

Interacting Particle Systems Treated as Non Homogeneous Markov Chains

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Abstract

Interacting particle systems is a mature area of probability theory. In this work we consider several kinds of Hamiltonians in one dimensional models. We explain how they can be treated as non-homogeneous Markov chains.

Keywords: Hamiltonian, Gibbs State, Phase Transition, Markov chains.

1. INTRODUCTION

Many authors dealt with the problem of phase transitions. In usual cases, there can be no phase transitions. But we have several models with different properties which admit phase transitions, for example [2-10].

In Kerimov [3,4], the random fields take values in a countably infinite set. Also the potential function of nearest neighbors is symmetric with respect to the two arguments and symmetric with respect to the point $x = -1/2$ and the external field is symmetric with respect to the point $x = 1/2$.

In Kerimov and Mallak [7], we constructed a one dimensional model with two spins $\{0,1\}$ and a unique ground state having infinitely many extreme limit Gibbs states. The interaction potential was between the spin variable at a point and the restriction of the configuration to an interval. The model was interesting since it disproved a conjecture formulated before [3]. In Mallak [8], we considered the case when the spin space is countably infinite. In Mallak [9], we improved somehow the interaction potential in [7]. Finally, in Mallak [10], we constructed a wide class of models that admit phase transitions. The interaction potential was between the spin variable at a point and the restriction of the configuration to an interval, the spin space was an arbitrary countably infinite set.

In this work we explain how these models are interpreted as non-homogeneous Markov chains.

In section 2 we give some mathematical definitions while interpreting the models as non-homogeneous Markov chains is presented in section 3. We conclude the paper with some comments.

2. MATHEMATICAL INTRODUCTION: DEFINITIONS

Definition 1. Let S be a discrete set, finite or countably infinite. Suppose to each pair $i, j \in S$ there is assigned a non-negative number p_{ij} such that these numbers satisfy the constraint $\sum_{j \in S} p_{ij} = 1, \forall i \in S$. Let X_0, X_1, \dots be a sequence of random variables whose range is in S . The sequence (X_n) is a Markov chain if $P[X_{n+1} = j | X_0 = i_0, \dots, X_n = i_n] = P[X_{n+1} = j | X_n = i_n] = p_{i_n j}^{n+1}$ for all n and every sequence $i_0, \dots, i_n \in S$ for which $P[X_0 = i_0, \dots, X_n = i_n] > 0$. A Markov chain is stationary (homogeneous) if $p_{i_n j}^{n+1}$ does not depend on n , that is $p_{i_n j}^{n+1} = p_{i_n j}$, otherwise it is non-stationary (non-homogeneous or inhomogeneous).

Definition 2. Let S be a countably infinite set and (Φ, E) any measurable space. A family $(\varphi(x))_{x \in S}$ of random variables which are defined on some probability space and take values in Φ is called a random field, or a spin system. S is called the parameter set, Φ is called the state space, or spin space, and $\varphi(x)$ is called the spin at state x .

Definition 3. Let $\Omega = \Phi^S = \{(\varphi(x))_{x \in S} : \varphi(x) \in \Phi\}$. Then $w \in \Omega$ is called a configuration and Ω is called the set of all possible configurations.

Definition 4. Let $\Gamma = \{A \subset S : A \neq \varnothing, |A| < \infty\}$. An interaction potential is a family $U = (U_A)_{A \in \Gamma}$ of measurable functions (with respect to the product sigma algebra) where $U_A : \Omega \rightarrow R$.

For all $A \in \Gamma$ and $w \in \Omega, H_A^U(w) = \sum_{\Lambda \in \Gamma, A \cap \Lambda = \varnothing} U_\Lambda(w)$ is called the total energy of w in A for U . It is also called Hamiltonian.

Definition 5. On the space $\Omega_\Lambda = \Phi^\Lambda = \{(\varphi(x))_{x \in \Lambda} : \varphi(x) \in \Phi\}$ we introduce a probability distribution defining the probability of a configuration by $P_\Lambda(w^\Lambda) = Z_\Lambda^{-1} \exp[-\beta H_\Lambda^\Phi(w^\Lambda)]$ where Z_Λ is a normalizing factor defined by the condition $\sum_{w^\Lambda \in \Omega_\Lambda} P_\Lambda(w^\Lambda) = 1$, that is $Z_\Lambda = \sum_{w^\Lambda \in \Omega_\Lambda} \exp[-\beta H_\Lambda^\Phi(w^\Lambda)]$. $\beta = (kT)^{-1}$, where k is a constant we consider it to be 1 and T is the temperature.

This Probability distribution is called a Gibbs probability distribution in Λ corresponding to the given Hamiltonian.

Definition 6. Let $\Lambda \in \Gamma$ and $A, B \subseteq \Lambda$ be such that $A \cap B = \varnothing$ and $\text{bd}(A) \subset B$ where $\text{bd}(A)$ denotes the boundary of A . Let $P_\Lambda^A(w^{-A} | w^{\sim B}) = P_\Lambda^A(\varphi(x) = \varphi^-(x) | \varphi(y) = \varphi^{\sim}(y), y \in B)$ denote the conditional probability that w^Λ equals w^{-A} on the set A under the condition that its values on the set B equals $w^{\sim B}$. A probability distribution P on the space Ω is said to determine a Gibbs measure (it is also called Gibbs state,

Gibbs random field or DLR state) if the conditional distribution $P_{\Lambda}^A(w^{-A} | w^{-B})$ generated by the distribution P coincides with the Gibbs distribution in A with the boundary configuration $w^{\sim bd(A)}$ for arbitrary finite subsets $A, B \subseteq \Lambda$ such that $A \cap B = \varnothing$ and $bd(A) \subset B$. If P is not unique, the given Hamiltonian is said to exhibit phase transition.

Definition 7. Let $\{P_{\Lambda_i}\}_{i=1}^{\infty}$ be a sequence of probability measures. If $P_{\Lambda_n}(A) \rightarrow P(A)$ for each cylinder event A , then P is called the weak limit of the sequence of probability measures.

Definition 8. An element μ of a convex subset of A is said to be extreme if $\mu \neq \alpha\mu_1 + (1 - \alpha)\mu_2, \forall 0 < \alpha < 1, \mu_1, \mu_2 \in A$. An extreme limit Gibbs state is the weak limit of finite volume Gibbs state. It is well-known that the set of all limit Gibbs states coincides with the closed convex hull of the set of weak limits of finite volume Gibbs states.

Definition 9. A configuration w^{gr} is said to be a ground state if for any finite perturbation w^{\sim} of the configuration w^{gr} the expression $H(w^{\sim}) - H(w^{gr})$ is non negative.

More details can be found in many books, for example [1] and [11].

3. INTERPRETING THE MODELS AS INHOMOGENEOUS MARKOV CHAINS

In Mallak [9], It was shown that the model has a unique ground state and has a countable number of extreme limit Gibbs states.

In this work we prove that the model has a countable number of extreme limit Gibbs states in a simple way. With boundary condition $\frac{\#\alpha}{c_l - c_{l-1}} \in I_k$, consider the inhomogeneous Markov chain with two states α & γ and with transition matrix

$P_k = \begin{bmatrix} S_k & 1 - S_k \\ p & q \end{bmatrix}$ in a block in Z^- . This matrix is obtained from the interaction potential (at $\beta = 1$) as follows:

$$p \left(\xi(x) = \alpha \mid \frac{\#\alpha}{c_l - c_{l-1}} \in I_k \right) = \frac{\exp\{-\beta \ln(S_k)\}}{\exp\{-\beta \ln(S_k)\} + \exp\{-\beta \ln(1 - S_k)\}} = S_k$$

$$p \left(\xi(x) = \gamma \mid \frac{\#\alpha}{c_l - c_{l-1}} \in I_k \right) = \frac{\exp\{-\beta \ln(1 - S_k)\}}{\exp\{-\beta \ln(S_k)\} + \exp\{-\beta \ln(1 - S_k)\}} = 1 - S_k$$

$$p \left(\xi(x) = \alpha \mid \frac{\#\alpha}{c_l - c_{l-1}} \notin I_k \right) = \frac{\exp\{-\beta \ln(p)\}}{\exp\{-\beta \ln(p)\} + \exp\{-\beta \ln(q)\}} = p$$

$$p \left(\xi(x) = \gamma \mid \frac{\#\alpha}{c_l - c_{l-1}} \notin I_k \right) = \frac{\exp\{-\beta \ln(q)\}}{\exp\{-\beta \ln(p)\} + \exp\{-\beta \ln(q)\}} = q$$

Then the probability of moving from the state α to the state α in n steps (with boundary condition $\frac{\#\alpha}{c_l - c_{l-1}} \in I_k$) $\geq \prod_{k=1}^n p_k \geq \delta > 0$ and as $n \rightarrow \infty$,

$$\prod_{k=1}^{\infty} p_k \geq \delta > 0.$$

This is from the fact that the probability of moving from the state α to the state α in n steps is the entry $p_{\alpha\alpha}$ in the matrix $P_1 P_2 \dots P_n$ and

$\sum s_k < \infty \Leftrightarrow \prod(1 - s_k)$ is convergent (s_k) positive numbers.

Now for each natural number k , this probability is $\geq \delta > 0$, if it were the same probability measure then these values must sum to one, but this is impossible since the infinite sum of a constant greater than zero is divergent.

Hence the model has a countable number of extreme limits Gibbs States j

4. COMMENTS

The theory of Markov chains is a useful tool to simplify the interacting particle systems in statistical mechanics.

We will try to apply this method for interacting in fuzzy environment.

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