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A Three-Step Iterative Method to Solve A Nonlinear Equation via an Undetermined Coefficient Method

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Abstract

This article discusses a three-step iterative method which is a combination of the iterative method that contains the function and first derivative with Newton's method. The process of the combination uses the principle of an undetermined coefficient method by allowing only one additional function evaluation that may occur in the process. By combining the methods proposed by Khattri and Abbasbandy [Math. Vesn., 63 (2011), 67-72] with Newton's method, then through the convergence analysis it was shown that the proposed method has a sixth order convergence and requires two function and two first derivative evaluations in each iteration. Hence its efficiency index is 1.56508. Some examples of nonlinear equations are solved using the proposed method. Then the obtained solutions are compared with those of other mention methods to see the effectiveness of proposed methods.

AMS Subject classification: 65H05

Keywords: Efficiency index, three-step iterative method, order of convergence

1. INTRODUCTION

Finding the root of the nonlinear equation in the form of

$$f(x) = 0 \tag{1}$$

is one of the important problems in numerical analysis. To find the simple root of the equation (1) many methods can be used and the most famous classical method is Newton's method with the following iteration:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0,1,2, \ldots,$$

where x_0 is the given initial approximation. This method has second order convergence [1, h.58].

Many researchers have modified Newton's method which produces high orders convergence such as [3] - [6], [8], [10], [11], [13] dan [15]. Fang et al. [8] modified Newton's method with five evaluation functions and produced a sixth order convergence method having the following iterations:

$$y_n = x_n - \frac{f(x_n)}{\alpha_n f(x_n) + f'(x_n)},$$
$$z_{n+1} = y_n - \frac{f(y_n)}{\beta_n f(y_n) + f'(y_n)},$$
$$x_{n+1} = z_n - \frac{f(z_n)}{\gamma_n f(z_n) + f'(y_n)},$$

where is $\alpha_n, \beta_n, \gamma_n$ real numbers chosen in such a way that $0 \le |\alpha_n|, |\beta_n|, |\gamma_n| \le 1$, and

$$\operatorname{sign}(\alpha_n f(x_n)) = \operatorname{sign}(f'(x_n)),$$
$$\operatorname{sign}(\beta_n f(y_n)) = \operatorname{sign}(f'(y_n)),$$
$$\operatorname{sign}(\gamma_n f(z_n)) = \operatorname{sign}(f'(y_n)),$$

where $n = 1,2,\ldots,$ and $\operatorname{sign}(x)$ is a sign function. Sharma and Guha [16] modified Ostrowski's method having four function evaluations and a sixth order convergence. Their formula is given as follows:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$z_n = y_n - \frac{f(y_n)}{f'(x_n)} \frac{f(x_n)}{f(x_n) - 2f(y_n)},$$
$$x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)} \frac{f(x_n) + af(y_n)}{f(x_n) + bf(y_n)},$$

where a and b are parameters, with $b = a - 2$. Grau and Diaz-Barero [10] introduced a new method by correcting Ostrowski's method. The method has a sixth order convergence with the following iterations:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$z_n = y_n - \frac{f(x_n)}{f(x_n) - 2f(y_n)} \frac{f(y_n)}{f'(x_n)},$$
$$x_{n+1} = z_n - \frac{f(x_n)}{f(x_n) - 2f(y_n)} \frac{f(z_n)}{f'(x_n)}.$$

Khattri and Abbasbandy [12] introduced an iterative method having a fourth order convergence with the following formula:

$$y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = y_n - [1 + (\frac{21}{8})(\frac{f'(y_n)}{f'(x_n)}) + (\frac{-9}{2})(\frac{f'(y_n)}{f'(x_n)})^2 + (\frac{15}{8})(\frac{f'(y_n)}{f'(x_n)})^3]\frac{f(x_n)}{f'(x_n)}.$$

In this article, the authors present a combination of the Khattri and Abbasbandy method [12] with Newton's method using the principle of an undetermined coefficient method. The derivation and analysis of convergence of the proposed method are presented in section two. In section three numerical computations are carried out on the four test functions. Then the results are compared to the obtained results from the some known methods to see the efficiency of the proposed method.

2. DEVELOPMENT OF METHOD AND CONVERGENCE ANALYSIS

Suppose that $\phi(x; f(x), f'(x), f'(y))$ is a function from $\mathbb{R}$ to $\mathbb{R}$. Consider the modification of Newton's method given as follows:

$$\left.\begin{aligned} z_n &= \phi(x_n; f(x_n), f'(x_n), f'(y_n)), \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(z_n)}. \end{aligned}\right\} \tag{2}$$

to eliminate $f'(z_n)$ in (2), first we consider the following equation:

$$f'(z_n) = Af(x_n) + Bf'(x_n) + Cf'(y_n) + Df(z_n) + Ef'(\frac{x_n+y_n}{2}). \tag{3}$$

Next we expand $f(x)$, $f'(x)$ about x_n to the fourth derivative, and evaluate $f'(y_n)$, $f(z_n)$, $f'(\frac{x_n+y_n}{2})$, and $f'(z_n)$, then substitute the resulting equations into equation (3). On comparing the coefficient of derivative of f at x_n, we obtain

$$A + D = 0, \tag{4}$$

$$B + C + D(z_n - x_n) + E = 1, \tag{5}$$

$$C(y_n - x_n) + \frac{D}{2}(z_n - x_n)^2 + \frac{E}{2}(y_n - x_n) = (z_n - x_n), \tag{6}$$

$$\frac{C}{2}(y_n - x_n)^2 + \frac{D}{6}(z_n - x_n)^3 + \frac{E}{8}(y_n - x_n)^2 = \frac{(z_n - x_n)^2}{2}, \tag{7}$$

$$\frac{C}{6}(y_n - x_n)^3 + \frac{D}{24}(z_n - x_n)^4 + \frac{E}{48}(y_n - x_n)^3 = \frac{(z_n - x_n)^3}{6}. \tag{8}$$

Then on solving the system equation (4)-(8), we have

$$A = \frac{-2(2\alpha-\beta)}{(-\beta+\alpha)\alpha}, \tag{9}$$

$$B = \frac{(3\beta^3-9\beta^2\alpha-2\alpha^3+7\beta\alpha^2)}{3\beta^2(-\beta+\alpha)}, \tag{10}$$

$$C = \frac{-\alpha^2(2\alpha-\beta)}{3\beta^2(-\beta+\alpha)}, \tag{11}$$

$$D = \frac{2(2\alpha-\beta)}{(-\beta+\alpha)\alpha}, \tag{12}$$

$$E = \frac{4\alpha^2(\alpha-2\beta)}{3\beta^2(-\beta+\alpha)}, \tag{13}$$

where $\alpha = z_n - x_n$ and $\beta = y_n - x_n$. On substituting equations (9)-(13) into equation (3), we end up with

$$\begin{aligned} f'(z_n) = & \frac{-2(2\alpha-\beta)}{(-\beta+\alpha)\alpha} f(x_n) + \frac{\eta}{3\beta^2(-\beta+\alpha)} f'(x_n) + \frac{-\alpha^2(2\alpha-\beta)}{3\beta^2(-\beta+\alpha)} f'(y_n) \\ & + \frac{2(2\alpha-\beta)}{(-\beta+\alpha)\alpha} f(z_n) + \frac{4\alpha^2(\alpha-2\beta)}{3\beta^2(-\beta+\alpha)} f'(\frac{x_n+y_n}{2}), \end{aligned} \tag{14}$$

with $\eta = 3\beta^3 - 9\beta^2\alpha - 2\alpha^3 + 7\beta\alpha^2$.

Replacing the value $f'(\frac{x_n+y_n}{2})$ by the arithmetic mean of $f'(x_n)$ and $f'(y_n)$, we can rewrite (14) as

$$\begin{aligned} f'(z_n) = & \frac{-2(2\alpha-\beta)}{(-\beta+\alpha)\alpha} f(x_n) + (\frac{\beta^2-3\beta\alpha+\alpha^2}{\beta(-\beta+\alpha)}) f'(x_n) + (\frac{-\alpha^2}{\beta(-\beta+\alpha)}) f'(y_n) \\ & + (\frac{2(2\alpha-\beta)}{(-\beta+\alpha)\alpha}) f(z_n), \end{aligned} \tag{15}$$

with $\eta = 3\beta^3 - 9\beta^2\alpha - 2\alpha^3 + 7\beta\alpha^2$.

Then on substituting equation (15) into the second step of equation (2), we obtain

$$x_{n+1} = z_n - \frac{\alpha\beta(-\beta+\alpha)f(z_n)}{\mu f\prime(x_n)+(-\alpha^3)f\prime(y_n)+2\beta(2\alpha-\beta)(f(z_n)-f(x_n))}, \tag{16}$$

with $\mu = \alpha(\beta^2 - 3\beta\alpha + \alpha^2)$.

From equation (16) we suggest a new family of iterative method of the form

$$\left.\begin{aligned} z_n &= \phi(x_n; f(x_n), f'(x_n), f'(y_n)), \\ x_{n+1} &= z_n - \frac{\alpha\beta(-\beta+\alpha)f(z_n)}{\mu f\prime(x_n)+(-\alpha^3)f\prime(y_n)+2\beta(2\alpha-\beta)(f(z_n)-f(x_n))}, \end{aligned}\right\} \tag{17}$$

where $\mu = \alpha(\beta^2 - 3\beta\alpha + \alpha^2)$, $\alpha = z_n - x_n$, $\beta = y_n - x_n$ and $\phi(x_n; f(x_n), f'(x_n), f'(y_n))$ is any fourth order method. Furthermore, using the fourth order method derived by Khattri and Abbasbandy [12], we propose the following iterative method.

$$y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f\prime(x_n)}, \tag{18}$$

$$z_n = x_n - [1 + (\frac{21}{8})(\frac{f\prime(y_n)}{f\prime(x_n)}) + (\frac{-9}{2})(\frac{f\prime(y_n)}{f\prime(x_n)})^2 + (\frac{15}{8})(\frac{f\prime(y_n)}{f\prime(x_n)})^3]\frac{f(x_n)}{f\prime(x_n)}, \tag{19}$$

$$x_{n+1} = z_n - \frac{\alpha\beta(-\beta+\alpha)f(z_n)}{\mu f\prime(x_n)+(-\alpha^3)f\prime(y_n)+2\beta(2\alpha-\beta)(f(z_n)-f(x_n))}, \tag{20}$$

with $\mu = \alpha(\beta^2 - 3\beta\alpha + \alpha^2)$, $\alpha = z_n - x_n$, $\beta = y_n - x_n$. The method defined in equation (18)-(20) is called the method MKA whose convergence analysis is given in Theorema 1.

Teorema 1 (Order of Convergence) Suppose that $\lambda \in I$ is a simple zero of a function $f: I \subset \mathbb{R} \to \mathbb{R}$ sufficiently differentiable at open interval I. If x_0 is close enough to λ, the iterative method defined in equation (18)-(20) has a sixth order convergence and satisfies the error equation:

$$e_{n+1} = (-\frac{85}{9}A_2^3A_3 - \frac{1}{9}A_3A_4 + A_2A_3^2)e_n^6 + O(e_n^7),$$

with

$$A_j = \frac{f^{(j)}(\lambda)}{j!f\prime(\lambda)}, \qquad j = 2,3,\cdots,8,$$

and $e_n = x_n - \lambda$.

Proof. Suppose that λ is a simple root of $f(x) = 0$ then $f(\lambda) = 0$, and $f'(\lambda) \neq 0$. Then by Taylor's expansion [2, h.216] of $f(x)$ about λ until the sixth order derivative, we get

$$\begin{aligned} f(x) = f(\lambda) + f'(\lambda)(x_n - \lambda) + f^{(2)}(\lambda)\frac{(x-\lambda)^2}{2!} + f^{(3)}(\lambda)\frac{(x-\lambda)^3}{3!} \\ + f^4(\lambda)\frac{(x-\lambda)^4}{4!} \\ + f^5(\lambda)\frac{(x-\lambda)^5}{5!} + f^{6)}(\lambda)\frac{(x-\lambda)^6}{6!} + O((x-\lambda)^7). \end{aligned} \tag{21}$$

By evaluating equation (21) at $x = x_n$, and recalling $e_n = x_n - \lambda$ and $f(\lambda) = 0$, equation (21) can be written as

$$f(x_n) = f'(\lambda)(e_n + A_2e_n^2 + A_3e_n^3 + A_4e_n^4 + A_5e_n^5 + A_6e_n^6) + O(e_n^7), \tag{22}$$

with

$$A_j = \frac{f^{(j)}(\lambda)}{j!f'(\lambda)}, \qquad j = 2,3,\cdots,8.$$

Furthermore by the same way, we obtain

$$f'(x_n) = f'(\lambda)(1 + 2A_2e_n + 3A_3e_n^2 + 4A_4e_n^3 + 5A_5e_n^4 + 6A_6e_n^5 + 7A_7e_n^6) + O(e_n^7). \tag{23}$$

Then using equation (22), equation (23) and the aid of geometric series [17], we have

$$\frac{f(x_n)}{f'(x_n)} = e_n - A_2e_n^2 + (2A_2^2 - 2A_3)e_n^3 + (-4A_2^3 - 3A_4 + 7A_2A_3)e_n^4 + (6A_3^2 -4A_5 - 20A_2^2A_3 + 10A_2A_4 + 8A_2^4)e_n^5 + (-16A_2^5 + 13A_2A_5 + 17A_3A_4 +52A_2^3A_3 - 5A_6 - 28A_2^2A_4 - 33A_2A_3^2)e_n^6 + O(e_n^7). \tag{24}$$

Next equation (24) is substituted to equation (18) and simplifying the resulting equation we get

$$y_n = \lambda + \frac{1}{3}e_n + \frac{2}{3}A_2e_n^2 - \frac{2}{3}(2A_2^2 - 2A_3)e_n^3 - \frac{2}{3}(-4A_2^3 - 3A_4 + 7A_2A_3)e_n^4 - \frac{2}{3}(6A_3^2 -4A_5 - 20A_2^2A_3 + 10A_2A_4 + 8A_2^4)e_n^5 - \frac{2}{3}(-16A_2^5 + 13A_2A_5 + 17A_3A_4 +52A_2^3A_3 - 5A_6 - 28A_2^2A_4 - 33A_2A_3^2)e_n^6 + O(e_n^7). \tag{25}$$

Following the idea how to obtain $f'(x_n)$, we have $f'(y_n)$ by considering equation (25) that is

$$f'(y_n) = f'(\lambda)(1 + \frac{2}{3}A_2e_n + (\frac{4}{3}A_2^2 + \frac{1}{3}A_3)e_n^2 + (4A_2A_3 + \frac{4}{27}A_4 - \frac{8}{3}A_2^3)e_n^3 +(\frac{44}{9}A_2A_4 + \frac{16}{3}A_2^4 - \frac{32}{3}A_2^2A_3 + \frac{8}{3}A_3^2 + \frac{5}{81}A_5)e_n^4 + (\frac{52}{9}A_3A_4 -12A_2A_3^2 + \frac{2}{81}A_6 + \frac{472}{81}A_2A_5 + \frac{80}{3}A_2^3A_3 - \frac{40}{3}A_2^2A_4 - \frac{32}{3}A_2^5)e_n^5 +(-64A_2^4A_3 + \frac{124}{3}A_2^2A_3^2 - \frac{8}{3}A_3^3 + \frac{512}{81}A_3A_5 - \frac{244}{9}A_2A_3A_4 + \frac{560}{81}A_2A_6 + \frac{64}{3}A_2^6 - \frac{1364}{81}A_2^2A_5 + \frac{944}{27}A_2^3A_4 + \frac{7}{729}A_7 + \frac{8}{3}A_4^2)e_n^6) + O(e_n^7). \tag{26}$$

From equation (23) and equation (26) and with the aid of the geometry series [17], we obtain

$$\begin{aligned}\frac{f'(y_n)}{f'(x_n)} = 1 &- \frac{4}{3}A_2e_n + (-\frac{8}{3}A_3 + 4A_2^2)e_n^2 + (-\frac{104}{27}A_4 - \frac{32}{3}A_2^3 + \frac{40}{3}A_2A_3)e_n^3 \\ &+(\tfrac{32}{3}A_3^2 - \tfrac{400}{81}A_5 + \tfrac{80}{3}A_2^4 - \tfrac{148}{3}A_2^2A_3 + \tfrac{484}{27}A_2A_4)e_n^4 + (-\tfrac{220}{3}A_2A_3^2 \\ &+28A_3A_4 + \tfrac{604}{27}A_2A_5 - \tfrac{484}{81}A_6 - 64A_2^5 - \tfrac{1760}{27}A_2^2A_4 + \tfrac{472}{3}A_2^3A_3)e_n^5 \\ &+(-\tfrac{1376}{3}A_2^4A_3 + \tfrac{2176}{81}A_2A_6 + 208A_2^3A_4 + \tfrac{2792}{81}A_3A_5 - \tfrac{1712}{9}A_2A_3A_4 \\ &-\tfrac{5096}{729}A_7 - \tfrac{104}{3}A_3^3 + 336A_2^2A_3^2 + \tfrac{488}{27}A_4^2 + \tfrac{448}{3}A_2^6 \\ &-\tfrac{6608}{81}A_2^2A_5)e_n^6 + O(e_n^7). \end{aligned} \tag{27}$$

Next equation (24) and equation (27) are substituted to equation (19) and simplifying the resulting equation, we end up with

$$\begin{aligned} z_n = \lambda &+ (-A_2A_3 + \frac{85}{9}A_2^3 + \frac{1}{9}A_4)e_n^4 + (-2A_3^2 - \frac{724}{9}A_2^4 + \frac{176}{3}A_2^2A_3 + \frac{8}{27}A_5 \\ &- \tfrac{20}{9}A_2A_4)e_n^5 + (\tfrac{2306}{27}A_2^2A_4 - \tfrac{5318}{9}A_2^3A_3 - \tfrac{22}{3}A_3A_4 + \tfrac{358}{3}A_2A_3^2 - \tfrac{10}{3}A_2A_5 \\ &+ \tfrac{4010}{9}A_2^5 + \tfrac{14}{27}A_6)e_n^6 + O(e_n^7). \end{aligned} \tag{28}$$

Then by evaluating equation (21) at $x = z_n$ as in equation (28), we get

$$\begin{aligned} f(z_n) = f'(\lambda)((&-A_2A_3 + \frac{85}{9}A_2^3 + \frac{1}{9}A_4)e_n^4 + (-2A_3^2 - \frac{724}{9}A_2^4 + \frac{176}{3}A_2^2A_3 \\ &+ \tfrac{8}{27}A_5 - \tfrac{20}{9}A_2A_4)e_n^5 + (\tfrac{2306}{27}A_2^2A_4 - \tfrac{5318}{9}A_2^3A_3 - \tfrac{22}{3}A_3A_4 + \tfrac{358}{3}A_2A_3^2 \\ &- \tfrac{10}{3}A_2A_5 + \tfrac{4010}{9}A_2^5 + \tfrac{14}{27}A_6)e_n^6 + O(e_n^7)). \end{aligned} \tag{29}$$

Since $\alpha = z_n - x_n$, $\beta = y_n - x_n$, from equation (28) and equation (25), we have respectively

$$\begin{aligned} \alpha = -e_n &+ (\frac{1}{9}A_4 + \frac{85}{9}A_2^3 - A_2A_3)e_n^4 + (-\frac{20}{9}A_2A_4 + \frac{176}{3}A_2^2A_3 + \frac{8}{27}A_5 - 2A_3^2 \\ &- \tfrac{724}{9}A_2^4)e_n^5 + (-\tfrac{5318}{9}A_2^3A_3 - \tfrac{10}{3}A_2A_5 + \tfrac{14}{27}A_6 + \tfrac{4010}{9}A_2^5 - \tfrac{22}{3}A_3A_4 \\ &+ \tfrac{358}{3}A_2A_3^2 + \tfrac{2306}{27}A_2^2A_4)e_n^6 + O(e_n^7), \end{aligned} \tag{30}$$

and

$$\begin{aligned} \beta = -\frac{2}{3}e_n &+ \frac{2}{3}A_2e_n^2 - \frac{2}{3}(-2A_3 + 2A_2^2)e_n^3 - \frac{2}{3}(-4A_2^3 + 7A_2A_3 - 3A_4)e_n^4 \\ &- \tfrac{2}{3}(-4A_5 + 10A_2A_4 + 6A_3^2 + 8A_2^4 - 20A_2^2A_3)e_n^5 - \tfrac{2}{3}(52A_2^3A_3 - 16A_2^5 \\ &-5A_6 + 13A_2A_5 - 33A_2A_3^2 - 28A_2^2A_4 + 17A_3A_4)e_n^6 + O(e_n^7). \end{aligned} \tag{31}$$

Using equation (30), equation (31) and equation (29) we get

$$\alpha\beta(-\beta+\alpha)f(z_n) = f'(\lambda)((\frac{2}{9}A_2A_3 - \frac{170}{81}A_2^3 - \frac{2}{81}A_4)e_n^7 + (-\frac{16}{243}A_5 + \frac{142}{9}A_2^4 - \frac{346}{27}A_2^2A_3 + \frac{38}{81}A_2A_4 + \frac{4}{9}A_3^2)e_n^8 + (-\frac{1964}{27}A_2^5 + \frac{164}{243}A_2A_5 + \frac{128}{81}A_3A_4 - \frac{28}{243}A_6 - \frac{4468}{243}A_2^2A_4 + \frac{3056}{27}A_2^3A_3 - \frac{692}{27}A_2A_3^2)e_n^9 + O(e_n^{10})). \quad (32)$$

Next using equation (22), (23), (26), (29), (30), and (31), the denominator of the equation (20) becomes

$$-2\beta(2\alpha-\beta)f(x_n) + \alpha(\beta^2 - 3\beta\alpha + \alpha^2)f'(x_n) + (-\alpha)^3 f'(y_n) + 2\beta(2\alpha - \beta)f(z_n) = (-\frac{2}{9}e_n^3 - \frac{2}{9}A_2e_n^4 + (-\frac{2}{9}A_3 + \frac{8}{9}A_2^2)e_n^5 + (\frac{464}{81}A_2^3 + \frac{4}{3}A_2A_3 + \frac{2}{81}A_4)e_n^6 + O(e_n^7))f'(\lambda). \quad (33)$$

Next by dividing (32) with (33) and with the aid of geometry series, we have

$$\frac{\alpha\beta(-\beta+\alpha)f(z_n)}{-2\beta(2\alpha-\beta)f(x_n) + \alpha(\beta^2 - 3\beta\alpha + \alpha^2)f'(x_n) + (-\alpha)^3 f'(y_n) + 2\beta(2\alpha-\beta)f(z_n)} = (-A_2A_3 + \frac{85}{9}A_2^3 + \frac{1}{9}A_4)e_n^4 + (-2A_3^2 - \frac{724}{9}A_2^4 + \frac{176}{3}A_2^2A_3 + \frac{8}{27}A_5 - \frac{20}{9}A_2A_4)e_n^5 + (-\frac{5233}{9}A_2^3A_3 + \frac{355}{3}A_2A_3^2 - \frac{10}{3}A_2A_5 - \frac{65}{9}A_3A_4 + \frac{4010}{9}A_2^5 + \frac{2306}{27}A_2^2A_4 + \frac{14}{27}A_6)e_n^6 + O(e_n^7). \quad (34)$$

On substituting equation (28) and equation (34) into equation (20), we obtain

$$x_{n+1} = \lambda + (-\frac{85}{9}A_2^3A_3 - \frac{1}{9}A_3A_4 + A_2A_3^2)e_n^6 + O(e_n^7). \quad (35)$$

Noting $e_{n+1} = x_{n+1} - \lambda$, then the equation (35) becomes

$$e_{n+1} = (-\frac{85}{9}A_2^3A_3 - \frac{1}{9}A_3A_4 + A_2A_3^2)e_n^6 + O(e_n^7). \quad (36)$$

From the definition of the order of convergence [14, h.75], we see that equation (18)-(20) is of order six and Theorem 1 is proven. ■

From formula equation (18)-(20) we recognize that the proposed method required four function evaluations per iteration, Thus based on the definition of the efficiency index [9, h.261], the efficiency index of MKA iterative method is $6^{\frac{1}{4}} \approx 1.56508$.

3. NUMERICAL SIMULATIONS

In this section numerical simulations are performed which aims to compare the number of iterations required by Fang's method (MF) defined in [8], Sharma's method (MSh) defined in [16], Grau's method (MG) which is defined in [10], and the MKA method. For this purpose we use the following nonlinear equations:

1. $f_1(x) = \cos(x) - x$, [3],
2. $f_2(x) = x^3 + 4x^2 - 10$, [5],
3. $f_3(x) = (x-1)^3 - 1$, [18],
4. $f_4(x) = \sin(x) - \frac{x}{2}$,[6].

In doing comparisons, tolerance allowed is 1.0×10^{-100}, and the stop criteria is $|x_{n+1} - x_n| + |f(x_{n+1})| \leq tol$ and a maximum iteration is 100. Whereas computational order of convergence [7] is estimated using a formula

$$\mathrm{COC} \approx \frac{\ln(|x_{k+1}-x_k|/|x_k-x_{k-1}|)}{\ln(|x_k-x_{k-1}|/|x_{k-1}-x_{k-2}|)}.$$

Tabel 1: Comparison of several iteration methods

$f_n(x)$	x_0	Method	n	COC	$\lvert f(x_n)\rvert$	$\lvert x_n - x_{n-1}\rvert$
f_1	1.7	MKA	4	6.00	$9.653555e-884$	$1.399506e-147$
		MF	5	6.00	$0.000000e+00$	$1.214996e-309$
		MSh	4	6.00	$1.945847e-853$	$1.608872e-142$
		MG	4	6.00	$3.007301e-865$	$1.869432e-144$
f_2	1.6	MKA	4	6.00	$1.200000e-998$	$3.718563e-218$
		MF	4	6.00	$1.837892e-735$	$1.686553e-123$
		MSh	4	6.00	$1.000000e-998$	$1.015452e-193$
		MG	4	6.00	$1.200000e-998$	$7.969299e-206$
f_3	3.5	MKA	5	6.00	$0.000000e+00$	$9.943515e-205$
		MF	6	6.00	$0.000000e+00$	$7.042344e-314$
		MSh	5	6.00	$0.000000e+00$	$1.831491e-175$
		MG	5	6.00	$0.000000e+00$	$1.098560e-201$
f_4	2.0	MKA	4	6.00	$4.000000e-1000$	$7.220656e-261$
		MF	4	6.00	$4.000000e-1000$	$9.815253e-185$
		MSh	4	6.00	$4.000000e-1000$	$4.178883e-241$
		MG	4	6.00	$4.000000e-1000$	$1.030113e-251$

In Table 1, $f_n(x)$ denotes the function of the nonlinear equation, n states the number of iterations obtained from all four methods, COC stand for the computational order of convergence, $|f(x_n)|$ is the absolute value of the value of function at x_n and $|x_n - x_{n-1}|$ denotes the absolute value of the difference between two consecutive approximation of the root.

Based on Table 1 it can be seen that for the function f_1, MKA requires the same number of iterations as MSh and MG and less than MF. Whereas for the functions f_2 and f_4 MKA requires the same number of iterations as MF, Msh and MG. For the f_3 function MKA requires the same number of iterations as Msh and MG and less than MF. Furthermore, the COC given in Table 1 is inagreement with the analytic result obtain in the previous section.

Overall based on the results of the numerical simulations as stated in Table 1 the new method (MKA) can compete with the existing methods and therefore it can be used as an alternative method to solve the nonlinear equation.

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