quad \forall (x,t) \in I \times (0,T), \quad i = 1,2,3 \\ \dfrac{\partial \rho}{\partial t} + \displaystyle\sum_{j=1}^{3} \dfrac{\partial \rho \nu_j}{\partial x_j} = 0, \ \ \rho \geq 0 \quad \forall (x,t) \in I \times (0,T), \\ \rho|_{t=0} = \rho_0(x), \quad \forall x \in I, \\ \nu|_{t=0} = \nu_0(x), \quad \forall x \in I, \\ \displaystyle\lim_{|x| \to \infty} (\nu, \rho) = (0,0) \quad \forall t \in (0; T), \end{cases} \tag{2.1}$$

where $L_{\lambda,\mu}(\nu)$ is an operator formally defined by

$$L_{\lambda,\mu}(\nu) \stackrel{\text{def}}{:=} \mu div(\nabla \nu) + (\lambda + \mu)\nabla(\nabla \nu) - \frac{\partial \pi_i}{\partial x_i}, \quad i = 1,2,3$$

with λ and μ which respectively represent the volumic and dynamic viscosity coefficients assumed to be constant.

We assume that on the I boundary of the domain, the velocity verifies:

$$\nu|_{\partial I} = 0, \quad \forall (x,t) \in \partial I \times (0,T). \tag{2.2}$$

For physical reasons, μ and λ meet the following conditions:

$$\mu > 0, \quad \lambda + 2\mu > 0. \tag{2.3}$$

This means that changes in volume, expansion or compression can be made without viscosity. The pressure π depending on the constant density, is given by the following state law:

$$\pi = \kappa \rho^{\alpha}, \quad \kappa \geq 1, \tag{2.4}$$

with α the adiabatic constant such that $\alpha > (d-1)/2$ ($d = 3$).

It should be mentioned that $\rho \nu \otimes \nu \in \mathbb{R}^3$ in $(2.1)_1$ is a tensor product of $\rho\nu$ and ν and that

$$\sum_{i,j=1}^{3} \frac{\partial}{\partial x_j}(\rho \nu_i \otimes \nu_i) = \sum_{i,j=1}^{3} \frac{\partial}{\partial x_j}(\rho \nu_i)\nu_i + \sum_{i,j=1}^{3} \rho \nu_j \frac{\partial \nu_i}{\partial x_j}. \tag{2.5}$$

Before announcing the results, it is necessary to define the areas in which we are working. In this sub-section, we introduce the notation that will be used throughout this document.

2.1. General terms and definitions

In this work, a couple of symbols and definitions are used, which are generally introduced when they are needed. However, some general notations that belong to the mathematical norm are given here for reference in advance. The following function spaces provide a norm framework for studying equation system stability (2.1) − (2.2).

The underlying domain. Let $I \subset \mathbb{R}^3$, a delimited domain ∂I its sufficiently smooth border. For $T > 0$, the interval $(0, T)$ defines the considered time interval and $I_t = I \times (0, T)$ a space-time domain with boundary $\partial I_t = \partial I \times (0, T)$.

Standard operators. $x = (x_1, x_2, x_3)$ is the space variable in $\mathbb{R}^3$. For x, $y \in \mathbb{R}^3$, $x \cdot y = \sum_{i=1}^{3} x_i y_i$. ∇ is the gradient ans Δ is the laplacian. When $\mathcal{G}(x) = (\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3)$ is an $\mathbb{R}^3$-valued function,

$$\nabla \cdot \mathcal{G} = \sum_{i=1}^{3} \frac{\partial \mathcal{G}_i}{\partial x_i}, \quad |\nabla \mathcal{G}|^2 = \sum_{i,j=1}^{3} \Big|\frac{\partial \mathcal{G}_i}{\partial x_j}\Big|^2, \quad \|\mathcal{G}\|_{L^p(I;\mathbb{R}^3)} = \Big(\sum_{i=1}^{3} \|\mathcal{G}\|^p_{L^p(I;\mathbb{R}^3)}\Big)^{1/p},$$

$$\|\nabla \mathcal{G}\|_{L^p(I;\mathbb{R}^3)} = \Big(\sum_{i,j=1}^{3} \Big\|\frac{\partial \mathcal{G}_i}{\partial x_j}\Big\|^p_{L^p(I;\mathbb{R}^3)}\Big)^{1/p}.$$

Standard Lebesgue spaces. Let m be a non-negative integer. We denote by $H^m(I;\mathbb{R}^3)$ the usual Sobolev space $W^{m,2}(I;\mathbb{R}^3)$ as defined in Lions and Magenes [21].

We note by $\mathcal{D}(I)$, the space of infinitely differentiable functions with compact support. Its closing in the norm $W^{m,p}(I;\mathbb{R}^3)$ $(1 < p < s < +\infty)$ is noted by $W_0^{m,p}(I;\mathbb{R}^3)$. An alternate characteristic in the case where $m = 1$ and $p = 2$ is

$$ker\gamma_0 = W_0^{1,2}(I;\mathbb{R}^3) = \{\mathcal{G} \in H^1(I;\mathbb{R}^3) : \gamma_0 \mathcal{G} = 0\},$$

where γ_0 is the $\mathcal{G}$ trace operator. We also note by $L^p(I)^3 = L^p(I;\mathbb{R}^3)$, the lebesgue space on I provided with the norm $\|.\|_p$ and by $\|.\|_{E_1}$ the norm associated with a space E_1. If E_1 is a Banach space, $L^p(0,T;E_1)$ is the Banach space composed of functions, measurable on $(0,T)$ which value in E_1. For details concerning these spaces, see, e.g., [22] or [23].

We consider zero divergence spaces introduced for the problem (2.1)–(2.2).

$$K^1_{div} := \big\{\mathcal{G} \in L^2(I;\mathbb{R}^3) : \sum_{j=1}^{3} \frac{\partial \mathcal{G}_j}{\partial x_j} = 0,\ \mathcal{G}\cdot \mathbf{n}|_{\partial I} = 0\big\},$$

$$K^0_{div} := \big\{\mathcal{G} \in H^1_0(I;\mathbb{R}^3) : \sum_{j=1}^{3} \frac{\partial \mathcal{G}_j}{\partial x_j} = 0\big\},$$

$$C^\infty_{0,\sigma}(I;\mathbb{R}^3) = \big\{\mathcal{G} \in \mathcal{D}(I;\mathbb{R}^3) :\ \sum_{j=1}^{3} \frac{\partial \mathcal{G}_j}{\partial x_j} = 0\big\},$$

where K^1_{div} and K^0_{div} the respective closure of $C^\infty_{0,\sigma}(I;\mathbb{R}^3)$ in $L^2(I;\mathbb{R}^3)$ and $H^1_0(I;\mathbb{R}^3)$.

2.1.1 Definition

Let I be a bounded domain in $\mathbb{R}^3$ with smooth boundary, and assume that the data $\nu_0(x)$, $\rho_0(x)$, $\xi_e(x,t)$ satisfy the regularity conditions $\nu_0(x) \in K^1_{div}$, $\rho_0(x) \in W^{1,2}(I;\mathbb{R})$, $\xi_e(x,t) \in L^1((0,T);L^{\frac{2s}{s-1}}(I;\mathbb{R}^3))(1 < p < s < \infty)$. Then (ν,ρ) is a solution of the problem $(2.1) - (2.2)$ on $(0,T)$ corresponding to the initial conditions ν_0 and ρ_0 if the following conditions are met:
i)

$$\begin{cases} \nu(x,t) > 0\ et\ \nu(x,t) \in L^2((0,T);K^0_{div}) \cap L^\infty((0,T);\ K^1_{div}), \\ \rho(x,t) > 0\ et\ \rho(x,t) \in L^\infty((0,T);W^{1,2}(I;\mathbb{R})), \\ \rho\nu \in \mathcal{C}((0,T);L_w^{\frac{2s}{s+1}}(I;\mathbb{R}^3)). \end{cases} \tag{2.6}$$

ii) For all $\mathcal{G} \in \mathcal{C}^1((0,T);K^0_{div})$

$$-\int_0^T\int_I (\rho\nu\cdot\frac{\partial\mathcal{G}}{\partial t})dxdt - \sum_{j=1}^{3}\int_0^T\int_I \rho\nu_j\nu\cdot\frac{\partial\mathcal{G}}{\partial x_j}\,dxdt$$
$$+ (\lambda+2\mu)\sum_{j=1}^{3}\int_0^T\int_I \frac{\partial\nu}{\partial x_j}\cdot\frac{\partial\mathcal{G}}{\partial x_j}dxdt$$

$$= -\underbrace{\int_I (\nu(x,T)\mathcal{G}(x,T))dx}_{=0} + \int_I (\rho_0\nu_0 \cdot \mathcal{G}(x,0))dx + \int_0^T \int_I (\rho\xi_e \cdot \mathcal{G})dxdt. \quad (2.7)$$

iii) For any $\mathcal{Y} \in \mathcal{C}^1((0,T); W^{1,2}(I;\mathbb{R}^3))$

$$-\int_0^T \int_I \rho \frac{\partial \mathcal{Y}}{\partial t} dxdt - \sum_{j=1}^{3} \int_0^T \int_I \rho\nu_j . \frac{\partial \mathcal{Y}}{\partial x_j} dxdt = \int_I \rho_0 \mathcal{Y}(x,0)dx. \quad (2.8)$$

Let us now present our nonlinear energy stability results obtained in convection problems using an integral inequality technique.

3. Conditional nonlinear energy stability

3.1. Nonlinear energy stability of the non-zero solution

The physical situation here is that of a cancerous tumour whose volume density is constant over time. Let's say at time $t = 0$, gangrene is fixed. The system (2.1) thus becomes:

$$\begin{cases} \rho\dfrac{\partial \nu_i}{\partial t} + \displaystyle\sum_{j=1}^{3} \nu_j \rho \dfrac{\partial \nu_i}{\partial x_j} = \rho\xi_e + L_{\lambda,\mu}(\nu), \quad \forall (x,t) \in I \times (0,T), \quad i = 1,2,3 \\ \displaystyle\sum_{j=1}^{3} \dfrac{\partial \nu_j}{\partial x_j} = 0, \ \forall (x,t) \in I \times (0,T), \\ \rho|_{t=0} = \rho|_{t=T} \quad \forall x \in I, \\ \nu|_{t=0} = \nu_0(x), \ \forall x \in I, \\ \displaystyle\lim_{|x|\to\infty} \nu(x,t) = 0, \ \forall t \in (0;T). \end{cases} \quad (3.1)$$

We study the nonlinear energy stability of the solution to the initial limit value problem (3.1) on a region I with ν given on the border ∂I, i.e:

$$\nu|_{\partial I} = \bar{\nu}|_{\partial I} \quad \forall (x,t) \in \partial I \times (0,T). \quad (3.2)$$

Mathematically, the problem is to study the ν behavior in the (3.1) resolution with $\bar{\nu} = 0$ in (3.2), subject to the initial condition $(3.1)_4$.

Let us now make some useful hypothesis for the continuation of our study:

Hypothesis 1 : Let $I \subset \mathbb{R}^3$ be strictly lipschitzian, uniform and regular $\mathcal{C}^2$ class, with a fairly smooth border ∂I.

Hypothesis 2 : Let $1 < p < s < \infty$, for all $T > 0$ (fixed), the initial flow $\nu_0 = \nu_0(x) \in K^1_{div}$, the volume density $\rho(.,t) \in W^{1,2}(I;\mathbb{R})$ and the external force $\xi_e = \xi_e(x,t) \in L^1((0,T); L^{\frac{2s}{s-1}}(I;\mathbb{R}^3))$

Theorem 3.1. Suppose the hypotheses 1 and 2 are verified, (μ, λ) and $\pi(\rho)$ satisfy respectively (2.3) and (2.4). Let $\nu(x,t) \in L^2((0,T); K^0_{div}) \cap L^\infty((0,T);\, K^1_{div})$ a solution of $(3.1) - (3.2)$. Moreover, if $\nabla\nu \in L^2((0,T); L^p(I;\mathbb{R}^3))$ and $\mathcal{R}^{-1}_{\mathcal{E}}(\lambda,\mu) < 1$ in the time interval $(0,t)$. Then there exists a constant $\eta(I) > 0$, which depends on the domain I such that

$$\mathcal{E}_I\nu(t) \leq \mathcal{E}_I\nu(0)\exp\left\{\frac{-2\,\eta(I)(\lambda+2\mu)t}{\rho}\big(1-\mathcal{R}^{-1}_{\mathcal{E}}(\lambda,\mu)\big)\right\}, \tag{3.3}$$

where $\mathcal{E}_I\nu(0)$ is the initial energy and $\mathcal{R}^{-1}_{\mathcal{E}}(\lambda,\mu) \overset{\text{def}}{:=} \max_{C_0^\infty}\{\frac{\Theta_p}{\Theta_d}\}$, Θ_p and Θ_d are production and dissipation terms of system (3.1)–(3.2).

Proof. Let $\nu(x,t)$ be the solution of (3.1)–(3.2). The functional $\mathcal{E}_I\nu$ defined by

$$\mathcal{E}_I\nu(t) \overset{\text{def}}{:=} \frac{1}{2}\int_I \rho|\nu(x,t)|^2 dx. \tag{3.4}$$

Multiplying the equation $(3.1)_1$ by $\nu(x,t)$ and integrating on the I domain, we get: $\forall t \in (0,T)$

$$\int_I \rho\frac{\partial\nu^2}{\partial t}dx + \int_I\sum_{i,j=1}^{3}\rho\nu_j\frac{\partial\nu_i^2}{\partial x_j}dx + \int_I \nabla\pi\;\nu dx = \int_I \rho\xi_e\nu dx + \int_I \mu div(\nabla\nu)\nu dx$$

$$+\int_I(\lambda+\mu)\nabla(\nabla\nu)\nu dx \tag{3.5}$$

$$\frac{1}{2}\frac{d}{dt}\int_I\rho\nu^2dx + \int_I\nabla\pi\;\nu dx = \int_I\rho\xi_e\nu dx + \mu\int_I div(\nabla\nu)\nu dx + (\lambda+\mu)\int_I\nabla(\nabla\nu)\nu dx. \tag{3.6}$$

From (3.4) we obtain the energy equation:

$$\frac{d\,\mathcal{E}_I\nu(t)}{dt} = -\int_I\nabla\pi\;\nu dx + \int_I\rho\xi_e\nu dx + \mu\int_I div(\nabla\nu)\nu dx + (\lambda+\mu)\int_I\nabla(\nabla\nu)\nu dx \tag{3.7}$$

$$\frac{d\,\mathcal{E}_I\nu(t)}{dt} = \Theta_p + \Theta_d$$

where the production term Θ_p is defined by

$$\Theta_p = -\int_I\nabla\pi\;\nu dx + \int_I\rho\xi_e\nu dx$$

and Θ_d the dissipation one by

$$\Theta_d = \mu \int_I div(\nabla v) v dx + (\lambda + \mu) \int_I \nabla(\nabla v) v dx$$

Let's give the estimate of each term of these two expressions in order to establish (3.3).

- Estimate of $\mu \int_I div(\nabla v) v dx$

$$\mu \int_I div(\nabla v) v dx = \mu \int_{\partial I} y_0 v(\nabla v . \vec{n_I}) ds - \mu \int_I tr(\nabla v . \nabla' v) dx$$

(where y_0 is the one continuous linear application from $W^{1,2}(I;\mathbb{R}^3) \longrightarrow L^2(I;\mathbb{R}^3)$ such that $y_0 v = 0$, $\vec{n_I}$ is the normal to the border of I noted ∂I and ds its surface element). Then, we have:

$$\mu \int_I (\Delta v) v dx = \mu \sum_{i,j=1}^{3} \int_I \frac{\partial v_i \partial v_j}{\partial x_i \partial x_j} dx \leq \mu \int_I \| \frac{Dv}{Dt} \|^2 dx \leq \mu \|\nabla v\|^2_{L^p(I;\mathbb{R}^3)} \tag{3.8}$$

- Estimate of: $\int_I \nabla(\nabla v) v \, dx$. By calculations similar to the previous ones, we obtain:

$$\int_I \nabla div(v) v dx = \int_I \nabla(v div(v)) dx - \int_I \Delta v^2 dx \leq \int_I \| \frac{Dv}{Dt} \|^2 dx \leq \|\nabla v\|^2_{L^p(I;\mathbb{R}^3)} \tag{3.9}$$

- Estimate of: $\int_I \nabla \pi(\rho) v \, dx$

$$\int_I \nabla \pi(\rho) v dx = \int_I \nabla \kappa \, \rho^\alpha v \, dx$$

after integration by parts we have:

$$\int_I \nabla k \rho^\alpha v dx = \underbrace{\int_{\partial I} \kappa \rho^\alpha y_0 v . \vec{n}_I ds}_{=0} - \underbrace{\frac{\kappa \, \alpha}{\alpha - 1} \int_I \sum_{j=1}^{3} \frac{\partial v_j}{\partial x_j} \rho^\alpha dx = 0}_{=0} \tag{3.10}$$

- Estimate of: $\int_I \rho \xi_e v \, dx$. From generalized Hälder inequality, we obtain

$$\begin{aligned} \int_I \rho \xi_e v \, dx &\leq \|\rho v\|_{L^{\frac{2s}{s+1}}(I;\mathbb{R}^3)} \|\xi_e\|_{L^{\frac{2s}{s-1}}(I;\mathbb{R}^3)} \\ &\leq \|\sqrt{\rho}\|_{W^{1,2}(I;\mathbb{R})} \|\sqrt{\rho} v\|_{L^2(I;\mathbb{R}^3)} \|\xi_e\|_{L^{\frac{2s}{s-1}}(I;\mathbb{R}^3)} \quad , \forall t \in (0;T) \end{aligned} \tag{3.11}$$

From (3.8) to (3.11), we get:

$$\begin{aligned}
\frac{d\,\mathcal{E}_I v(t)}{dt} &\leq -\mu\|\nabla v\|^2_{L^p(I;\mathbb{R}^3)} - (\lambda+\mu)\|\nabla v\|^2_{L^p(I;\mathbb{R}^3)} \\
&\quad + \|\sqrt{\rho}\|_{W^{1,2}(I)}\|\sqrt{\rho}v\|_{L^2(I;\mathbb{R}^3)}\|\xi_e\|_{L^{\frac{2s}{s-1}}(I;\mathbb{R}^3)} \\
\frac{d\,\mathcal{E}_I v(t)}{dt} &\leq -(\lambda+2\mu)\|\nabla v\|^2_{L^p(I;\mathbb{R}^3)} + \|\sqrt{\rho}\|_{W^{1,2}(I;\mathbb{R})}\|\sqrt{\rho}v\|_{L^2(I;\mathbb{R}^3)}\|\xi_e\|_{L^{\frac{2s}{s-1}}(I;\mathbb{R}^3)} \\
\frac{d\,\mathcal{E}_I v(t)}{dt} &\leq -(\lambda+2\mu)\|\nabla v\|^2_{L^p(I;\mathbb{R}^3)}\left(1 - \frac{\|\sqrt{\rho}\|_{W^{1,2}(I;\mathbb{R})}\|\sqrt{\rho}v\|_{L^2(I;\mathbb{R}^3)}\|\xi_e\|_{L^{\frac{2s}{s-1}}(I;\mathbb{R}^3)}}{(\lambda+2\mu)\|\nabla v\|^2_{L^p(I;\mathbb{R}^3)}}\right)
\end{aligned}$$

$$\frac{d\,\mathcal{E}_I v(t)}{dt} \leq -(\lambda+2\mu)\|\nabla v\|^2_{L^p(I;\mathbb{R}^3)}\left(1 - \max_{C_0^\infty}\frac{\Theta_p}{\Theta_d}\right) \tag{3.12}$$

Let's put: $\mathcal{R}_{\mathcal{E}}^{-1}(\lambda,\mu) = \max_{C_0^\infty}\frac{\Theta_p}{\Theta_d}$.

Then the previous inequality can be rewritten

$$\frac{d\,\mathcal{E}_I v(t)}{dt} \leq -(\lambda+2\mu)\|\nabla v\|^2_{L^p(I;\mathbb{R}^3)}(1-\mathcal{R}_{\mathcal{E}}^{-1}(\lambda,\mu)). \tag{3.13}$$

If $\mathcal{R}_{\mathcal{E}}(\lambda,\mu) < 1$, then $1-\mathcal{R}_{\mathcal{E}}^{-1}(\lambda,\mu) > 0$ and by virtue of the poincar inequality there exists a constant $\eta(I)$, which depends on the domain I, such that

$$\int_I |v|^2 dx \leq \eta(I)\int_I |\nabla v|^2 dx,$$

we have:

$$\frac{d\,\mathcal{E}_I v(t)}{dt} \leq -\eta(I)(\lambda+2\mu)\left(1-\mathcal{R}_{\mathcal{E}}^{-1}(\lambda,\mu)\right)\|v\|^2. \tag{3.14}$$

A simple integration of (3.14) over $(0,t)$ leads to

$$\mathcal{E}_I v(t) \leq \mathcal{E}_I v(0)\exp\left\{\frac{-2\,\eta(I)(\lambda+2\mu)t}{\rho}\left(1-\mathcal{R}_{\mathcal{E}}^{-1}(\lambda,\mu)\right)\right\}. \tag{3.15}$$

And from Poincar's inequality, nonlinear stability is inferred by the condition:

$$\mathcal{R}_{\mathcal{E}}^{-1}(\lambda,\mu) < 1 \quad when \quad t \longrightarrow \infty.$$

From (3.13) we see that the optimal stability threshold is reached for: $\mathcal{R}_{\mathcal{E}}^{-1}(\lambda,\mu) = 1$.■

We have shown that all solutions decrease as physically expected. Nevertheless, it is a result of conditional stability. In the following section we look at the stability of the perturbed system (3.1)–(3.2).

3.2. Nonlinear energy stability for the perturbed solution

The idea now is to study the stability of the solution $v(x,t)$ when it is perturbed. In this order, we consider $v^{\epsilon}(x,t)$, or solution of the following system

$$\begin{cases} \rho \dfrac{\partial v_i^{\epsilon}}{\partial t} + \displaystyle\sum_{j=1}^{3} v_j^{\epsilon} \rho \dfrac{\partial v_i^{\epsilon}}{\partial x_j} = \rho \xi_e + L_{\lambda,\mu}(v^{\epsilon}), \quad \forall (x,t) \in I \times (0,T), \quad i = 1,2,3 \\ \displaystyle\sum_{j=1}^{3} \dfrac{\partial v_j^{\epsilon}}{\partial x_j} = 0, \ \forall (x,t) \in I \times (0,T), \\ \rho|_{t=0} = \rho|_{t=T} \quad \forall x \in I, \\ v^{\epsilon}|_{t=0} = v_0^{\epsilon}(x), \ \forall x \in I. \end{cases} \tag{3.16}$$

Since we are looking for a stability result, it is natural to introduce a perturbation $u(x,t) = v^{\epsilon}(x,t) - v(x,t)$ and $\Pi = \pi^{\epsilon} - \pi$, a pressure perturbation that also verifies the law (2.4). Then u satisfies the following coupling system:

$$\begin{cases} \rho \dfrac{\partial u}{\partial t} + \displaystyle\sum_{i,j=1}^{3} v_j \rho \dfrac{\partial u_i}{\partial x_j} + \displaystyle\sum_{i,j=1}^{3} u_j \rho \dfrac{\partial v_i^{\epsilon}}{\partial x_j} = \rho \xi_e - \nabla \Pi + \mu div(\nabla u) + (\lambda + \mu) \nabla^2 u, \\ \displaystyle\sum_{j=1}^{3} \dfrac{\partial u_j}{\partial x_j} = 0, \ \forall (x,t) \in I \times (0,T), \\ u(x,0) = \bar{u}(x), \quad \forall x \in I, \\ \rho|_{t=0} = \rho|_{t=T} \quad \forall x \in I. \end{cases} \tag{3.17}$$

The objective is to determine a sufficient condition on $v(x,t)$ such that all perturbations $u(x,t)$ tend to zero when $t \longrightarrow \infty$.

Theorem 3.2. Suppose the hypotheses 1 and 2 are verified, (μ, λ) and $\pi(\rho)$ satisfy respectively (2.3) and (2.4). Let $v(x,t)$ a solution of (3.1) $-$ (3.2) and $v^{\epsilon}(x,t)$ a solution of (3.16). Moreover, if $\nabla \widetilde{u} \in L^2((0,T); L^p(I;\mathbb{R}^3))$ and

$$\left[\frac{(\lambda + 2\mu)}{\sigma^2 l} - \mathcal{Q}^{-1} \right] > 0$$

where

$$\mathcal{Q}^{-1} = \max_{C_0^{\infty}} \frac{\frac{1}{2} \int_I \rho \left(\frac{\partial \widetilde{v}}{\partial \widetilde{x}_i} + \frac{\partial \widetilde{v}}{\partial \widetilde{x}_j} \right) \widetilde{u}_i \widetilde{u}_j dx}{\|\nabla \widetilde{u}\|^2_{L^p(I;\mathbb{R}^3)}}, \quad (\|\nabla \widetilde{u}\|^2_{L^p(I;\mathbb{R}^3)} \neq 0).$$

Then there exists a constant $\mathcal{L}(I) > 0$, which depends on the domain I such that

$$\mathcal{E}_I u(t) \leq \mathcal{E}_I u(0) \exp\left\{ -\frac{2\sigma l t}{(\lambda + 2\mu)\rho} \mathcal{L}(I) \left[\frac{(\lambda + 2\mu)}{\sigma l} - \mathcal{Q}^{-1} \right] \right\}, \tag{3.18}$$

where σ and l dimensionless variables for ν and x respectively.

Proof. Let $\nu(x,t)$ be the solution of (3.17). The functional of the perturbation $u(x,t) = \nu^{\epsilon}(x,t) - \nu(x,t)$ defined by

$$\mathcal{E}_I u(t) \stackrel{\text{def}}{:=} \frac{1}{2} \int_I \rho |u(x,t)|^2 dx. \tag{3.19}$$

By multiplying the equation $(3.17)_1$ by $u(x,t)$ and by integrating on I, we get: $\forall t \in (0,T)$

$$\int_I \rho \frac{\partial u^2}{\partial t} dx + \int_I \sum_{i,j=1}^{3} u_j u_i \rho \frac{\partial \nu_i^{\epsilon}}{\partial x_j} dx + \int_I \sum_{i,j=1}^{3} \nu_j \rho \frac{\partial u_i^2}{\partial x_j} dx + \int_I \nabla \Pi u dx - \mu \int_I \Delta u^2 dx$$

$$-(\lambda + \mu) \int_I \nabla(\nabla u) u dx = 0. \tag{3.20}$$

Let's derive $\mathcal{E}_I u(t)$, then from (3.20) we find equation:

$$\frac{d\, \mathcal{E}_I u(t)}{dt} = -\int_I \sum_{i,j=1}^{3} \rho \nu_j^{\epsilon} u_i u_{i,j} dx - \int_I \sum_{i,j=1}^{3} \rho u_j u_i \nu_{i,j} dx + \mu \int_I u\, \Delta u dx$$

$$+(\lambda + \mu) \int_I u \nabla(\nabla u) dx - \int_I \nabla \Pi u\, dx. \tag{3.21}$$

After integrating each term of (3.21) by parts, using the divergence theorem and the boundary conditions, we obtain the following equalities:

$$\int_I \sum_{i,j=1}^{3} \rho \nu_j^{\epsilon} u_i u_{i,j} dx = \frac{1}{2} \int_{\partial I} \rho \nu_j^{\epsilon} u^2 \vec{n}_I ds - \frac{1}{2} \int_I \rho \frac{\partial \nu_j^{\epsilon}}{\partial x_j} u^2 dx = 0, \tag{3.22}$$

$$\int_I \nabla \Pi u dx = \int_{\partial I} u \Pi \cdot \vec{n}_I ds - \int_I \sum_{j=1}^{3} \frac{\partial u_j}{\partial x_j} \Pi dx = 0, \tag{3.23}$$

$$\int_I u\, \Delta u dx = -\int_I \frac{\partial^2 u}{\partial x^2} dx + \int_{\partial I} u \frac{\partial u}{\partial x} dx = -\int_I \frac{\partial^2 u}{\partial x^2} dx, \tag{3.24}$$

$$\int_I \sum_{i,j=1}^{3} u_j u_i v_{i,j} dx = -\frac{1}{2}\int_I (\frac{\partial v}{\partial x_i} + \frac{\partial v}{\partial x_j}) u_i u_j dx, \tag{3.25}$$

$$\int_I u\nabla div(u) dx = -\int_I \frac{\partial^2 u}{\partial x^2} dx + \int_{\partial I} u\frac{\partial u}{\partial x} dx = -\int_I \frac{\partial^2 u}{\partial x^2} dx. \tag{3.26}$$

From (3.22) to (3.26), we get:

$$\frac{d\,\mathcal{E}_I u(t)}{dt} = \frac{1}{2}\int_I \rho(\frac{\partial v}{\partial x_i} + \frac{\partial v}{\partial x_j}) u_i u_j dx - \mu\int_I \frac{\partial^2 u}{\partial x^2} dx - (\lambda+\mu)\int_I \frac{\partial^2 u}{\partial x^2} dx. \tag{3.27}$$

Let us define typical scales to the phenomenon we are studying. We introduce variables without dimensions in the equation (3.27)

$$\begin{cases} x = \widetilde{x} l \\ v = \widetilde{v}\sigma \\ u = \widetilde{u}\sigma \\ t = \dfrac{l^2}{(\lambda+2\mu)}\widetilde{t} \end{cases}$$

With these transformations, the terms of equation (3.27) thus become:

$$\frac{1}{2}\int_I \rho(\frac{\partial v}{\partial x_i} + \frac{\partial v}{\partial x_j}) u_i u_j dx = \frac{1}{2}\sigma^3 l^2 \int_I \rho(\frac{\partial \widetilde{v}}{\partial \widetilde{x}_i} + \frac{\partial \widetilde{v}}{\partial \widetilde{x}_j}) \widetilde{u}_i \widetilde{u}_j d\widetilde{x}$$

$$-\mu\int_I \frac{\partial^2 u}{\partial x^2} dx - (\lambda+\mu)\int_I \frac{\partial^2 u}{\partial x^2} dx = -\mu\sigma^2 l \int_I \frac{\partial^2 \widetilde{u}}{\partial \widetilde{x}^2} d\widetilde{x} - (\lambda+\mu)\sigma^2 l \int_I \frac{\partial^2 \widetilde{u}}{\partial \widetilde{x}^2} d\widetilde{x}.$$

The following modified equation (3.28) is deducted:

$$(\lambda+2\mu)\sigma^2 l\,\frac{d\,\mathcal{E}_I u(t)}{d\widetilde{t}} = \sigma^3 l^2 \int_I \rho(\frac{\partial \widetilde{v}}{\partial \widetilde{x}_i} + \frac{\partial \widetilde{v}}{\partial \widetilde{x}_j}) \widetilde{u}_i \widetilde{u}_j d\widetilde{x} - (\lambda+2\mu)\sigma^2 l \int_I \frac{\partial^2 \widetilde{u}}{\partial \widetilde{x}^2} d\widetilde{x} \tag{3.28}$$

$$\frac{d\,\mathcal{E}_I u(t)}{d\widetilde{t}} = \frac{\sigma l}{(\lambda+2\mu)}\int_I \rho(\frac{\partial \widetilde{v}}{\partial \widetilde{x}_i} + \frac{\partial \widetilde{v}}{\partial \widetilde{x}_j}) \widetilde{u}_i \widetilde{u}_j d\widetilde{x} - \int_I \frac{\partial^2 \widetilde{u}}{\partial \widetilde{x}^2} d\widetilde{x}$$

$$\frac{d\,\mathcal{E}_I u(t)}{d\widetilde{t}} = \frac{\sigma l}{(\lambda+2\mu)}\int_I \rho(\frac{\partial \widetilde{v}}{\partial \widetilde{x}_i} + \frac{\partial \widetilde{v}}{\partial \widetilde{x}_j}) \widetilde{u}_i \widetilde{u}_j d\widetilde{x} - \|\nabla\widetilde{u}\|^2_{L^p(I;\mathbb{R}^3)}$$

$$\frac{d\,\mathcal{E}_I u(t)}{d\widetilde{t}} = -\frac{\sigma l}{(\lambda+2\mu)}\|\nabla\widetilde{u}\|^2_{L^p(I;\mathbb{R}^3)}\left[\frac{(\lambda+2\mu)}{\sigma l} - \frac{\int_I \rho(\frac{\partial\widetilde{\nu}}{\partial\widetilde{x}_i}+\frac{\partial\widetilde{\nu}}{\partial\widetilde{x}_j})\widetilde{u}_i\widetilde{u}_j d\widetilde{x}}{\|\nabla\widetilde{u}\|^2_{L^p(I;\mathbb{R}^3)}}\right]$$

$$\frac{d\,\mathcal{E}_I u(t)}{d\widetilde{t}} = -\frac{\sigma l}{(\lambda+2\mu)}\|\nabla\widetilde{u}\|^2_{L^p(I;\mathbb{R}^3)}\left[\frac{(\lambda+2\mu)}{\sigma l} - \mathcal{Q}^{-1}\right] \tag{3.29}$$

where $\mathcal{Q}^{-1} = \max_{\mathcal{C}_0^\infty} \frac{\frac{1}{2}\int_I \rho(\frac{\partial\widetilde{\nu}}{\partial\widetilde{x}_i}+\frac{\partial\widetilde{\nu}}{\partial\widetilde{x}_j})\widetilde{u}_i\widetilde{u}_j d\widetilde{x}}{\|\nabla\widetilde{u}\|^2_{L^p(I;\mathbb{R}^3)}}$.

Indeed if $\left[\frac{(\lambda+2\mu)}{\sigma l} < \mathcal{Q}\right]$, we have : $\frac{(\lambda+2\mu)}{3\sigma l} - \mathcal{Q}^{-1} > 0$

So for everything $\mathcal{L}(I) > 0$, $\|\nabla u\|^2_{L^p(I;\mathbb{R}^3)} \geq \mathcal{L}(I)\|u\|^2_{L^2(I;\mathbb{R}^3)}$ (PoincarrÃ©'s inequality), the inequality (3.29) becomes :

$$\frac{d\,\mathcal{E}_I u(t)}{dt} \leq -\frac{2\,\sigma l t}{(\lambda+2\mu)\rho}\mathcal{L}(I)\left[\frac{(\lambda+2\mu)}{\sigma l} - \mathcal{Q}^{-1}\right]\mathcal{E}_I u(t) \tag{3.30}$$

So,

$$\mathcal{E}_I u(t) \leq \mathcal{E}_I u(0)\exp\left\{-\frac{2\,\sigma l t}{(\lambda+2\mu)\rho}\mathcal{L}(I)\left[\frac{(\lambda+2\mu)}{\sigma l} - \mathcal{Q}^{-1}\right]\right\}$$

This gives an unconditional nonlinear stability and (3.18) is completely proved. ■

4. Asymptotic stability

In this section, we study asymptotic stability for solving the partial differential equation when the initial data and external force are perturbed.

If $\nu(x,t)$ is initially perturbed by $a \in L^2(I;\mathbb{R}^3)$ then the system perturbed by $\nu^\epsilon(x,t)$ is governed by the following equations:

$$\begin{cases} \rho\frac{\partial\nu^\epsilon}{\partial t} + \sum_{i,j=1}^{3}\nu_j^\epsilon\rho\frac{\partial\nu_i}{\partial x_j} = \rho\xi_e - \nabla\pi + \mu div(\nabla\nu^\epsilon) + (\lambda+\mu)\nabla(\nabla\nu^\epsilon), \\ \sum_{j=1}^{3}\frac{\partial\nu_j^\epsilon}{\partial x_j} = 0, \quad \forall(x,t)\in I\times(0,T), \\ \nu^\epsilon(x,0) = \nu(x,0) + a \quad \forall x \in I, \end{cases} \tag{4.1}$$

To show asymptotic stability, we have to show that the difference between $\nu^\epsilon(x,t)$ and $\nu(x,t)$ tends toward zero when time tends toward infinity in a Sobolev norm.

Let's start by stating the main result of this section.

Theorem 4.1. (asymptotic solution behaviour) Let I be as in Hypothesis 1 and ν_0, ξ_e as in Hypothesis 2. Suppose that $\nu(x,t)$ is a solution of (3.16) and $\nu^\epsilon(x,t)$ a solution of the perturbed problem (4.1) with $a \in L^2(I;\mathbb{R}^3)$, moreover, if $\nabla\nu$ also lies in the following regular class

$$\nabla\nu \in L^\gamma((0,T);L^p(I;\mathbb{R}^3)), \quad \frac{2}{\gamma}+\frac{3}{p} \leqslant 2, \quad 2 \leq p < \infty. \tag{4.4}$$

Then we have the property of asymptotic stability

$$\|\nu^\epsilon(x)-\nu(x)\|_{L^2(I;\mathbb{R}^3)} \longrightarrow 0 \qquad (t \longrightarrow \infty)$$

The demonstration of theorem 4.1 uses the following lemma 4.1.

Lemma 4.2. Let I be as in Hypothesis 1 and ν_0, ξ_e as in Hypothesis 2. Suppose that $\nu(x,t)$ is a solution of (3.16) and $\nu^\epsilon(x,t)$ a solution of the perturbed problem (4.1). If the perturbation $u(x,t)=\nu(x,t)-\nu^\epsilon(x,t)$ satisfies the perturbed system, then by the Fourier Plancherel transformation:

$$k_\rho\left|\widehat{(\nu^\epsilon-\nu)}\right|^2 \leqslant 2\,k_\rho|\widehat{a_0^\epsilon}|^2\exp(-2\,k_\rho\zeta^2 t)+4\big(1+\rho^{-1}\big)^2\big(\|\nu^\epsilon\|^4_{K^1_{div}}+\|\nu\|^4_{K^1_{div}}\big), \quad \zeta \in I$$

where $u(\zeta,0)=a_0^\epsilon$ and $\quad k_\rho=\rho^{-1}(\lambda+2\mu)$.

Proof. Let $\nu_0 \in K^1_{div}$ and $\xi_e=\xi_e(x,t) \in L^1((0,T);L^{\frac{2s}{s-1}}(I;\mathbb{R}^3))$. Let $(\nu\,;\pi)$ and $(\nu^\epsilon;\,\pi^\epsilon)$ be two solutions corresponding respectively to $(\nu_0\,;\xi_e)$ and $(\nu_0^\epsilon\,;\xi_e)$.

The perturbation $u(x,t)=\nu^\epsilon(x,t)-\nu(x,t)$ checks the coupling equation:

$$\begin{aligned}&\frac{\partial(\nu^\epsilon-\nu)}{\partial t}+\sum_{i,j=1}^{3}u_j\frac{\partial(u+\nu^\epsilon)_i}{\partial x_j}-\sum_{i,j=1}^{3}(\nu-u)_j\frac{\partial\nu_i^\epsilon}{\partial x_j}+\rho^{-1}\nabla(\pi^\epsilon-\pi)\\&=\rho^{-1}(\lambda+2\mu)\Delta(\nu^\epsilon-\nu)\end{aligned}$$

Which gives us:

$$\begin{aligned}&\frac{\partial(\nu^\epsilon-\nu)}{\partial t}+\sum_{i,j=1}^{3}\nu_j^\epsilon\frac{\partial(\nu^\epsilon-\nu)_i}{\partial x_j}+\sum_{i,j=1}^{3}(\nu^\epsilon-\nu)_j\frac{\partial\nu_i}{\partial x_j}+\rho^{-1}\nabla(\pi^\epsilon-\pi)\\&=\rho^{-1}(\lambda+2\mu)\Delta(\nu^\epsilon-\nu)\end{aligned} \tag{4.5}$$

The perturbed system thus becomes:

$$\left\{\begin{aligned}&\frac{\partial(\nu^\epsilon-\nu)}{\partial t}+\sum_{i,j=1}^{3}\nu_j^\epsilon\frac{\partial(\nu^\epsilon-\nu)_i}{\partial x_j}+\sum_{i,j=1}^{3}(\nu^\epsilon-\nu)_j\frac{\partial\nu_i}{\partial x_j}+\rho^{-1}\nabla(\pi^\epsilon-\pi)\\&=\rho^{-1}(\lambda+2\mu)\Delta(\nu^\epsilon-\nu)\\&\sum_{i=1}^{3}(\nu^\epsilon-\nu)_i=0, \quad (x,t)\in I\times(0,T),\ u(x,0)=a_0^\epsilon \quad x\in I.\end{aligned}\right. \tag{4.6}$$

Let's multiply the equation $(4.6)_1$ by $2(\nu - \nu^\epsilon)$ and integrate on the domain I, we obtain:

$$
\begin{aligned}
&2\int_I \frac{\partial(\nu^\epsilon-\nu)^2}{\partial t}dx + 2\int_I \sum_{i,j=1}^{3} \nu_j^\epsilon \frac{\partial(\nu^\epsilon-\nu)_i}{\partial x_j}(\nu^\epsilon-\nu)_i dx \\
&\quad + 2\int_I \sum_{i,j=1}^{3} (\nu^\epsilon-\nu)_j \frac{\partial \nu_i}{\partial x_j}(\nu^\epsilon-\nu)_i dx \\
&\quad + 2\rho^{-1}\int_I \nabla(\pi^\epsilon-\pi)(\nu^\epsilon-\nu)dx = 2\rho^{-1}(\lambda+2\mu)\int_I \Delta(\nu^\epsilon-\nu)^2 dx. \qquad (4.7)
\end{aligned}
$$

According to $(4.6)_2$, and by integration by parts we get (4.7) becomes:

$$
\begin{aligned}
&\frac{d}{dt}\int_I (\nu^\epsilon-\nu)^2 dx + 2\int_I \sum_{i,j=1}^{3} \nu_j^\epsilon \frac{\partial(\nu^\epsilon-\nu)_i}{\partial x_j}(\nu^\epsilon-\nu)_i dx \\
&\quad + 2\int_I \sum_{i,j=1}^{3} (\nu^\epsilon-\nu)_j \frac{\partial \nu_i}{\partial x_j}(\nu^\epsilon-\nu)_i dx \\
&\quad = -2\rho^{-1}(\lambda+2\mu)\int_I \nabla(\nu^\epsilon-\nu)^2 dx, \qquad (4.8)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{d}{dt}\|\nu^\epsilon-\nu\|^2_{L^2(I;\mathbb{R}^3)} + 2\rho^{-1}(\lambda+2\mu)\|\nabla(\nu^\epsilon-\nu)\|^2_{L^p(I;\mathbb{R}^3)} \\
&\quad = -2\int_I \sum_{i,j=1}^{3} \nu_j^\epsilon \frac{\partial(\nu^\epsilon-\nu)_i}{\partial x_j}(\nu^\epsilon-\nu)_i dx \\
&\quad\quad -2\int_I \sum_{i,j=1}^{3} (\nu^\epsilon-\nu)_j \frac{\partial \nu_i}{\partial x_j}(\nu^\epsilon-\nu)_i dx. \qquad (4.9)
\end{aligned}
$$

Using Hälder's inequality we deduce the following estimates:

$$
\begin{aligned}
\left|-2\int_I \sum_{i,j=1}^{3} (\nu^\epsilon-\nu)_j \frac{\partial \nu_i}{\partial x_j}(\nu^\epsilon-\nu)_i dx\right| &\leqslant 2\,\|\nu^\epsilon-\nu\|_{L^2(I;\mathbb{R}^3)}\|\nabla(\nu^\epsilon-\nu)\|_{L^p(I;\mathbb{R}^3)}\|\nu\|_{K^1_{div}} \\
&\leqslant 2\,\|\nu\|^2_{K^1_{div}}\|\nu^\epsilon-\nu\|^2_{L^2(I;\mathbb{R}^3)} + \frac{1}{2}\,\|\nabla(\nu^\epsilon-\nu)\|^2_{L^p(I;\mathbb{R}^3)}
\end{aligned}
$$

$$
\begin{aligned}
\left|-2\int_I \sum_{i,j=1}^{3} \nu_j^\epsilon \frac{\partial(\nu^\epsilon-\nu)_i}{\partial x_j}(\nu^\epsilon-\nu)_i dx\right| &\leqslant 2\,\|\nu^\epsilon-\nu\|_{L^2(I;\mathbb{R}^3)}\|\nabla(\nu^\epsilon-\nu)\|_{L^p(I;\mathbb{R}^3)}\|\nu^\epsilon\|_{K^1_{div}} \\
&\leqslant 2\,\|\nu^\epsilon\|^2_{K^1_{div}}\|\nu^\epsilon-\nu\|^2_{L^2(I;\mathbb{R}^3)} + \frac{1}{2}\,\|\nabla(\nu^\epsilon-\nu)\|^2_{L^p(I;\mathbb{R}^3)}
\end{aligned}
$$

After these estimates the equation (4.9) becomes:

$$\frac{d}{dt}\|v^\epsilon - v\|^2_{L^2(I;\mathbb{R}^3)} + 2\,\rho^{-1}(\lambda + 2\mu)\|\nabla(v^\epsilon - v)\|^2_{L^p(I;\mathbb{R}^3)}$$

$$\leqslant 2\,\|v\|^2_{K^1_{div}}\,\|v^\epsilon - v\|^2_{L^2(I;\mathbb{R}^3)} + \|\nabla(v^\epsilon - v)\|^2_{L^p(I;\mathbb{R}^3)} + 2\,\|v^\epsilon\|^2_{K^1_{div}}\,\|v^\epsilon - v\|^2_{L^2(I;\mathbb{R}^3)} \tag{4.10}$$

Therefore, we get

$$\frac{d}{dt}\|v^\epsilon - v\|^2_{L^2(I;\mathbb{R}^3)} + \big[2\,\rho^{-1}(\lambda + 2\mu) - 1\big]\|\nabla(v^\epsilon - v)\|^2_{L^p(I;\mathbb{R}^3)}$$

$$\leqslant 2\|v^\epsilon - v\|^2_{L^p(I;\mathbb{R}^3)}\big(\|v\|^2_{K^1_{div}} + \|v^\epsilon\|^2_{K^1_{div}}\big). \tag{4.11}$$

Let's introduce the Fourier space transform of h in relation with $x \in I$ denoted $\mathcal{F}[h]$:

$$\mathcal{F}[h] = \hat{h}(\zeta) = \int_I h(x)\exp(-ix\zeta)dx, \quad \zeta \in I,$$

$h(x) \in S(I)$ where $S(I)$ is a vector space defined by:

$$S(I) = \{h \in C^\infty(I) : |\,h\,|_{m,n} = sup|x^m(\partial^n h)(x)| < \infty,\ m,n \in \mathbb{N}\}.$$

By applying the Fourier Plancherel theorem the equation (4.11) becomes:

$$\frac{d}{dt}\int_I \Big|\widehat{(v^\epsilon - v)}\Big|^2 d\zeta + \big[2\,\rho^{-1}(\lambda + 2\mu) - 1\big]\int_I |\zeta|^2\big|\widehat{(v^\epsilon - v)}\big|^2 d\zeta \leqslant$$

$$2\,\|\widehat{v^\epsilon - v}\|^2_{L^p(I;\mathbb{R}^3)}\big(\|v\|^2_{K^1_{div}} + \|v^\epsilon\|^2_{K^1_{div}}\big). \tag{4.12}$$

Consider the equation $(4.6)_1$ and apply the Fourier spatial transform.

$$\mathcal{F}[(v^\epsilon - v)_t + (v^\epsilon\cdot\nabla)(v^\epsilon - v) + (v^\epsilon - v)\cdot\nabla v + \rho^{-1}\nabla(\pi^\epsilon - \pi)$$
$$= \rho^{-1}(\lambda + 2\mu)\Delta(v^\epsilon - v)]$$

Which gives us

$$(\widehat{v^\epsilon - v})_t + k\,|\zeta|^2\rho^{-1}(\lambda + 2\mu)(\widehat{v^\epsilon - v}) = \widehat{f}(x,t), \tag{4.13}$$

where

$$\widehat{f}(x,t) = \mathcal{F}\left[-\sum_{i,j=1}^{3} v^\epsilon_j\frac{\partial(v^\epsilon - v)_i}{\partial x_j} - \sum_{i,j=1}^{3}(v^\epsilon - v)_j\frac{\partial v_i}{\partial x_j} - \rho^{-1}\nabla(\pi^\epsilon - \pi)\right].$$

Let's now estimate each term of $\widehat{f}(x,t)$. We recall that,

$$(u\cdot\nabla)\nu = u_j\nu_{i,j=1} = u_j\frac{\partial \nu_i}{\partial x_j} = \sum_{i,j=1}^{3} u_j\frac{\partial \nu_i}{\partial x_j}$$

$$\left|\mathcal{F}\left[-\sum_{i,j=1}^{3}(\nu^{\epsilon}-\nu)_j\frac{\partial \nu_i}{\partial x_j}\right]\right| = \left|\int_I\sum_{i,j=1}^{3}(\widehat{\nu^{\epsilon}-\nu})_j\frac{\partial \nu_i}{\partial x_j}\exp(-ix\zeta)dx\right|$$

$$\leqslant \int_I\sum_{i,j=1}^{3}|(\nu^{\epsilon}-\nu)_j||\nu_{i,j}||\zeta_j|dx \leqslant \|\nu^{\epsilon}-\nu\|_{L^2(I;\mathbb{R}^3)}\|\nu\|_{K^1_{div}}|\zeta| \tag{4.14}$$

$$\left|\mathcal{F}\Big[-\sum_{i,j=1}^{3}\nu_j^{\epsilon}(\nu^{\epsilon}-\nu)_{xi}\Big]\right| = \left|\int_I\sum_{i,j=1}^{3}\nu_j^{\epsilon}(\widehat{\nu^{\epsilon}-\nu})_{xi}\exp(-ix\zeta)dx\right|$$

$$\leqslant \int_I\sum_{i,j=1}^{3}|\nu_j^{\epsilon}||(\nu^{\epsilon}-\nu)_i||\nu_{i,j}^{\epsilon}||\zeta_j|dx \leqslant \|\nu^{\epsilon}-\nu\|_{L^2(I;\mathbb{R}^3)}\|\nu^{\epsilon}\|_{K^1_{div}}|\zeta| \tag{4.15}$$

For the estimation of $\mathcal{F}\left[-\rho^{-1}\nabla(\pi^{\epsilon}-\pi)\right]$, consider the perturbation equation $(4.6)_1$

$$-\rho^{-1}\nabla(\pi^{\epsilon}-\pi) = \frac{\partial(\nu^{\epsilon}-\nu)}{\partial t} + \sum_{i,j=1}^{3}\nu_j^{\epsilon}\frac{\partial(\nu^{\epsilon}-\nu)_i}{\partial x_j}$$
$$+\sum_{i,j=1}^{3}(\nu^{\epsilon}-\nu)_j\frac{\partial \nu_i}{\partial x_j} - \rho^{-1}(\lambda+2\mu)\Delta(\nu^{\epsilon}-\nu)$$

By applying the operator ∇, we obtain:

$$-\rho^{-1}\nabla^2(\pi^{\epsilon}-\pi) = \sum_{i,j=1}^{3}\frac{\partial}{\partial x_i}\left(\frac{\partial \nu_i}{\partial x_j}(\nu^{\epsilon}-\nu)_j\right) + \sum_{i,j=1}^{3}\frac{\partial}{\partial x_i}\left((\nu^{\epsilon}-\nu)_{xj}\nu_j^{\epsilon}\right)$$

$$-\rho^{-1}\nabla^2(\pi^{\epsilon}-\pi) = \sum_{i,j}^{3}\left(\frac{\partial^2\nu_i(\nu^{\epsilon}-\nu)_j}{\partial x_i\partial x_j} + \frac{\partial^2(\nu^{\epsilon}-\nu)_i\nu_j^{\epsilon}}{\partial x_i\partial x_j}\right).$$

The Fourier transformation gives

$$\mathcal{F}\big[-\rho^{-1}\nabla^2(\pi^{\epsilon}-\pi)\big] = \mathcal{F}\left[\sum_{i,j=1}^{3}\left(\frac{\partial^2\nu_i(\nu^{\epsilon}-\nu)_j}{\partial x_i\partial x_j} + \frac{\partial^2(\nu^{\epsilon}-\nu)_i\nu_j^{\epsilon}}{\partial x_i\partial x_j}\right)\right]$$

$$-\rho^{-1}\int_I \widehat{\Delta\pi}\exp(-ix\zeta)d\zeta = \int_I \sum_{i,j=1}^{3}\left(\frac{\partial^2 \nu_i(\widehat{\nu^\epsilon-\nu})_j}{\partial x_i \partial x_j} + \frac{\partial^2 (\widehat{\nu^\epsilon-\nu})_i \nu_j^\epsilon}{\partial x_i \partial x_j}\right)\exp(-ix\zeta)d\zeta. \tag{4.16}$$

So from (4.16) we get

$$\begin{aligned}\rho^{-1}\widehat{\pi}|\zeta| &\leqslant \left(\|\nu^\epsilon-\nu\|_{L^2(I;\mathbb{R}^3)}\|\nu_i\|_{K^1_{div}} + \|\nu^\epsilon-\nu\|_{L^2(I;\mathbb{R}^3)}\|\nu_j^\epsilon\|_{K^1_{div}}\right)|\zeta|^2\\ &\leqslant \left(\|\nu_i\|_{K^1_{div}} + \|\nu_j^\epsilon\|_{K^1_{div}}\right)\|\nu^\epsilon-\nu\|_{L^2(I;\mathbb{R}^3)}|\zeta|^2.\end{aligned}$$

We get more simply

$$\widehat{\pi} \leqslant \rho^{-1}|\zeta|\left(\|\nu\|_{K^1_{div}} + \|\nu^\epsilon\|_{K^1_{div}}\right)\|\nu^\epsilon-\nu\|_{L^2(I;\mathbb{R}^3)} \tag{4.17}$$

From (4.14) to (4.17), we get

$$\widehat{f}(x,t) \leqslant \|\nu^\epsilon-\nu\|_{L^2(I;\mathbb{R}^3)}\|\nu^\epsilon\|_{K^1_{div}}|\zeta| + \|\nu^\epsilon-\nu\|_{L^2(I;\mathbb{R}^3)}\|\nu\|_{K^1_{div}}|\zeta|$$

$$+\rho^{-1}|\zeta|\left(\|\nu\|_{K^1_{div}} + \|\nu^\epsilon\|_{K^1_{div}}\right)\|\nu^\epsilon-\nu\|_{L^2(I;\mathbb{R}^3)}$$

$$\widehat{f}(x,t) \leqslant \|\nu^\epsilon-\nu\|_{L^2(I;\mathbb{R}^3)}|\zeta|\left(\|\nu\|_{K^1_{div}} + \|\nu^\epsilon\|_{K^1_{div}}\right) + \rho^{-1}|\zeta|\left(\|\nu\|_{K^1_{div}} + \|\nu^\epsilon\|_{K^1_{div}}\right)$$

$$\leqslant |\zeta|\|\nu^\epsilon-\nu\|_{L^2(I;\mathbb{R}^3)}\left(1+\rho^{-1}\right)\left(\|\nu\|_{K^1_{div}} + \|\nu^\epsilon\|_{K^1_{div}}\right) \tag{4.18}$$

Like

$$\|\nu^\epsilon-\nu\|_{L^2(I;\mathbb{R}^3)} \leqslant \|\nu^\epsilon\|_{K^1_{div}} + \|\nu\|_{K^1_{div}},$$

then (4.18) becomes:

$$\widehat{f}(x,t) \leqslant |\zeta|\left(1+\rho^{-1}\right)\left(\|\nu^\epsilon\|_{K^1_{div}} + \|\nu\|_{K^1_{div}}\right)^2$$

$$\leqslant 2|\zeta|\left(1+\rho^{-1}\right)\left(\|\nu^\epsilon\|^2_{K^1_{div}} + \|\nu\|^2_{K^1_{div}}\right), \quad \zeta \in I \tag{4.19}$$

Multiply the equation (4.13) by the term $\exp(k_\rho\zeta^2\tau)$ and integrate on $(0\,;t)$

$$\begin{aligned}&\int_0^t \exp(k_\rho\zeta^2\tau)(\widehat{\nu^\epsilon-\nu})_t d\tau + k_\rho\int_0^t \zeta^2(\widehat{\nu^\epsilon-\nu})\exp(k_\rho\zeta^2\tau)d\tau\\ &= \int_0^t \widehat{f}(\zeta,\tau)\exp(k_\rho\zeta^2\tau)d\tau\end{aligned}$$

After integration by parts one arrives at the following result:

$$\widehat{(\nu^\epsilon - \nu)} \exp(k_\rho \zeta^2 t) - C_0 = \int_0^t \exp(k_\rho \zeta^2 \tau) \widehat{f}(\zeta, \tau) d\tau$$

where $k_\rho = \rho^{-1}(\lambda + 2\mu)$ and $C_0 = \widehat{a_0^\epsilon}$

$$\left|\widehat{(\nu^\epsilon - \nu)}\right| \leqslant \widehat{|a_0^\epsilon|} \exp(-k_\rho \zeta^2 t) + \left|\exp(-k_\rho \zeta^2 t) \int_0^t \exp(k_\rho \zeta^2 \tau) \widehat{f}(\zeta, \tau)\right| d\tau$$

$$\left|\widehat{(\nu^\epsilon - \nu)}\right|^2 \leqslant 2\widehat{|a_0^\epsilon|}^2 \exp(-2k_\rho \zeta^2 t) + 2\left|\exp(-2k_\rho \zeta^2 t) \int_0^t \exp(2k_\rho \zeta^2 \tau) \widehat{f^2}(\zeta, \tau)\right| d\tau \tag{4.20}$$

We get from (4.19) that

$$\left|\widehat{(\nu^\epsilon - \nu)}\right|^2 \leqslant 2\widehat{|a_0^\epsilon|}^2 \exp(-2k_\rho \zeta^2 t) + 4|\zeta|^2 \big(1 + \rho^{-1}\big)^2 \big(\|\nu^\epsilon\|^4_{K^1_{div}} + \|\nu\|^4_{K^1_{div}}\big)$$

$$\times \frac{1}{k_\rho \zeta^2} \big(1 - \exp(-2k_\rho \zeta^2 t)\big)$$

$$\left|\widehat{(\nu^\epsilon - \nu)}\right|^2 \leqslant 2\widehat{|a_0^\epsilon|}^2 \exp(-2k_\rho \zeta^2 t) + \frac{4}{k_\rho} \big(1 + \rho^{-1}\big)^2 \big(\|\nu^\epsilon\|^4_{K^1_{div}} + \|\nu\|^4_{K^1_{div}}\big) \tag{4.21}$$

$$k_\rho \left|\widehat{(\nu^\epsilon - \nu)}\right|^2 \leqslant 2k_\rho \widehat{|a_0^\epsilon|}^2 \exp(-2k_\rho \zeta^2 t) + 4\big(1 + \rho^{-1}\big)^2 \big(\|\nu^\epsilon\|^4_{K^1_{div}} + \|\nu\|^4_{K^1_{div}}\big) \tag{4.22}$$

The lemma is thus proved. ∎

Remark 4.3. We use the Fourier transform because the equation transformed in "Fourier space", because the expression (4.14), is much simpler than the equation $(4.7)_2$ in physical space.

Proof of Theorem 4.1. Consider again the equation (4.12)

$$\frac{d}{dt} \int_I \left|\widehat{(\nu^\epsilon - \nu)}\right|^2 d\zeta + \big[2\rho^{-1}(\lambda + 2\mu) - 1\big] \int_I |\zeta|^2 \big|\widehat{(\nu^\epsilon - \nu)}\big|^2 d\zeta \leqslant$$

$$2\,\|\widehat{\nu^\epsilon - \nu}\|^2_{L^p(I;\mathbb{R}^3)} \big(\|\nu\|^2_{K^1_{div}} + \|\nu^\epsilon\|^2_{K^1_{div}}\big).$$

Let $I = \Omega(t) \cup \Omega(t)^c$ where Ω is a domain such as:

$$\Omega(t) = \left\{\zeta \in I \quad telque \quad (1+t)^r|\zeta|^2 \leq r(1+t)^{r-1}, \quad r > 1\right\}. \tag{4.23}$$

Let's estimate the 2nd term of the equation (4.12) by multiplying it by $(1+t)^r$. We have:

$$\left[2\,\rho^{-1}(\lambda+2\mu)-1\right](1+t)^r \int_I |\zeta|^2|\widehat{(\nu^\epsilon-\nu)}|^2 d\zeta =$$

$$\left[2\,\rho^{-1}(\lambda+2\mu)-1\right]\left[(1+t)^r \int_{\Omega^c} |\zeta|^2|\widehat{(\nu^\epsilon-\nu)}|^2 d\zeta + (1+t)^r \int_{\Omega} |\zeta|^2|\widehat{(\nu^\epsilon-\nu)}|^2 d\zeta\right] \tag{4.24}$$

$$\left[2\,\rho^{-1}(\lambda+2\mu)-1\right]\left[(1+t)^r \int_{\Omega^c} |\zeta|^2|\widehat{(\nu^\epsilon-\nu)}|^2 d\zeta + (1+t)^r \int_{\Omega} |\zeta|^2|\widehat{(\nu^\epsilon-\nu)}|^2 d\zeta\right]$$

$$\geqslant r\left[2\,\rho^{-1}(\lambda+2\mu)-1\right](1+t)^{r-1} \int_{\Omega^c} |\widehat{(\nu^\epsilon-\nu)}|^2 d\zeta \tag{4.25}$$

$$r\left[2\,\rho^{-1}(\lambda+2\mu)-1\right](1+t)^{r-1} \int_{\Omega^c} |\widehat{(\nu^\epsilon-\nu)}|^2 d\zeta =$$

$$r\left[2\,\rho^{-1}(\lambda+2\mu)-1\right]\left[(1+t)^{r-1} \int_I |\widehat{(\nu^\epsilon-\nu)}|^2 d\zeta - (1+t)^{r-1} \int_{\Omega} |\widehat{(\nu^\epsilon-\nu)}|^2 d\zeta\right] \tag{4.26}$$

From (4.26), the equation (4.24) therefore becomes,

$$\left[2\,\rho^{-1}(\lambda+2\mu)-1\right](1+t)^r \int_I |\zeta|^2|\widehat{(\nu^\epsilon-\nu)}|^2 d\zeta =$$

$$r\left[2\,\rho^{-1}(\lambda+2\mu)-1\right]\left[(1+t)^{r-1} \int_I |\widehat{(\nu^\epsilon-\nu)}|^2 d\zeta - (1+t)^{r-1} \int_{\Omega} |\widehat{(\nu^\epsilon-\nu)}|^2 d\zeta\right] \tag{4.27}$$

$$\left[2\,\rho^{-1}(\lambda+2\mu)-1\right] \int_I |\zeta|^2|\widehat{(\nu^\epsilon-\nu)}|^2 d\zeta =$$

$$r\left[2\,\rho^{-1}(\lambda+2\mu)-1\right]\left[(1+t)^{-1} \int_I |\widehat{(\nu^\epsilon-\nu)}|^2 d\zeta - (1+t)^{-1} \int_{\Omega} |\widehat{(\nu^\epsilon-\nu)}|^2 d\zeta\right] \tag{4.28}$$

Substituting in the equation (4.12) we find:

$$\frac{d}{dt}\int_I \left|\widehat{(\nu^\epsilon-\nu)}\right|^2 d\zeta + r\big[2\,\rho^{-1}(\lambda+2\mu)-1\big]\big[(1+t)^{-1}\int_I |\widehat{(\nu^\epsilon-\nu)}|^2 d\zeta$$

$$-(1+t)^{-1}\int_\Omega |\widehat{(\nu^\epsilon-\nu)}|^2 d\zeta\big] \leqslant 2\,\|\widehat{\nu^\epsilon-\nu}\|^2_{L^2(I;\mathbb{R}^3)}\big(\|\nu\|^2_{K^1_{div}}+\|\nu^\epsilon\|^2_{K^1_{div}}\big) \tag{4.29}$$

$$\frac{d}{dt}\int_I \left|\widehat{(\nu^\epsilon-\nu)}\right|^2 d\zeta + r\big[2\,\rho^{-1}(\lambda+2\mu)-1\big]\big[(1+t)^{-1}\|(\nu^\epsilon-\nu)\|^2_{L^2(I;\mathbb{R}^3)}$$

$$(1+t)^{-1}\int_\Omega \left|\widehat{(\nu^\epsilon-\nu)}\right|^2 d\zeta\big] \leqslant 2\,\|\widehat{\nu^\epsilon-\nu}\|^2_{L^2(I;\mathbb{R}^3)}\big(\|\nu\|^2_{K^1_{div}}+\|\nu^\epsilon\|^2_{K^1_{div}}\big) \tag{4.30}$$

Multiplying (4.30) by $(1+t)^r$ we get:

$$\frac{d}{dt}\big((1+t)^r\int_I \left|\widehat{(\nu^\epsilon-\nu)}\right|^2 d\zeta\big) + r\big[2\,\rho^{-1}(\lambda+2\mu)-1\big]\big[(1+t)^{r-1}\|(\nu^\epsilon-\nu)\|^2_{L^2(I;\mathbb{R}^3)}$$

$$-(1+t)^{r-1}\int_\Omega \left|\widehat{(\nu^\epsilon-\nu)}\right|^2 d\zeta\big] \leqslant 2\,(1+t)^r\,\|\widehat{\nu^\epsilon-\nu}\|^2_{L^2(I;\mathbb{R}^3)}\big(\|\nu\|^2_{K^1_{div}}+\|\nu^\epsilon\|^2_{K^1_{div}}\big) \tag{4.31}$$

$$\frac{d}{dt}\big((1+t)^r\int_I \left|\widehat{(\nu^\epsilon-\nu)}\right|^2 d\zeta\big) \leqslant -r\big[2\,\rho^{-1}(\lambda+2\mu)-1\big](1+t)^{r-1}\|(\nu^\epsilon-\nu)\|^2_{L^2(I;\mathbb{R}^3)}$$

$$+r\big[2\,\rho^{-1}(\lambda+2\mu)-1\big](1+t)^{r-1}\int_\Omega \left|\widehat{(\nu^\epsilon-\nu)}\right|^2 d\zeta$$

$$+2\,(1+t)^r\,\|\widehat{\nu^\epsilon-\nu}\|^2_{L^2(I;\mathbb{R}^3)}\big(\|\nu\|^2_{K^1_{div}}+\|\nu^\epsilon\|^2_{K^1_{div}}\big) \tag{4.32}$$

Let's determine the estimate of:

$$r\,(1+t)^{r-1}\int_\Omega \left|\widehat{(\nu^\epsilon-\nu)}\right|^2 d\zeta$$

We have:

$$(1+t)^r\frac{d}{dt}\int_I \left|\widehat{(\nu^\epsilon-\nu)}\right|^2 d\zeta \leqslant r(1+t)^{r-1}\int_\Omega \left|\widehat{(\nu^\epsilon-\nu)}\right|^2 d\zeta \tag{4.33}$$

And according to the Lemme 4.1 we have:

$$r(1+t)^{r-1}\int_\Omega \left|\widehat{(\nu^\epsilon-\nu)}\right|^2 d\zeta \leqslant$$

$$r(1+t)^{r-1}\int_{\Omega}\big\{2\,\widehat{|a_0^{\epsilon}|}^2\exp\big(-2\,k_\rho\zeta^2 t\big)+\frac{4}{k_\rho}\big(1+\rho^{-1}\big)^2\big(\|\nu^\epsilon\|^4_{K^1_{div}}+\|\nu\|^4_{K^1_{div}}\big)\big\}d\zeta \tag{4.34}$$

Going back to the equation (4.32), we obtain:

$$\frac{d}{dt}\Big((1+t)^r\int_I\Big|\widehat{(\nu^\epsilon-\nu)}\Big|^2 d\zeta\Big)\leqslant -r\big[2\,\rho^{-1}(\lambda+2\mu)-1\big]\Big[(1+t)^{r-1}\|(\nu^\epsilon-\nu)\|^2_{L^2(I;\mathbb{R}^3)}$$
$$-(1+t)^{r-1}\int_{\Omega}\big\{2\,\widehat{|a_0^{\epsilon}|}^2\exp\big(-2\,k_\rho\zeta^2 t\big)+\frac{4}{k_\rho}\big(1+\rho^{-1}\big)^2\big(\|\nu^\epsilon\|^4_{K^1_{div}}+\|\nu\|^4_{K^1_{div}}\big)\big\}d\zeta\Big]$$
$$+2\,(1+t)^r\|\widehat{\nu^\epsilon-\nu}\|^2_{L^2(I;\mathbb{R}^3)}\big(\|\nu\|^2_{K^1_{div}}+\|\nu^\epsilon\|^2_{K^1_{div}}\big) \tag{4.35}$$

$$\frac{d}{dt}\Big((1+t)^r\int_I\Big|\widehat{(\nu^\epsilon-\nu)}\Big|^2 d\zeta\Big)\leqslant 2\,(1+t)^r\|\widehat{\nu^\epsilon-\nu}\|^2_{L^2(I;\mathbb{R}^3)}\big(\|\nu\|^2_{K^1_{div}}+\|\nu^\epsilon\|^2_{K^1_{div}}\big)$$
$$-r\big(2\,k_\rho-1\big)\Big[(1+t)^{r-1}\|(\nu^\epsilon-\nu)\|^2_{K^1_{div}}-(1+t)^{r-1}\int_{\Omega}\widehat{|a_0^{\epsilon}|}^2\exp\big(-2\,k_\rho\zeta^2 t\big)d\zeta$$
$$+\frac{4}{k_\rho}\big(1+\rho^{-1}\big)^2\big(\|\nu^\epsilon\|^4_{K^1_{div}}+\|\nu\|^4_{K^1_{div}}\big)\int_{\Omega}d\zeta\Big] \tag{4.36}$$

$$\frac{d}{dt}\Big((1+t)^r\int_I\Big|\widehat{(\nu^\epsilon-\nu)}\Big|^2 d\zeta\Big)\leqslant 2\,(1+t)^r\|\widehat{\nu^\epsilon-\nu}\|^2_{L^2(I;\mathbb{R}^3)}\big(\|\nu\|^2_{K^1_{div}}+\|\nu^\epsilon\|^2_{K^1_{div}}\big)$$
$$-r\big(2\,k_\rho-1\big)\Big[(1+t)^{r-1}\|(\nu^\epsilon-\nu)\|^2_{L^2(I;\mathbb{R}^3)}-(1+t)^{r-1}\int_{\Omega}\widehat{|a_0^{\epsilon}|}^2\exp\big(-2\,k_\rho\zeta^2 t\big)d\zeta\Big]$$
$$+4\sqrt{r}(1+t)^{r-\frac{3}{2}}\big(2-\frac{1}{k_\rho}\big)\big(1+\rho^{-1}\big)^2\big(\|\nu^\epsilon\|^4_{K^1_{div}}+\|\nu\|^4_{K^1_{div}}\big) \tag{4.37}$$

$$\frac{d}{dt}\Big((1+t)^r\int_I\Big|\widehat{(\nu^\epsilon-\nu)}\Big|^2 d\zeta\Big)\leqslant r\big(2\,k_\rho-1\big)(1+t)^{r-1}\int_{\Omega}\widehat{|a_0^{\epsilon}|}^2\exp\big(-2\,k_\rho\zeta^2 t\big)d\zeta$$
$$+(1+t)^r\|\widehat{\nu^\epsilon-\nu}\|^2_{L^2(I;\mathbb{R}^3)}\Big[2\,\big(\|\nu\|^2_{K^1_{div}}+\|\nu^\epsilon\|^2_{K^1_{div}}\big)-r\big(2\,k_\rho-1\big)(1+t)^{-1}\Big]$$
$$+4\sqrt{r}(1+t)^{r-\frac{3}{2}}\big(2-\frac{1}{k_\rho}\big)\big(1+\rho^{-1}\big)^2\big(\|\nu^\epsilon\|^4_{K^1_{div}}+\|\nu\|^4_{K^1_{div}}\big) \tag{4.38}$$

By integrating on the interval $(0,t)$, we obtain:

$$(1+t)^r\|\widehat{(\nu^\epsilon-\nu)}\|^2_{L^2(I;\mathbb{R}^3)}\leqslant r\big(2\,k_\rho-1\big)\Big[\int_0^t(1+\tau)^{r-1}\int_{\Omega}\widehat{|a_0^{\epsilon}|}^2\exp\big(-2k_\rho\zeta^2\tau\big)d\zeta\,d\tau\Big]$$

$$+8\sqrt{r}\big(1+\rho^{-1}\big)^2\big(2-\frac{1}{k_\rho}\big)\big(\|\nu^\epsilon\|^4_{L^2(I;\mathbb{R}^3)}+\|\nu\|^4_{L^2(I;\mathbb{R}^3)}\big)\int_0^t(1+\tau)^{r-\frac{3}{2}}d\tau+\|\widehat{a_0^\epsilon}\|^2_{L^2(I;\mathbb{R}^3)}$$

$$+\|\widehat{(\nu^\epsilon-\nu)}\|^2_{L^2(I;\mathbb{R}^3)}\int_0^t(1+\tau)^r\Big[2\left(\|\nu\|^2_{K^1_{div}}+\|\nu^\epsilon\|^2_{K^1_{div}}\right)-r\big(2k_\rho-1\big)(1+\tau)^{-1}\Big]d\tau \tag{4.39}$$

By applying Gronwall's lemma, we obtain:

$$\|\widehat{(\nu^\epsilon-\nu)}\|^2_{L^2(I;\mathbb{R}^3)}\leqslant(1+t)^{-r}\Big[r\big(2k_\rho-1\big)\int_0^t(1+\tau)^{r-1}\int_\omega|\widehat{a_0^\epsilon}|^2\exp(-2k_\rho\zeta^2\tau)d\zeta d\tau$$

$$+4\sqrt{r}\big(1+\rho^{-1}\big)^2\big(2-\frac{1}{k_\rho}\big)\big(\|\nu^\epsilon\|^4_{L^2(I;\mathbb{R}^3)}+\|\nu\|^4_{L^2(I;\mathbb{R}^3)}\big)\int_0^t(1+\tau)^{r-\frac{3}{2}}d\tau+\|\widehat{a_0^\epsilon}\|^2_{L^2(I;\mathbb{R}^3)}\Big]$$

$$\times\exp\left\{\int_0^t(1+\tau)^{-r}\Big[2\left(\|\nu\|^2_{K^1_{div}}+\|\nu^\epsilon\|^2_{K^1_{div}}\right)-r\big(2k_\rho-1\big)(1+\tau)^{-1}\Big]d\tau\right\} \tag{4.40}$$

where $k_\rho=\rho^{-1}(\lambda+2\mu)$ Therefore, the proof of Theorem 4.1 is complete for $t\longrightarrow\infty$,

$$\|\widehat{(\nu-\nu^\epsilon)}\|^2_{L^2(I;\mathbb{R}^3)}\longrightarrow 0$$

■

The mathematical model illustrated in this paper is a system of nonlinear partial differential reaction-diffusion equations with dynamic and volume viscosity coefficients. It describes the evolution of a tumour cell density, which depends on the availability of an essential nutrient. It is assumed that cells move due to nonlinear scattering. The results we presented concern the asymptotic behaviour of a nonlinear tridimensional model. We have proved a convergence theorem of solutions for a constant density in L^2-norm. As the main tool, we used the Fourrier Plancherel transform. Our nonlinear stability result is proved by the energy estimated by Sobolev and holds for small perturbations of the initial data of constant states. Theoretical studies of biological phenomena may test an existing hypothesis or indicate other processes that produce the observed results. Studies of the asymptotic behavior of mathematical model solutions of tumor growth can be informative while planning a cancer treatment. Its suspension depends on the strength of the drugs and the effectiveness of their administration in the tissues. Numerical simulations of specific treatment solutions can provide essential information before applying it to the human body. Our work shows that under specific conditions on system parameters, solutions converge asymptotically towards stationary states, which can be associated with tumor suspension.

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