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# On a class of Perturbed Discrete Nonlinear Systems

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**Abstract**

We study a class of nonlinear two-time-scale discrete system called *D-model*. Conditions for the existence and uniqueness of a BVP's solution are established and a perturbation method is used to find approximate solutions at any order. We discuss the IVP and for the FVP the method reduces the computational load since it provides reliable results with few iterations.

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## 1. Introduction

Singular perturbation techniques were mainly applied to continuous systems, while few mathematical methods have been used for perturbed discrete time models, especially the nonlinear case [1, 6, 8, 13]. By singularly perturbed discrete system, is meant a system in which the deletion of a small parameter implies a reduction of its dimension. This interpretation has led to different models characterizing these types of systems, among others we can cite the *D-model*, the *C-model*, the *R-model* [10, 16, 17, 18, 19]. In this paper we present the nonlinear extension of the linear *D-model* studied in [10, 17] which is similar to the model used in the continuous case described by a system of ODEs, regarding the position of the small parameter. We focus on a BVP associated with this model. Lately, there has been much research activity concerning the solvability of BVPs for nonlinear discrete systems. For some of theses works, we refer the reader to [2, 3, 5, 7, 11, 12]. The purpose of this paper is to develop iterative methods for the construction of asymptotic solutions of the nonlinear *D-model* satisfying two-point boundary conditions:

$$\begin{cases} x(t+1) = f\,(x(t), y(t), \varepsilon, t)\,, \\ \varepsilon y(t+1) = g\,(x(t), y(t), \varepsilon, t)\,, \end{cases} \quad t = \in I_{N-1}, \tag{1}$$

$$x(t=0) = \alpha(\varepsilon), \qquad y(t=N) = \beta(\varepsilon), \tag{2}$$

where $I_N = \{0, 1, \cdots, N\}$, $N$ a positive integer, $F(I_N, X)$ and $G(I_N, Y)$ denote the space of all mappings of $I_N$ into the Banach spaces $(X, \|.\|)$ and $(Y, \|.\|)$, respectively. Let $\mathcal{U} := F(I_N, X) \times G(I_N, Y) \times (-1, 1) \times I_N$, the mappings $f : \mathcal{U} \to X$ and $g : \mathcal{U} \to Y$ are supposed to be $n$-differentiable in their arguments, and for $|\varepsilon| < \delta \leq 1$, we suppose that $\alpha_k(\varepsilon)$ and $\beta(\varepsilon)$ have asymptotic representations

$$\alpha(\varepsilon) = \alpha^{(0)} + \varepsilon\alpha^{(1)} + \cdots + \varepsilon^n \alpha^{(n)}, \quad \beta(\varepsilon) = \beta^{(0)} + \varepsilon\beta^{(1)} + \cdots + \varepsilon^n \beta^{(n)}. \tag{3}$$

The paper is structured as follows. The next Section is devoted to the solvability of BVP (1)–(2), we use an iterative technique developed recently for nonlinear perturbed difference equations in [20, 21, 22], and earlier for linear problems in [14, 15, 16, 17, 18, 19]. We study the existence and uniqueness of a solution $(x(t,\varepsilon), y(t,\varepsilon))$, $t \in I_N$, and we show how to recursively calculate the coefficients of asymptotic expansions

$$\begin{aligned} x(t,\varepsilon) &= x(t)^{(0)} + \varepsilon x(t)^{(1)} + \cdots + \varepsilon^n x(t)^{(n)} + \mathcal{O}(\varepsilon^{n+1}), \\ y(t,\varepsilon) &= y(t)^{(0)} + \varepsilon y(t)^{(1)} + \cdots + \varepsilon^n y(t)^{(n)} + \mathcal{O}(\varepsilon^{n+1}). \end{aligned} \tag{4}$$

In Section 3 we dicuss the IVP and we study the FVP. Finally we end this work with a brief Conclusion. Throughout this paper, for an abbreviated writing, $D_1^{k_1} D_2^{k_2} \cdots D_p^{k_p} f$ denotes the partial derivative $\dfrac{\partial^{k_1+k_2+\cdots+k_p} f\left(x_1, x_2, \cdots, x_p\right)}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_p^{k_p}}$.

## 2. Main Result

### 2.1. Reduced Problem

The reduced problem is obtained by setting the small parameter to zero:

$$\begin{cases} x^{(0)}(t+1) = f\left(x^{(0)}(t), y^{(0)}(t), 0, t\right), \\ \qquad\quad 0 = g\left(x^{(0)}(t), y^{(0)}(t), 0, t\right), \end{cases} \quad t \in I_{N-1}, \tag{5}$$

$$x^{(0)}(0) = \alpha(0), \qquad y^{(0)}(N) = \beta(0). \tag{6}$$

It is seen that in (5), the first equation defines a recurrence, and the second is an algebraic equation. A boundary layer appears at the final point $y(N)$ which is dissociated from the system. If we assume that second equation has a unique root, then the values $y^{(0)}(0), \ldots,$ $y^{(0)}(N-1)$ can be automatically fixed from the difference equation in (5) using only the initial value $x^{(0)}(0) = \alpha(0)$. To guarantee this condition, we use the following hypothesis.

**H1** Suppose for each $t = 0, \ldots, N-1$, the range of $g$ contains the value 0, and

$$D_2 g\,(x(t), y(t), 0, t) \neq 0, \quad \forall (x, y) \in X \times Y.$$

**Proposition 2.1.** If H1 holds, then problem (5)−(6) has a unique solution.

### 2.2. Preliminaries

To establish the existence of a solution as well to find approximate solutions, the BVP (1)−(2) is reformulated into a system of equations depending on a parameter. Using the notation

$$\chi = (x(0), y(0), x(1), y(1), \ldots, x(N), y(N)),$$

we can write the system (1)−(2) in the form $\mathcal{F}(\varepsilon, \chi) = 0$, where

$$\mathcal{F} : (-1, 1) \times X^{2N+2} \longrightarrow X^{2N+2}, \tag{7}$$

$$\mathcal{F}(\varepsilon, \chi) = (f_0(\varepsilon, \chi), g_0(\varepsilon, \chi), f_1(\varepsilon, \chi), g_1(\varepsilon, \chi), \ldots, f_N(\varepsilon, \chi), g_N(\varepsilon, \chi)),$$

$$\begin{cases} f_0(\varepsilon, \chi) = x(0) - \alpha(\varepsilon), & \\ f_t(\varepsilon, \chi) = x(t+1) - f\,(x(t), y(t), \varepsilon, t), & \\ g_t(\varepsilon, \chi) = \varepsilon y(t+1) - g\,(x(t), y(t), \varepsilon, t), & t = 0, \cdots, N-1, \\ g_N(\varepsilon, \chi) = y(N) - \beta(\varepsilon). & \end{cases}$$

Under appropriate assumptions, we can apply the classical Implicit Function Theorem [4]. If the hypothesis H1 is satisfied, for a sufficiently small parameter $\varepsilon$, we can determine a function

$$\Phi(\varepsilon) = (\phi_0(\varepsilon), \psi_0(\varepsilon), \phi_1(\varepsilon), \psi_1(\varepsilon), \ldots, \phi_N(\varepsilon), \psi_N(\varepsilon)),$$

with same regularity as $\mathcal{F}$, i.e., of class $C^n$, such that $\mathcal{F}(\varepsilon, \Phi(\varepsilon)) = 0$. Hence,

$$\begin{cases} \phi_{t+1}(\varepsilon) = f\,(\phi_t(\varepsilon), \psi_t(\varepsilon), \varepsilon, t), \\ \varepsilon\psi_{t+1}(\varepsilon) = g\,(\phi_t(\varepsilon), \psi_t(\varepsilon), \varepsilon, t), \end{cases} \quad t \in I_{N-1}, \tag{8}$$

$$\phi_0(\varepsilon) = \alpha(\varepsilon), \quad \psi_N(\varepsilon), = \beta(\varepsilon). \tag{9}$$

To obtain an approximate value for $\Phi_k(\varepsilon)$, we use the Maclaurin polynomial expansion

$$\begin{aligned} \phi_t(\varepsilon) &= \phi_t(0) + \frac{\varepsilon}{1!}\frac{d\phi_t}{d\varepsilon}(0) + \frac{\varepsilon^2}{2!}\frac{d^2\phi_t}{d\varepsilon^2}(0) + \cdots + \frac{\varepsilon^n}{n!}\frac{d^n\phi_t}{d\varepsilon^n}(0) + \mathcal{O}(\varepsilon^{n+1}), \\ \psi_t(\varepsilon) &= \psi_t(0) + \frac{\varepsilon}{1!}\frac{d\psi_t}{d\varepsilon}(0) + \frac{\varepsilon^2}{2!}\frac{d^2\psi_t}{d\varepsilon^2}(0) + \cdots + \frac{\varepsilon^n}{n!}\frac{d^n\psi_t}{d\varepsilon^n}(0) + \mathcal{O}(\varepsilon^{n+1}). \end{aligned} \tag{10}$$

To explicitly find the sequential differentiation of (8) we apply the formula of Faa di Bruno [9]. In the following lemma, for a brief write, we omit the arguments for $f$ and $g$.

**Lemma 2.2.** Suppose that $\phi$ and $\psi$ satisfy (8), and that all necessary derivatives are defined. Then we have for $n \geq 2$,

$$\frac{d^n\phi_{t+1}(\varepsilon)}{d\varepsilon^n} - D_1 f \frac{d^n\phi_t(\varepsilon)}{d\varepsilon^n} - D_2 f \frac{d^n\psi_t(\varepsilon)}{d\varepsilon^n} = \underline{\sum_0 \sum_1 \cdots \sum_n} \frac{n! D_1^{p_1} D_2^{p_2} D_3^{p_3} f \times \prod_{i=1}^n (\frac{d^i\phi_t(\varepsilon)}{d\varepsilon^i})^{q_{i1}} (\frac{d^i\psi_t(\varepsilon)}{d\varepsilon^i})^{q_{i2}} (\varepsilon^{(i)})^{q_{i3}}}{\prod_{i=1}^n (i!)^{k_i} \prod_{i=1}^n \prod_{j=1}^3 q_{ij}!}, \tag{11}$$

$$n\frac{d^{n-1}\psi_{t+1}(\varepsilon)}{d\varepsilon^{n-1}} - D_1 g \frac{d^n\phi_t(\varepsilon)}{d\varepsilon^n} - D_2 g \frac{d^n\psi_t(\varepsilon)}{d\varepsilon^n} = \underline{\sum_0 \sum_1 \cdots \sum_n} \frac{n! D_1^{p_1} D_2^{p_2} D_3^{p_3} g \times \prod_{i=1}^n (\frac{d^i\phi_t(\varepsilon)}{d\varepsilon^i})^{q_{i1}} (\frac{d^i\psi_t(\varepsilon)}{d\varepsilon^i})^{q_{i2}} (\varepsilon^{(i)})^{q_{i3}}}{\prod_{i=1}^n (i!)^{k_i} \prod_{i=1}^n \prod_{j=1}^3 q_{ij}!}, \tag{12}$$

where the coefficients $k_i$, $q_{ij}$ and $p_j$, $i = 0, \cdots, n$, $j = 1, 2, 3$, are all nonnegative integer solutions of the Diophantine equations

$$\begin{aligned} \sum_0 &\rightarrow k_1 + 2k_2 + \cdots + nk_n = n, \\ \sum_i &\rightarrow q_{i1} + q_{i2} + q_{i3} = k_i, \quad i = 1, 2, \cdots, n, \\ p_j &= q_{1j} + q_{2j} + \cdots + q_{nj}, \quad j = 1, 2, 3, \\ k &= p_1 + p_2 + p_3 = k_1 + k_2 + \cdots + k_n, \end{aligned} \tag{13}$$

in $\underline{\sum_0 \cdots \sum_n}$ we fix $k_n = 0$; the case $k_n = 1$ is removed to the left side,

$$\left(\varepsilon^{(i)}\right)^{q_{i3}} := \begin{cases} 1, & i = 1 \quad \vee \quad q_{i3} = 0, \\ 0, & i \geq 2 \quad \wedge \quad q_{i3} \neq 0. \end{cases} \tag{14}$$

*Proof.* Notice that for $i \geq 1$,

$$\frac{d^n\left(\varepsilon\phi_k(\varepsilon)\right)}{d\varepsilon^n} = i\frac{d^{n-1}\phi_k(\varepsilon)}{d\varepsilon^{n-1}} + \varepsilon\frac{d^n\phi_k(\varepsilon)}{d\varepsilon^n}, \tag{15}$$

$$\frac{d^n\left(\varepsilon\psi_k(\varepsilon)\right)}{d\varepsilon^n} = i\frac{d^{n-1}\psi_k(\varepsilon)}{d\varepsilon^{n-1}} + \varepsilon\frac{d^n\psi_k(\varepsilon)}{d\varepsilon^n}, \tag{16}$$

we expand Faa Di Bruno's Formula into (8) and we shift the terms that correspond to $k_n = 1$ on the left hand side. ■

### 2.3. Description of the Method

In this section, we describe an iterative process giving the coefficients of (4). The *zero* order approximation coefficients are the solution sequence of the *reduced problem* (5)−(6). Knowing that

$$y^{(0)}(N) = \beta(0), \tag{17}$$

and for $t = 0, \cdots, N-1$, we have

$$0 = g\left(x^{(0)}(t), y^{(0)}(t), 0, t\right), \tag{18}$$

once we set the values $y^{(0)}(t)$ by means of $x^{(0)}(t)$ from (18), then we just need to replace them into

$$x^{(0)}(t+1) = f\left(x^{(0)}(t), y^{(0)}(t), 0, t\right),\ t \in I_{N-1}, \tag{19}$$

which will be a recurrence equation for $x^{(0)}(t)$ only, starting from

$$x^{(0)}(0) = \alpha(0). \tag{20}$$

To get higher order coefficients, we substitute, for all $0 \leq t \leq N$, $l = 0, 1$, into (11) and (12) by

$$x^{(i)}(t+l) := \frac{1}{i!}\frac{d^i\phi_{t+l}}{d\varepsilon^i}(0),\ y^{(i)}(t+l) := \frac{1}{i!}\frac{d^i\psi_{t+l}}{d\varepsilon^i}(0). \tag{21}$$

For *first* order approximation, we fix

$$y^{(1)}(N) = \beta^{(1)}, \tag{22}$$

and we express the coefficients $y^{(1)}(t)$, $t \in I_{N-1}$, according to $x^{(1)}(t)$,

$$y^{(1)}(t) = [D_2g_t]^{-1}[y^{(0)}(t+1) - D_1g_tx^{(1)}(t) - D_3g_t], \tag{23}$$

then introduce into

$$x^{(1)}(t+1) = D_1f_tx^{(1)}(t) + D_2f_ty^{(1)}(t) + D_3f_t,\ t \in I_{N-1}, \tag{24}$$

$$x^{(1)}(0) = \alpha^{(1)}. \tag{25}$$

Agree that

$$f_t := f(x^{(0)}(t), y^{(0)}(t), 0, t), \quad g_t := g\left(x^{(0)}(t), y^{(0)}(t), 0, t\right), \tag{26}$$

and let

$$\mathcal{G}_t = D_1f_t - D_2f_t[D_2g_t]^{-1}D_1g_t. \tag{27}$$

Therefore, we obtain the difference equation for $x^{(1)}(t)$ exclusively,

$$x^{(1)}(t+1) = \mathcal{G}_t x^{(1)}(t) + D_2 f_t [D_2 g_t]^{-1} y^{(0)}(t+1) - D_2 f_t [D_2 g_t]^{-1} D_3 g_t + D_3 f_t, \tag{28}$$

which must be solved with (25), and where $y^{(0)}(t+1)$ are known from the previous step, then in a next step, we can set $y^{(1)}(t)$ from (23).
For *second* order development, we use the same calculation technique as the previous steps. We have

$$y^{(2)}(N) = \beta^{(2)}, \tag{29}$$

and

$$\begin{aligned} y^{(2)}(t) = [D_2 g_t]^{-1}[y^{(1)}(t+1) - D_1 g_t x^{(2)}(t) - \frac{1}{2!} D_1^2 g_t \left(x^{(1)}(t)\right)^2 \\ -\frac{1}{2!} D_2^2 g_t \left(y^{(1)}(t)\right)^2 - \frac{1}{2!} D_3^2 g_t - D_1 D_3 g_t x^{(1)}(t) - D_2 D_3 g_t y^{(1)}(t) \\ -D_1 D_2 g_t x^{(1)}(t) y^{(1)}(t)]. \end{aligned} \tag{30}$$

If we replace Formula (30) in

$$\begin{aligned} x^{(2)}(t+1) = D_1 f_t x^{(2)}(t) + D_2 f_t y^{(2)}(t) + \frac{1}{2!} D_1^2 f_t \left(x^{(1)}(t)\right)^2 + \frac{1}{2!} D_2^2 f_t \left(y^{(1)}(t)\right)^2 \\ +\frac{1}{2!} D_3^2 f_t + D_1 D_2 f_t x^{(1)}(t) y^{(1)}(t) + D_1 D_3 f_t x^{(1)}(t) + D_2 D_3 f_t y^{(1)}(t), \end{aligned} \tag{31}$$

$$x^{(2)}(0) = \alpha^{(2)}, \tag{32}$$

then (31) will be a recursion formula for just $x^{(2)}(t)$,

$$x^{(2)}(t+1) = \mathcal{G}_t x^{(2)}(t) + D_2 f_t [D_2 g_t]^{-1} y^{(1)}(t+1) + \mathcal{H}(t), \tag{33}$$

beginning from (32), where

$$\begin{aligned} \mathcal{H}(t) = \frac{1}{2!} D_1^2 f_t \left(x^{(1)}(t)\right)^2 + \frac{1}{2!} D_2^2 f_t \left(y^{(1)}(t)\right)^2 + \frac{1}{2!} D_3^2 f_t + D_1 D_2 f_t x^{(1)}(t) y^{(1)}(t) + \\ D_1 D_3 f_t x^{(1)}(t) + D_2 D_3 f_t y^{(1)}(t) - D_2 f_t [D_2 g_t]^{-1} [D_1 D_2 g_t x^{(1)}(t) y^{(1)}(t) + \\ D_1 D_3 g_t x^{(1)}(t) + D_2 D_3 g_t y^{(1)}(t) + \frac{1}{2!} D_1^2 g_t \left(x^{(1)}(t)\right)^2 + \frac{1}{2!} D_2^2 g_t \left(y^{(1)}(t)\right)^2 + \frac{1}{2!} D_3^2 g_t]. \end{aligned} \tag{34}$$

In general, to find the $n$ order coefficients, we set

$$y^{(n)}(N) = \beta^{(n)}, \tag{35}$$

we incorporate the term

$$y^{(n)}(t) = [D_2 g_t]^{-1}\Big[y^{(n-1)}(t+1) - D_1 g_t x^{(n)}(t) - \underline{\sum_0 \sum_1 \cdots \sum_n} \frac{D_1^{p_1} D_2^{p_2} D_3^{p_3} g_t \times \prod_{i=1}^n (x^{(i)}(t))^{q_{i1}} (y^{(i)}(t))^{q_{i2}} (\delta_i)^{q_{i3}}}{\prod_{i=1}^n \prod_{j=1}^3 q_{ij}!}\Big], \tag{36}$$

which implicates $x^{(n)}(t)$ and coefficients of smaller order, into

$$x^{(n)}(t+1) = D_1 f_t x^{(n)}(t) + D_2 f_t y^{(n)}(t) + \underline{\sum_0 \sum_1 \cdots \sum_n} \frac{D_1^{p_1} D_2^{p_2} D_3^{p_3} f_t \times \prod_{i=1}^n (x^{(i)}(t))^{q_{i1}} (y^{(i)}(t))^{q_{i2}} (\varepsilon^{(i)})^{q_{i3}}}{\prod_{i=1}^n \prod_{j=1}^3 q_{ij}!}, \tag{37}$$

therefore, we obtain the recurrence equation

$$\begin{aligned} x^{(n)}(t+1) &= \mathcal{G}_t x^{(n)}(t) + D_2 f_t [D_2 g_t]^{-1}\Big[y^{(n-1)}(t+1) \\ &+ \underline{\sum_0 \sum_1 \cdots \sum_n} \frac{D_1^{p_1} D_2^{p_2} D_3^{p_3} f_t \times \prod_{i=1}^n (x^{(i)}(t))^{q_{i1}} (y^{(i)}(t))^{q_{i2}} (\varepsilon^{(i)})^{q_{i3}}}{\prod_{i=1}^n \prod_{j=1}^3 q_{ij}!} \\ &- D_2 f_t [D_2 g_t]^{-1} \underline{\sum_0 \sum_1 \cdots \sum_n} \frac{D_1^{p_1} D_2^{p_2} D_3^{p_3} g_t \times \prod_{i=1}^n (x^{(i)}(t))^{q_{i1}} (y^{(i)}(t))^{q_{i2}} (\delta_i)^{q_{i3}}}{\prod_{i=1}^n \prod_{j=1}^3 q_{ij}!}\Big]. \end{aligned} \tag{38}$$

which must be treated first to determine $x^{(n)}(t)$, $t \in I_{N-1}$, where only the initial value

$$x^{(n)}(0) = \alpha^{(n)}, \tag{39}$$

is needed, the values $y^{(n)}(t)$, $t \in I_{N-1}$, can be deduced in next step from (35). The following theorem illustrates the effectiveness of the suggested algorithm.

**Theorem 2.3.** If H1 holds, there exists $\epsilon > 0$, such that for all $|\varepsilon| < \epsilon$, the boundary value problem (1)−(2) has a unique solution which satisfy (4); the coefficients $x^{(0)}(t)$, $y^{(0)}(t)$, $x^{(1)}(t)$, $y^{(1)}(t)$, $x^{(2)}(t)$, $y^{(2)}(t)$, $x^{(n)}(t)$, $y^{(n)}(t)$, are found following the ordered iterative process (5)−(6), (25)−(28), (22)−(23), (32)−(33), (29)−(30), (38)−(39), (35)−(36), respectively.

*Proof.* Suppose $\tilde{\chi} = (\varepsilon, \chi)$, $|\varepsilon| \leq \delta < 1$, and let $\mathrm{F}(\tilde{\chi}) = (\varepsilon, \mathcal{F}(\tilde{\chi}))$ where $\mathcal{D}\mathrm{F}$ denotes its jacobian matrix. From hypothesis H1 we have

$$\det \mathcal{D}\mathrm{F}\left(\tilde{\chi}^{(0)}\right) = \prod_{t=0}^{N-1} \mathcal{D}_2 g\left(x^{(0)}(t), y^{(0)}(t), 0, t\right) \neq 0.$$

what confirms that $\mathcal{D}$F is invertible at

$$\tilde{\chi}^{(0)} = \left(0, x^{(0)}(0), y^{(0)}(0), x^{(0)}(1), y^{(0)}(1), \ldots, x^{(0)}(N), y^{(0)}(N)\right)$$

and the *reduced problem* (5)−(6) has a unique solution. The continuity of $\mathcal{D}$F allows to choose $\xi > 0$ such that, if $\|\tilde{\chi} - \tilde{\chi}^{(0)}\| < \xi$, we have

$$\|\mathcal{D}\mathrm{F}\left(\tilde{\chi}\right) - \mathcal{D}\mathrm{F}\left(\tilde{\chi}^{(0)}\right)\| < \frac{1}{2}\| \left(\mathcal{D}\mathrm{F}\left(\tilde{\chi}^{(0)}\right)\right)^{-1} \|^{-1}. \tag{40}$$

Let $\epsilon = \dfrac{\xi}{2}\| \left(\mathcal{D}\mathrm{F}\left(\tilde{\chi}^{(0)}\right)\right)^{-1} \|^{-1}$, we can easily verify that the mapping

$$\Theta_\tau(\tilde{\chi}) = \tilde{\chi} - \left(\mathcal{D}\mathrm{F}\left(\tilde{\chi}^{(0)}\right)\right)^{-1} (\mathrm{F}(\tilde{\chi}) - \tau)$$

is a contraction that maps $B\left(\tilde{\chi}^{(0)}, \xi\right)$ to itself, when $|\varepsilon| < \epsilon$ and $\|\tau\| < \epsilon$. Then $\Theta_\tau$ has a unique fixed point $\tilde{\chi}$. Accordingly, for $\tau$ fixed, $\|\tau\| < \epsilon$, there exists a unique $\tilde{\chi}$ such that $\|\tilde{\chi} - \tilde{\chi}^{(0)}\| < \xi$, and $\tau = \mathrm{F}(\tilde{\chi})$, i.e., F is 1-to-1 from $\mathrm{F}^{-1}\left(B(0,\epsilon)\right)$ into $B(0,\epsilon)$. Assuming that $|\varepsilon| < \epsilon$, obviously we have $(\varepsilon, 0, \cdots, 0) \in B(0,\epsilon)$, there exists a unique $(\varepsilon, \Phi(\varepsilon))$ which belongs to $B\left(\tilde{\chi}^{(0)}, \xi\right)$, such that $(\varepsilon, 0, \cdots, 0) = \mathrm{F}(\varepsilon, \Phi(\varepsilon))$, where $\Phi(\varepsilon) = (\phi_0(\varepsilon), \psi_0(\varepsilon), \cdots, \phi_N(\varepsilon), \psi_N(\varepsilon))$. We justified that $|\varepsilon| < \epsilon$, there exists a unique $\phi(\varepsilon)$ such that $\mathcal{F}(\varepsilon, \Phi(\varepsilon)) = 0$, then the BVP (1)−(2) has a unique solution. Furthermore, the function $\Phi$ is $C^n(-\epsilon, \epsilon)$, as are F and $\mathrm{F}^{-1}$, and its derivatives are given in Lemma 2.2. The proof is completed. ■

The iterative problems described above are defined for any order in case $f$ and $g$ are *smooth* functions and the asymptotic developments for the boundary conditions are convergent.

**H2** Assume that $\|\alpha_k^{(i)}\| \leq \dfrac{A}{\delta^i}$, $\|\beta^{(i)}\| \leq \dfrac{B}{\delta^i}$, $A$ and $B$ are constants.

**Theorem 2.4.** If assumptions H1 and H2 hold, and $f$ is a *smooth* function, then there exists $\epsilon > 0$, for all $|\varepsilon| < \epsilon$, the boundary value problem (1)−(2) has a unique solution which satisfy

$$x(t,\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n x^{(n)}(t), \quad y(t,\varepsilon)) = \sum_{n=0}^{\infty} \varepsilon^n y^{(n)}(t),$$

where $x^{(0)}(t), y^{(0)}(t), x^{(1)}(t), y^{(1)}(t), x^{(2)}(t), y^{(2)}(t), x^{(n)}(t), y^{(n)}(t)$, are the solutions of the problems (5)−(6), (25)−(28), (22)−(23), (32)−(33), (29)−(30), (38)−(39), (35)−(36), respectively.

# 3. Other problems

## 3.1. Initial Value Problem

By cancelling the small parameter in the system (1) combined with the initial values $x(t=0)=\alpha_1(\varepsilon), y(t=0)=\alpha_2(\varepsilon)$, results the reduced problem

$$\begin{cases} x(t+1) = f\left(x(t), y(t), 0, t\right), \quad t \in I_{N-1}, \\ \quad\quad 0 = g\left(x(t), y(t), 0, t\right), \quad t \in I_{N-1}, \\ \\ \quad x(0) = \alpha_1(0), \qquad y(0) = \alpha_2(0). \end{cases} \tag{41}$$

Substituting $t = 0$ into (41), we have

$$0 = g\left(\alpha_1(0), \alpha_2(0), 0, 0\right).$$

This condition is not necessary verified, moreover there is no indication for computing the value $y_N^{(0)}$, it can be facultative. The *perturbation method* is irrelevant for the initial value problem.

## 3.2. Final Value Problem

We associate to the system (1) the end point values

$$x(t=N) = \beta_1(\varepsilon), \tag{42}$$

$$y(t=N) = \beta_2(\varepsilon), \tag{43}$$

where

$$\beta_1(\varepsilon) = \beta_1^{(0)} + \varepsilon\beta_1^{(1)} + \cdots + \varepsilon^n\beta_1^{(n)}, \quad \beta_2(\varepsilon) = \beta_2^{(0)} + \varepsilon\beta_2^{(1)} + \cdots + \varepsilon^n\beta_2^{(n)}. \tag{44}$$

By cancelling the small parameter in the FVP (1)−(42)−(43), results the reduced problem (5)−(45)−(46), where

$$x^{(0)}(N) = \beta_1^{(0)}, \tag{45}$$

$$y^{(0)}(N) = \beta_2^{(0)}. \tag{46}$$

The boundary layer is located at $y(N)$. The states of the system can be decoupled if the second equation in (5) has a unique root and (19)−(45) can be solved backward. We use the following hypothesis.

**H3** Suppose for each $t = 0, \ldots, N-1$, and for $\forall(x, y) \in X \times Y$, we have

$$\mathcal{G}_t = D_1 f_t - D_2 f_t [D_2 g_t]^{-1} D_1 g_t \neq 0.$$

**Proposition 3.1.** If H1 and H3 hold, then problem (5)–(45)–(46) has a unique solution.

By using Lemme 2.2 again and the (21) substitution, we immediately deduce for the final value problem (1)−(42)−(43) the same steps of the procedure given in Subsection 2.3. For any $n$ order approximation, $n \geq 1$, we iterate backward the difference equation

$$\begin{aligned} x^{(n)}(t) = [\mathcal{G}_t]^{-1}\Big(x^{(n)}(t+1) - D_2 f_t [D_2 g_t]^{-1}\Big[y^{(n-1)}(t+1) \\ -\underline{\sum_0 \sum_1 \cdots \sum_n} \frac{D_1^{p_1} D_2^{p_2} D_3^{p_3} f_t \times \prod_{i=1}^{n} (x^{(i)}(t))^{q_{i1}} (y^{(i)}(t))^{q_{i2}} (\varepsilon^{(i)})^{q_{i3}}}{\prod_{i=1}^{n} \prod_{j=1}^{3} q_{ij}!} \\ + D_2 f_t [D_2 g_t]^{-1} \underline{\sum_0 \sum_1 \cdots \sum_n} \frac{D_1^{p_1} D_2^{p_2} D_3^{p_3} g_t \times \prod_{i=1}^{n} (x^{(i)}(t))^{q_{i1}} (y^{(i)}(t))^{q_{i2}} (\delta_i)^{q_{i3}}}{\prod_{i=1}^{n} \prod_{j=1}^{3} q_{ij}!}\Big]\Big) \end{aligned} \tag{47}$$

with the final value

$$x^{(n)}(N) = \beta_1^{(n)} \tag{48}$$

knowing that all these calculations are achievable under assumptions H1 and H3. In next step, we can set from (36), the coefficients $y^{(n)}(t)$, $t \in I_{N-1}$, while

$$y^{(n)}(N) = \beta_2^{(n)}. \tag{49}$$

**Theorem 3.2.** If H1 and H3 hold, there exists $\epsilon > 0$, such that for all $|\varepsilon| < \epsilon$, the FVP (1)−(42)−(43) has a unique solution which satisfy (4); where $x^{(0)}(t)$, $y^{(0)}(t)$, $x^{(n)}(t)$, $y^{(n)}(t)$, are found solving backward (5)−(45)−(46), (47)−(48), (36)−(49), respectively.

**H4** Assume that $\|\beta_1^{(i)}\| \leq \frac{A}{\delta^i}$, $\|\beta_2^{(i)}\| \leq \frac{B}{\delta^i}$, $A$ and $B$ are constants.

**Theorem 3.3.** If assumptions H1, H3 and H4 hold, $f$ and $g$ are *smooth* functions, then there exists $\epsilon > 0$, for all $|\varepsilon| < \epsilon$, the FVP (1)−(42)−(43) has a unique solution which satisfy

$$x(t,\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n x^{(n)}(t), \quad y(t,\varepsilon)) = \sum_{n=0}^{\infty} \varepsilon^n y^{(n)}(t),$$

where $x^{(0)}(t)$, $y^{(0)}(t)$, $x^{(n)}(t)$, $y^{(n)}(t)$, are found solving backward (5)−(45)−(46), (47)−(48), (36)−(49), respectively.

## 4. Conclusion

This paper provides a generalization of the results in [17]. A nonlinear *D-model* of perturbed discrete-time systems is studied by a perturbation technique providing iterative algorithm to find asymptotic solutions up to any order. Our tool is the Faa Di Bruno formula and the contraction mapping principle. We have also consider the FVP, the solution may be found by straitforward computation, but the time-scale associated with the high dimension generate error accumulation and requires a large computation time. The proposed perturbation method is intended to reduce the computational load since it offers reliable results with few iterations. For the IVP, the *perturbation method* is not applicable. In forthcoming papers, we will study separately other models of singularly perturbed systems.

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