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Further Results on Edge Sum Labelings of Graphs

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Abstract

Let G=(V,E) be a given graph with p vertices and q edges. Edge sum labelings of a graph is a one-to-one function f: V(G)→ {0,1,2,…,P-1} that induces a labeling f^+: E(G)→ {1,2,…,q} of the edges of G defined by f^+(uv)=f(u)+f(v), for each edge e=uv of G. Then the value of an edge sum labeling of G is the sum of the edge values of G. In this paper, I study edge sum labelings of regular graph, star graph, lattice grid and show that Val(f^+) between f^+_{min}(G) and f^+_{max}(G) for a class of graphs form an arithmetic progression.

Keywords: Edge sum labelings, n-star, n-prism, generalized Petersen graph, lattice grid.

INTRODUCTION

G. Chartrand, D. Erwin, D.W. Vanderjagt and P. Zhang [4,5] have defined γ-labelings of a graph *G* as a one-to-one function *f: V(G)→ {0,1,2,…,q}* that induces a labeling $f': E(G) \to \{1,2,\dots,q\}$ of the edges of *G* defined by $f'(e) = |f(u) - f(v)|$, for each edge *e=uv* of *G*. Then the value of γ-labeling *f* is $Val(f) = \sum_{e\in E(G)} f'(e)$. The minimum value of γ-labeling of *G* is defined as $Val_{min}(G)$=*min*{*Val(f): f* is *γ*-labeling of *G* } and the maximum value of γ-labeling of *G* is $Val_{max}(G)$ =*max*{*Val(f): f* is γ-labeling of *G* }. Motivated by this labeling, S.M. Hegde and P. Shankaran[7] introduced edge sum labelings of graphs.

An edge sum labeling of a graph G is a one-to-one function *f: V(G)→ {0,1,2,…,P-1}* that induces a labeling $f^+: E(G) \to \{1,2,\dots,q\}$ of the edges of *G* defined by $f^+(uv) = f(u) + f(v)$, for each edge *e=uv* of *G*. Then the value of an edge sum labeling of *G*, denoted by $Val(f^+)$ is the sum of the edge values of *G*.

In this paper, I study edge sum labelings of regular graph, star graph, lattice grid, n-prism, generalized Petersen graph and show that $Val(f^+)$ between $f^+_{min}(G)$ and $f^+_{max}(G)$ for a class of graphs form an A.P.

For all terminology and notation in graph theory, refer Harary[6] and West[10].

Given a graph *G=(V,E)*, the set N of nonnegative integers, a finite subset A of N and a commutative binary operation $+: N \times N \to N,$ every vertex function $f: V(G) \to A$ induces an edge function $f^+: E(G) \to N$ such that $f^+(uv) = f(u) + f(v), \forall uv \in E(G)$. Such vertex functions are called additive vertex functions[Acharya and Hegde][1].

The following notations are adopted throughout this paper.

$$f(G) = \{f(u)/u \in V(G)\}$$

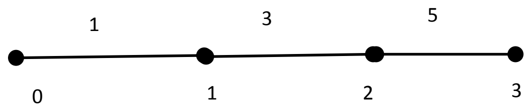
$$f^+(G) = \{f^+(e)/e \in E(G)\}$$

$$f^+_{min} = \min\{\sum_{i=1}^{q} f^+(e_i)\}$$

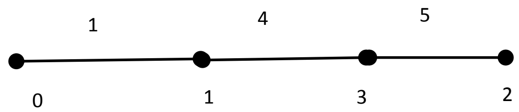
$$f^+_{max} = \max\{\sum_{i=1}^{q} f^+(e_i)\}.$$

Illustration: Consider the path P_4 of order 4. One can label this path using the values 0,1,2,3 in 24 different ways. But it can be observed that when sum of the values of the edges are calculated, it gives only five different values as in Figure 1.

In Figure 1, $Val(f_1^+)$ for (a) is 9, $Val(f_2^+)$ for (b) is 10, $Val(f_3^+)$ for (c) is 11, $Val(f_4^+)$ for (d) is 8, $Val(f_5^+)$ for (e) is 7. Therefore, $f^+_{min}(P_4) = 7$ and $f^+_{max}(P_4) = 11$.



(a)



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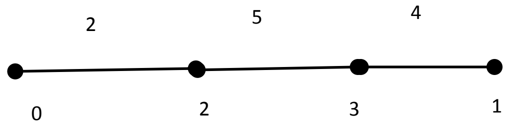

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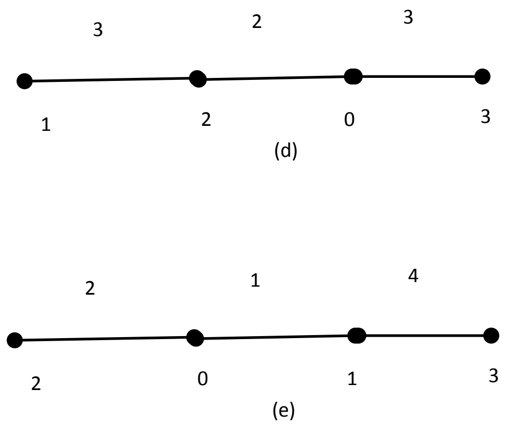

Figure 1. Edge sum labelings of P_4.

RESULTS

Theorem 1[Acharya and Hegde][1]. For any graph $G=(V,E)$ and for any additive vertex function f,

$$\sum_{e\in E(G)} f^+(e)=\sum_{v\in V(G)} f(u)d(u).$$

Theorem 2. For an r-regular graph G,

$$f^+_{min}(G) = f^+_{max}(G) = q(p-1).$$

Proof. For an r-regular graph G, by Theorem 1

$$\sum_{e\in E(G)} f^+(e)=r(1+2+\ldots+p-1)$$

$$= \frac{rp(p-1)}{2} \text{ ………………..(1)}$$

But by handshaking lemma,

$$\sum_{i=1}^{p} d(u_i) = 2q$$

From which we get

$$rp=2q$$

or $$r = \frac{2q}{p}$$

On substituting this in equation (1), we get

$$\sum_{e\in E(G)} f^+(e)=q(p-1)$$

But for an r-regular graph

$$\sum_{e\in E(G)} f^+(e) = f^+_{min}(G) = f^+_{max}(G)$$

From the above results $f^{+}_{min}(G) = f^{+}_{max}(G) = q(p-1)$.

Hence the proof.

Corollary 1. For the n-prism, $G = K_2 \times C_n, n \geq 3$,

$$f^{+}_{min}(G) = f^{+}_{max}(G) = 3n(2n-1).$$

Definition 1[9]. The generalized Petersen graph *P(n,m), n≥3, 1≤m≤□(n-1)/2□* consists of an outer n-cycle $u_0, u_1, \ldots, u_{n-1}$, a set of n-spokes $u_i v_i$, $0 \leq i \leq n-1$ and n-inner edges $v_i v_{i+m}$ with indices taken modulo n.

Corollary 2. For the generalized Petersen graph *P(n,m), n≥3, 1≤m≤□(n-1)/2□*

$$f^{+}_{min}\{P(n,m)\} = f^{+}_{max}\{P(n,m)\} = 3n(2n-1).$$

Definition 2[3]. A complete r-partite graph is obtained by partitioning the vertex set into r sets, joining two vertices if and only if they lie in different sets, if all of these have size k, then the resulting graph is denoted by $K_r(k)$.

Corollary 3. For the complete r-partite graph $G=K_r(k)$,

$$f^{+}_{min}(G) = f^{+}_{max}(G) = C(r,2)k^2(rk-1).$$

Theorem 3. For any graph *G*,

$$f^{+}_{min}(G) = (j-1)n_{j-1}\sum_{r=j}^{\Delta} n_r + \frac{j\,n_j(n_j-1)}{2}$$

for $\delta \leq j \leq \Delta$

$$f^{+}_{max}(G) = \left\lceil \frac{(j-1)n_{j-1}}{2} \right\rceil [2\sum_{r=1}^{j-2} n_r + n_{j-1} - 1]$$

for $2 \leq j \leq \Delta + 1$

Proof. Let *G* be the given graph. Denote the vertices of degree j as v_j^i and the number of vertices of degree j as n_j for $1 \leq i \leq n_j$,□≤j≤Δ. Observe that $\sum_{j=\delta}^{\Delta} n_j = p$. Define the map $f\colon V(G) \to \{0,1,2,\ldots,\sum_{j=\delta}^{\Delta} n_j - 1\}$ such that

$f(v^i_\Delta) = i-1$, $1 \leq i \leq n_\Delta$

$f(v_j^i) = \sum_{r=j+1}^{\Delta} n_r + i - 1$, $1 \leq i \leq n_j$, □≤j≤Δ-1.

Since each of the values from the sets$\{0,1,2,\ldots, n_\Delta-1\}$,$\{n_\Delta,n_\Delta+1,\ldots,n_{\Delta-1}+n_\Delta-1\}$,…,$\{n_3+n_4+\ldots+n_\Delta,\ldots,n_2+n_3+\ldots$

$+n_\Delta-1\}$ and$\{n_2+n_3+n_4+\ldots+n_\Delta,\ldots,n_1+n_2+\ldots+n_\Delta-1\}$ is counted Δ, Δ-1,…2,1 times respectively, the sum of these is definitely minimum. Therefore, $f^+_{min}(G) =$ $(n_2+n_3+n_4+\ldots+n_\Delta)+(n_2+n_3+\ldots+n_\Delta+1)+\ldots+(n_2+n_3+\ldots+n_\Delta+n_1 1)+2[(n_2+n_3+n_4+\ldots+n_\Delta)+(n_3+n_4+\ldots+n_\Delta+1)+\ldots+(n_3+n_4+\ldots+n_\Delta+n_2 1)]+3[(n_4+n_5+\ldots+n_\Delta)+(n_4+n_5+\ldots+n_\Delta+1)+\ldots+(n_4+n_5+\ldots+n_\Delta+n_3-1)]+\ldots+(\Delta-1)[(n_\Delta+n_\Delta+1+\ldots+n_\Delta$

$+n_{\Delta-1}-1)+\Delta(\Delta+1+\ldots+n_\Delta-1)$

$$= (j-1)\sum_{r=j}^{\Delta} n_r + (n_{j-1}-1)(j-1)\sum_{r=j}^{\Delta} n_r + \frac{jn_j(n_j-1)}{2}$$

$$f^+_{min}(G) = (j-1)n_{j-1}\sum_{r=j}^{\Delta} n_r + \frac{j\,n_j(n_j-1)}{2}$$

$$\text{for } \delta \le j \le \Delta$$

Similarly define,

$f(v^i_{\square})=i-1,\ 1\le i\le n_{\square}$

$f(v^i_j) = \sum_{r=1}^{j-1} n_r + i - 1,\ 1\le i\le n_j,\ \square+1\le j\le\Delta.$

Since each of the values from the sets$\{0,1,2,\ldots,n_1-1\}$,$\{n_1,n_1+1,\ldots,n_1+n_2-1\}$,…,}$\{n_1+n_2+\ldots+n_{\Delta-2},\ldots, n_1+n_2+n_3$

$+\ldots+n_{\Delta-1}-1\}$ and$\{n_1+n_2+\ldots+n_{\Delta-1}, n_1+n_2+\ldots+n_\Delta-1\}$ is counted 1,2,…,Δ times respectively, the sum of these is definitely maximum. Therefore,

$f^+_{max}(G) =(1+2+\ldots+n_1-1+2(n_1+n_1+1+\ldots+n_1+n_2-1)+3(n_3+n_4+\ldots+n_\Delta+1)+\ldots+(n_3+n_4+\ldots+n_\Delta+n_2 1)]+3(n_1+n_2+n_1+n_2+1+\ldots+n_1+n_2+n_3-1)+\ldots+(\Delta-1)[(n_1+n_2+\ldots+n_{\Delta-2}+\ldots+n_1+\ldots+n_{\Delta-1}-1)+\Delta[(n_1+n_2+\ldots+n_{\Delta-1}+\ldots+n_\Delta-1)$

$$= (j-1)(n_j-1)\sum_{r=1}^{j-2} n_r + \frac{(j-1)(n_{j-1}-1)n_{j-1}}{2}$$

$$ie.\, f^+_{max}(G) = \left[\frac{(j-1)n_{j-1}}{2}\right][2\sum_{r=1}^{j-2} n_r + n_{j-1} - 1]$$

$for\ 2 \le j \le \Delta + 1.$

Hence the proof.

Corollary 4. For the lattice grid $G=P_m\times P_n$,

$$f^+_{min}(G) = 2mn(m-1)(n-1) + 2(m^2+n^2) - 7(m+n) + 16$$

$$f^+_{max}(G) = 2mn(mn-3) - 2(m^2+n^2) + 9(m+n) - 16.$$

An example is given in Figure 2, where

$f^+_{min}(P_3 \times P_4) = 161$ and $f^+_{max}(P_3 \times P_4) = 213.$

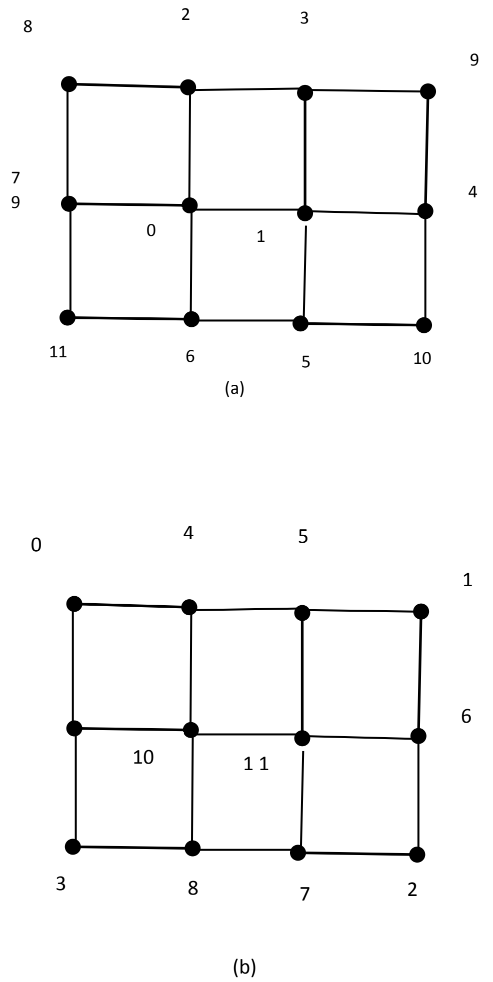

Figure 2. Edge sum labelings of the lattice grid $P_3 \times P_4$

Definition 3[8]. The n-star $St(m_1,m_2,\ldots,m_n)$ is a disjoint union of n-stars $K(_{1,m1}),K(_{1,m2}),\ldots,K(_{1,mn})$,where $m_1,m_2,\ldots,m_n$ is a sequence of increasing non negative integers.

Theorem 4. For the n-star $G=St(m_1,m_2,\ldots,m_n)$, $m_1\leq m_2\leq\ldots\leq m_n$,

$$f^+_{min}(G) = \frac{1}{2}[\sum_{i=1}^{n} m_i^2 + (4n-3)m_1 + (4n-5)m_2 + \cdots$$

$$+(2n+3)m_{n-2} + (2n+1)m_{n-1} + (2n-1)m_n]$$

$$+m_1\sum_{i=2}^{n} m_i + m_2\sum_{i=3}^{n} m_i + \cdots + m_{n-3}\sum_{i=n-2}^{n} m_i + m_{n-2}(m_{n-1}+m_n)$$

$$+m_{n-2}(m_{n-1}+m_n) + m_{n-1}m_n$$

$$f_{max}^{+}(G) = \frac{1}{2}\left[3\sum_{i=1}^{n} m_i^2 - m_1 + m_2 + 3m_3 + 5m_4 + \cdots + (2n-5)m_{n-1} + (2n-3)m_n\right]$$

$$+3m_1\sum_{i=2}^{n} m_i + 3m_2\sum_{i=3}^{n} m_i + \cdots + +3m_{n-1}m_n$$

Proof. Let G= St($m_1,m_2,\ldots,m_n$) be the n-star. Let d(u_i)=m_i, $1\leq i\leq n$ and u_i^js be the pendant vertices of the stars $K(_{1,m1})$,$K(_{1,m2})$,…,$K(_{1,mn})$ for$1\leq j\leq m_i$,$1\leq i\leq n$ respectively. Note that G contains $n+\sum_{i=1}^{n} m_i$ vertices.

Define,

$f(u_i)=n-i,\ 1\leq i\leq n$

$f(u_1^j)=n+j-1,\ 1\leq j\leq m_1$

$$f\left(u_{i+1}^{j}\right) = \sum_{r=1}^{i} m_r + n + j - 1, 1 \leq i \leq n-1,$$

$$1 \leq j \leq m_{i+1}$$

Since the numbers 0 to n-1 are assigned to the root vertices and the numbers from $\sum_{i=1}^{n} m_i + n - 1$ to n are assigned to the pendant vertices of the stars $K(_{1,mn})$,$K(_{1,mn-1})$,…,$K(_{1,m1})$ respectively, the resulting sum of the values of the edges of respective stars is clearly minimum. Therefore,

f_{min}^{+}(K(1,m_1))=n+n+1+…+n+m_1-1+m_1(n-1)

from which, we get

f_{min}^{+}(K(1,m_1))=$\frac{m_1^2-3m_1}{2}$+$2m_1n$

f_{min}^{+}(K(1,m_2))=n+m_1+n+m_1+1+…+n+m_1+m_2-1+m_2(n-2)

from which, we get

f_{min}^{+}(K(1,m_2))=$\frac{m_2^2-5m_2}{2}$+$2m_2n + m_1m_2$

Proceeding on similar lines,

f^{+}_{min}(K(1,m_n))=$\frac{m_n^2-m_n}{2}$+$m_n(m_1+m_2+\cdots+m_{n-1}+n)$

Adding all these results, we get

$$f^{+}_{min}(G)=\frac{1}{2}[\sum_{i=1}^{n}m_i^2+(4n-3)m_1+(4n-5)m_2+\cdots$$
$$+(2n+3)m_{n-2}+(2n+1)m_{n-1}+(2n-1)m_n]$$
$$+m_1\sum_{i=2}^{n}m_i+m_2\sum_{i=3}^{n}m_i+\cdots+m_{n-3}\sum_{i=n-2}^{n}m_i+m_{n-2}(m_{n-1}+m_n)$$
$$+m_{n-1}m_n.$$

Similarly define $f(u_i)=\sum_{r=1}^{i}m_r+i-1$, $1\leq i\leq n$

$f(u_1^j)=j-1$, $1\leq j\leq m_1$

$f(u_{i+1}^j)=\sum_{r=1}^{i}m_r+j-1$, $1\leq i\leq n-1$, $1\leq j\leq m_{i+1}$.

Since the numbers 0 to $\sum_{r=1}^{n}m_i-1$ are assigned to the pendant vertices and the numbers from $\sum_{r=1}^{n}m_i$ to $\sum_{r=1}^{n}m_i+n-1$ are assigned to the root vertices of the stars K($1,m_1$),K($1,m_2$),…,K($1,m_n$) respectively, the resulting sum of the values of the edges of respective stars is clearly maximum. Therefore,

$f^{+}_{max}(K(1,m_1))$=1+2+…+m_1-1+m_1(m_1+…+m_n)

from which, we get

$f^{+}_{max}(K(1,m_1))=\frac{3m_1^2-m_1}{2}$+$m_1m_2+m_1m_3+\cdots+m_1m_n$

$f^{+}_{max}(K(1,m_2))$=m_1+m_1+1+…+m_1+m_2-1+m_2(m_1+…+m_n+1)

from which, we get

$f^{+}_{max}(K(1,m_2))=\frac{3m_2^2+m_2}{2}$+$2m_1m_2+m_2m_3+\cdots+m_2m_n$

$$f^{+}_{max}(K(1,m_3))$$
$$=m_1+m_2+m_1+m_2+1+\cdots+m_1+m_2+m_3-1+m_3(m_1+\cdots+m_n+2)$$

from which, we get

$f^{+}_{max}(K(1,m_3))=\frac{3m_3^2+3m_3}{2}$+$2m_1m_3+2m_2m_3+\cdots+m_3(m_4+\cdots+m_n)$

Proceeding on similar lines,

$f^{+}_{max}(K(1,m_n))=\frac{3m_n^2-3m_n}{2}$+$2m_n(m_1+m_2+\cdots+m_{n-1})+nm_n$

Adding all these results, we get

$$f_{max}^{+}(G) = \frac{1}{2}\left[3\sum_{i=1}^{n} m_i^2 - m_1 + m_2 + 3m_3 + 5m_4 + \cdots + (2n-5)m_{n-1} + (2n-3)m_n\right]$$

$$+3m_1\sum_{i=2}^{n} m_i + 3m_2\sum_{i=3}^{n} m_i + \cdots + +3m_{n-1}m_n$$

Hence the proof.

Now we check for any path P_n, if $f_{min}^{+}(P_n) = f_{max}^{+}(P_n)$.

Let P_n be a path of order n. Denote the end vertices of P_n as u,v and the internal vertices as $v_1, v_2, \ldots, v_{n-2}$.

Define, $f(u)=n-2$

$f(v)=n-1$

$f(v_i)=i-1,\ 1\leq i\leq n-2.$

That means

$f(v_1)<f(v_2)<\ldots<f(v_{n-2})<f(u)<f(v).$

Note that even if we interchange the labelings of any pair on internal vertices, the values of edge sum labelings remain unaltered as each of them is counted twice. Also, since the two higher values are counted only once and each of the remaining values is counted twice, the sum of these is definitely the minimum out of edge sum values of all possible edge sum labelings. Therefore,

$$f_{min}^{+}(P_n) = n-2+n-1+2(0+1+\ldots n-3)$$

$$=n(n-3)+3.$$

Similarly define,

$f(u)=0$

$f(v)=1$

$f(v_i)=i+1,\ 1\leq i\leq n-2.$

That means

$f(u)<f(v)<f(v_1)<f(v_2)<\ldots<f(v_{n-2}).$

Note that even if we interchange the labelings of any pair on internal vertices, the values of edge sum labelings remain unaltered as each of them is counted twice. Also, since the two lower values are counted only once and each of the remaining values is counted twice, the sum of these is definitely the maximum out of edge sum values of all possible edge sum labelings. Therefore,

$f^{+}_{max}(P_n)$ =0+1+2(2+3+…n-1)

=*n(n-1)-1*.

From the above illustration, it is obvious that

$f^{+}_{min}(P_n) \neq f^{+}_{max}(P_n)$.)

However, we believe that there are 2n-3 different edge sum values for P_n. Therefore it is interesting to determine if there exists all the $Val(f^{+})$ between $f^{+}_{min}(G)$ and $f^{+}_{max}(G)$ for a graph G.

The next result is on this line.

Denote the vertices of P_n as explained above and label them so as to get $f^{+}_{min}(P_n)$. Now interchange the labels according to the following algorithm.

$f(u) \leftrightarrow f(v_{n-2})$ and denote the new value of u as $f_1(u)$

$f_1(u) \leftrightarrow f(v_{n-3})$ and denote the new value of u as $f_2(u)$

$f_2(u) \leftrightarrow f(v_{n-4})$ and denote the new value of u as $f_3(u)$ etc.

$f_{n-3}(u) \leftrightarrow f(v_1)$.

The process is continued until u receives the value 0. Then repeat the same process starting with the last labelings thus obtained for f(v). That is,

$f(v) \leftrightarrow f(v_{n-2})$ and denote the new value of u as $f_1(u)$

$f_1(v) \leftrightarrow f(v_{n-3})$ and denote the new value of u as $f_2(u)$

$f_2(v) \leftrightarrow f(v_{n-4})$ and denote the new value of u as $f_3(u)$ etc.

$f_{n-3}(v) \leftrightarrow f(v_1)$.

The process is continued until v receives the value 1. Then one can observe that in every interchange of labels two is added and one is subtracted by the above process. The net result is an addition of one to the previous $Val(f^{+})$. Hence all the $Val(f^{+})$ from $f^{+}_{min}(P_n)$ to $f^{+}_{max}(P_n)$ are generated and they form an arithmetic progression with a common difference d=1.

Next consider $K_{1,n}$. To check if there exists all the $Val(f^{+})$ from $f^{+}_{min}(K_{1,n})$ to $f^{+}_{max}(K_{1,n})$. Denote the central vertex of $K_{1,n}$ as u and the pendant vertices as $v_1, v_2, \ldots, v_n$. Define,

$f(u)=0$

$f(v_i)=i,\ 1 \leq i \leq n.$

Then $\sum_{e\in E(G)} f^+(e) = 1+2+\cdots+n$

$$=\frac{n(n+1)}{2}.$$

Since the root vertex, which is of degree n and value 0 and all other vertices are pendant vertices, the sum of the edge values is clearly the minimum out of all other possible edge sum labelings. Therefore

$$f^+_{min}(K_{1,n}) = \frac{n(n+1)}{2}.$$

Now start with the above labeling and apply the following algorithm.

$f(u)\leftrightarrow f(v_1)$ and denote the new value of u as $f_1(u)$

$f_1(u)\leftrightarrow f(v_2)$ and denote the new value of u as $f_2(u)$

$f_2(u)\leftrightarrow f(v_3)$ and denote the new value of u as $f_3(u)$ etc.

$f_{n-1}(u)\leftrightarrow f(v_n)$.

The process is continued until u receives the value $f(v_n)$. Then one can check that all $Val(f^+)$ from $f^+_{min}(K_{1,n})$ to $f^+_{max}(K_{1,n})$ are generated and they form an arithmetic progression with a common difference d=n-1. Note that the total number of different edge sum values is n+1.

Theorem 5. Let *G* be any graph with k vertices of degree d_1 and p-k vertices of degree d_2 such that $d_1>d_2$. Then $Val(f^+)$ from $f^+_{min}(G)$ to $f^+_{max}(G)$ form an arithmetic progression with common difference d_1-d_2.

Proof. Suppose that $V = V_1 \cup V_2$, where $V_1=\{u_i\ 1\le i\le k\}$, $V_2=\{v_j\ 1\le j\le p-k\}$, $d_1=deg(u_i)$, $d_2=deg(v_j)$ and $d_1>d_2$.

Define, $f(u_i)=i-1, 1\le i\le k$

$f(v_j)=k+j-1, 1\le j\le p-k$.

Then,

$\sum_{e\in E(G)} f^+(e) = d_1(1+2+\ldots+k-1)+d_2(k+k+1+\ldots+p-1)$

$$=\frac{d_1k(k-1)}{2}+d_2[k+k(p-k-1)+\frac{(p-k-1)(p-k)}{2}]$$

$$=\frac{k(d_1-d_2)(k-1)+d_2p(p-1)}{2}.$$

Since each of the numbers from 0 to k-1 is counted d_1 times and each of the remaining numbers from k to p-1 is counted d_2 times, the sum of these numbers is clearly the minimum out of all possible edge sum labelings of *G*.

Therefore,

$$f_{min}^{+}(G) = \frac{k(d_1-d_2)(k-1)+d_2p(p-1)}{2}.$$

Similarly define,

$$f(u_i)=p-k+i-1, 1\le i\le k$$

$$f(v_j)=j-1, 1\le j\le p-k.$$

Then,

$\sum_{e\in E(G)} f^{+}(e) = d_2(1+2+\ldots+p-k-1)+d_1(p-k+p-k+1+\ldots+p-1)$

$$= \frac{d_2(p-k)(p-k-1)}{2} + d_1[p-k+(p-k)(k-1)+(p-k-1)+\frac{k(k-1)}{2}]$$

$$= \frac{k(d_1-d_2)(2p-k-1)+d_2p(p-1)}{2}.$$

Since each of the numbers from 0 to p-k-1 is counted d_2 times and each of the remaining numbers from p-k to p-1 is counted d_1 times, the sum of these numbers is clearly the maximum out of all possible edge sum labelings of *G*.

Therefore,

$$f_{max}^{+}(G) = \frac{k(d_1-d_2)(2p-k-1)+d_2p(p-1)}{2}.$$

Now, to obtain all $Val(f^{+})$ from $f_{min}^{+}(G)$ to $f_{max}^{+}(G)$, apply the following algorithm.

Step 1: Interchange the labels of u_k with v_j for $1\le j\le p-k$.

Step 2: Start from the previous labeling and interchange the labels of u_{k-1} with v_j.

Step 3: Start from the previous labeling and interchange the labels of u_{k-2} with v_j etc.

Step 4: At last interchange the labels of u_1 with v_j.

From the resulting labelings, one can determine $f_{max}^{+}(G)$. Since, in every interchange of labels, replace a value I, $0\le i\le k-1$, which is counted d_1 times by a value i+1, which is counted d_2 times, the net result is an addition of d_1-d_2 to the previous $Val(f^{+})$. Hence, it is obvious that all $Val(f^{+})$ from $f_{min}^{+}(G)$ to $f_{max}^{+}(G)$ are generated and they form an arithmetic progression with a common difference d_1-d_2. Note that, since there are k vertices in V_1 and p-k vertices in V_2, there must be k(p-k) interchange of labels, so that *G* has k(p-k)+1 number of $Val(f^{+})$ values. Hence the proof.

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