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An Analytical Study of Mass Transfer in Doubly Connected Region

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Abstract

An eccentric annular region created by placing a catheter in an artery is considered. Mass transfer in the lumen region between the wall and catheter is considered by species transport equation. The eccentric annular region is mapped to a concentric annulus through a conformal mapping. Absorption of species at the wall is considered. The resulting equations are analytically solved and the results are depicted graphically.

AMS Subject Classification: 76Rxx.
Keywords: Doubly connected region, eccentric annulus, mass transfer, conformal mapping, species transport equation.

1. Introduction

Catheterization of artery causes the frictional resistance to increase which inturn modifies pressure gradient. The amount of blood flowing in artery during a particular duration of time is regulated by the need of nutrients to the surrounding tissue. Hence to maintain pressure the volume flow rate increases which causes catheter to create eccentric annulus. [1] has obtained theoretical estimates for corrections in pressure gradient in pulsatile blood flow when catheter is induced. [2] have considered narrow artery with inserted catheter and studied flow pattern. They have also estimated effects of frictional resistance and included non-Newtonian effect by considering casson fluid. [3] have considered steady, fully developed flow in eccentric annulus by using Newtonian fluid. [4] have extended above study to couple stress fluid.

Mass transfer in artery is influenced by permeability at the walls and near catheter. [5] has considered axial dispersion in presence of pulsatile flow. Dispersion of bolus in a rigid tube is considered by [6]. Dispersion of solute between concentric cylinders with effect of permeable wall is studied by [7].

In the present study, convective diffusive mass transfer of injected dye in an artery in the presence of catheter is considered. The eccentric annular region created is conformally mapped to concentric annulus and analytical solution is obtained.

2. Mathematical Formulation

Schematic diagram of eccentric annulus is given in figure 1. The annular domain D in the xy - plane bounded internally by a circle C_1 and externally by a circle C_2 is considered.

The mass transfer is governed by species equation,

$$\frac{\partial c}{\partial t} + w\frac{\partial c}{\partial z} = D\left[\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2}\right] - \gamma_0 c, \tag{2.1}$$

where w, D and γ are the constant velocity, diffusion coefficient and absorption coefficient respectively.

Non-dimensionalising the above equation using the quantities,

$$\left(x^*, y^*, z^*\right) = \frac{(x, y, z)}{h}, \ t^* = \frac{D}{h^2}t, \ c^* = \frac{c}{c_0}, \ w^* = \frac{wh}{2D}, \tag{2.2}$$

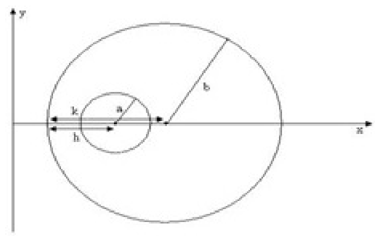

Figure 1: Physical configuration

and neglecting (*) we get,

$$\frac{\partial c}{\partial t} + 2w\frac{\partial c}{\partial z} = \left[\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2}\right] - \gamma c, \tag{2.3}$$

where $\gamma = \dfrac{\gamma_0 h^2}{D}$.

Using the transformation,

$$c\,(x, y, z, t) = \theta\,(x, y, z, t)\, e^{wz-(w^2+\gamma)t}, \tag{2.4}$$

and assuming axial diffusion to be very small compared to radial diffusion, we get

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}. \tag{2.5}$$

Assuming $\theta\,(x, y, t) = H\,(x, y)\, e^{-\alpha^2 t}$ the equation (2.5) becomes,

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \alpha^2 H = 0. \tag{2.6}$$

Expressing the above equation in terms of complex variables $z = x + iy$, $\bar{z} = x - iy$, we get

$$\frac{\partial^2 H}{\partial z \partial \bar{z}} + \frac{\alpha^2}{4} H = 0, \tag{2.7}$$

where H is the temperature which has real values, hence functions of $(z\bar{z})$ and $(z + \bar{z})$.

Therefore H can be expressed as,

$$H = F\,(z\bar{z}) + G\,(z + \bar{z}) = F\,(r) + G\,(s)\,, \tag{2.8}$$

where $r = z\bar{z}$, $s = z + \bar{z}$.

Substituting the equation (2.8) in equation (2.7), we get

$$\frac{\partial F}{\partial r} + r\frac{\partial^2 F}{\partial r^2} + \frac{\partial^2 G}{\partial s^2} + \frac{\alpha^2}{4}\,(F + G) = 0. \tag{2.9}$$

After separating the variables we have,

$$r\frac{\partial^2 F}{\partial r^2} + \frac{\partial F}{\partial r} + \frac{\alpha^2}{4} F(r) = 0,$$

and

$$\frac{\partial^2 G}{\partial s^2} + \frac{\alpha^2}{4} G(s) = 0. \tag{2.10}$$

The solution of equation (2.10) is given by

$$F(r) = A_1 J_0\left(\frac{\alpha}{2}\sqrt{z\bar{z}}\right) + A_2 Y_0\left(\frac{\alpha}{2}\sqrt{z\bar{z}}\right),$$

and

$$G(s) = A_3 cos\frac{\alpha}{2}\left(z+\bar{z}\right) + A_4 sin\frac{\alpha}{2}\left(z+\bar{z}\right). \tag{2.11}$$

Boundary Conditions:
Using flux conditions at both the walls, we have

$$\frac{\partial c}{\partial n} = -k_0 c \text{ on } \quad C_1, \ \frac{\partial c}{\partial n} = k\left(c - c_T\right) \text{ on } \quad C_2. \tag{2.12}$$

where n is the normal direction.

Non-dimensionalising and using the transformation given in the equation (2.4), boundary conditions becomes:

$$\frac{\partial H}{\partial n} = -kH \text{ on } \quad C_1, \ \frac{\partial H}{\partial \rho} = k\left(H - \phi_0 c_T\right) \text{ on } \quad C_2, \tag{2.13}$$

where $\phi_0 = e^{-wz+\left(w^2+\gamma+\alpha^2\right)t}$.

Conformal Mapping:
The eccentric circles $(x-h)^2 + y^2 = a^2$ and $(x-k)^2 + y^2 = b^2$ with $a < b$, $h < k$ are mapped to concentric circles $\zeta\bar{\zeta} = {\rho_1}^2$ and $\zeta\bar{\zeta} = {\rho_2}^2$ with $\rho_1 = \frac{a}{h}$, $\rho_2 = \frac{b}{k}$, $c = h - \frac{a^2}{h} = k - \frac{b^2}{k}$.

Using $z = \frac{c}{1-\zeta}$ and eccentricity $k - h = \frac{a^2}{h} - \frac{b^2}{k}$, the solution becomes,

$$\begin{aligned} H\left(\rho,\, \zeta+\bar{\zeta}\right) = {} & A_1 \sum \beta\left(i,\ j,\ k\right)\rho^{2k}\left(\zeta^j + \frac{\rho^{2j}}{\zeta^j}\right) \\ & + A_2 \sum \beta\left(i,\ k,\ j\right)\rho^{2k}\left(\zeta^j + \frac{\rho^{2j}}{\zeta^j}\right)\left[\psi_i - \sum \frac{1}{s}\left(\zeta^s + \frac{\rho^{2s}}{\zeta^s}\right)\right] \\ & + A_3 \sum \phi_1\left(i,\ j,\ k,\ l,\ m\right)\rho^{2(k+j)}\left(\zeta^j + \frac{\rho^{2j}}{\zeta^j}\right) \\ & + A_4 \sum \phi_2\left(i,\ j,\ k,\ l,\ m\right)\rho^{2(k+j)}\left(\zeta^j + \frac{\rho^{2j}}{\zeta^j}\right). \end{aligned} \tag{2.14}$$

Similarly the boundary conditions become,

$$\frac{\partial H}{\partial \rho} = -kH \text{ on } \quad \rho = \rho_1, \ \frac{\partial H}{\partial \rho} = k\left(H - \phi_0 c_T\right) \text{ on } \quad \rho = \rho_2. \tag{2.15}$$

Using equation (2.15), the constants in equation (2.14) are evaluated and listed in the appendix.

Therefore we have,

$$\theta(x, y, t) = \frac{k}{2} H\left(\rho,\ \zeta + \bar{\zeta}\right) e^{wz-\left(w^2+\beta\right)t}. \tag{2.16}$$

3. Numerical Results

Flux condition at the wall is considered. The analytical solution obtained are numerically calculated and graphically depicted. The velocity is assumed to be constant. The effect of reaction parameter γ, absorption coefficient k, eccentricity $(k - h)$, velocity and concentration are plotted in radial direction. Concentration showed a decreasing profile from the middle of annulus towards the wall.

The figures 2 and 3 shows plot of concentration in radial direction. As velocity increases the concentration decreases as convection increases and more solute convected along the axis. Figures 2, 4 and 5 shows the effect of reaction parameter. Variation of reaction parameter do not affect concentration as it is considered small.

Figure 5 shows slight increase in concentration as reaction at the wall effect more concentration to diffuse towards the wall. Figures 2, 6 and 7 shows the effect of absorption coefficient. As absorption coefficient increases, concentration increases in the middle of annulus. Absorption coefficient results in more concentration to diffuse towards the wall.

Figures 8 and 9 show the effect of eccentricity. Concentration decreases with eccentricity. As eccentricity increases, area of cross-sectional flow and convection increases. Hence there is a decrease in concentration.

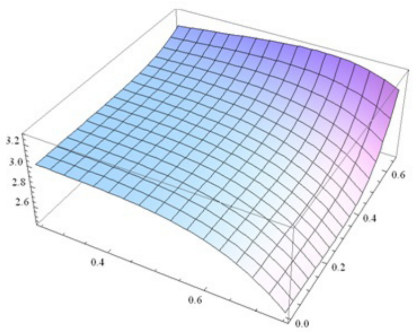

Figure 2: Radial concentration profile for $\omega = 0.1$, $\gamma = 0.001$, $k = 2$, $\epsilon = 0.3$

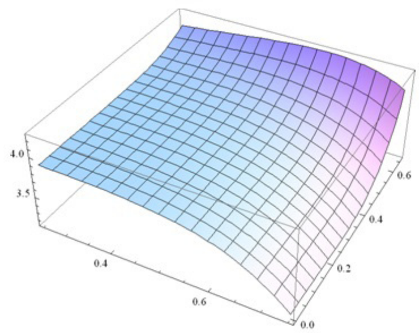

Figure 3: Radial concentration profile for $\omega = 0.01$, $\gamma = 0.001$, $k = 2$, $\epsilon = 0.3$

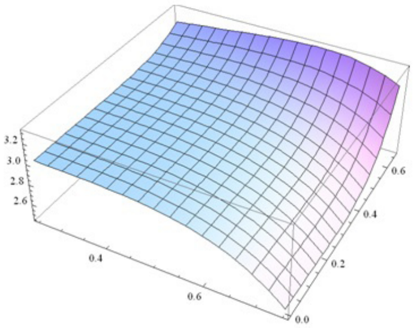

Figure 4: Radial concentration profile for $\omega = 0.1$, $\gamma = 0.01$, $k = 2$, $\epsilon = 0.3$

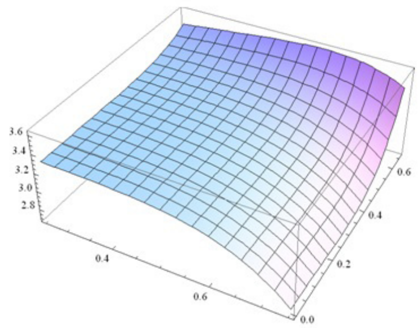

Figure 5: Radial concentration profile for $\omega = 0.1$, $\gamma = 0.1$, $k = 2$, $\epsilon = 0.3$

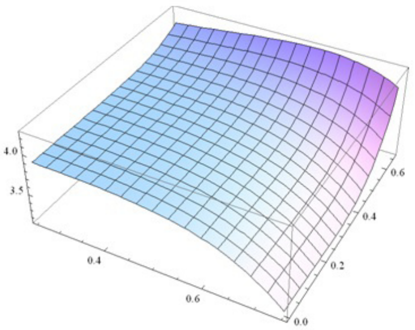

Figure 6: Radial concentration profile for $\omega = 0.01$, $\gamma = 0.001$, $k = 3$, $\epsilon = 0.3$

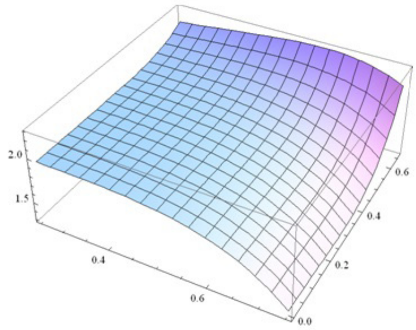

Figure 7: Radial concentration profile for $\omega = 0.01$, $\gamma = 0.001$, $k = 1$, $\epsilon = 0.3$

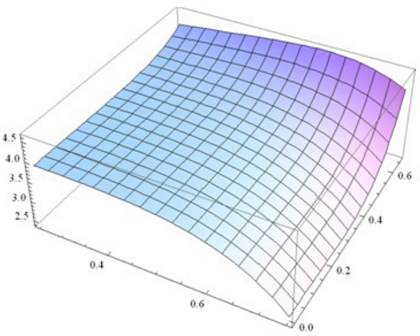

Figure 8: Radial concentration profile for $\omega = 0.01$, $\gamma = 0.001$, $k = 2$, $\epsilon = 0.0$

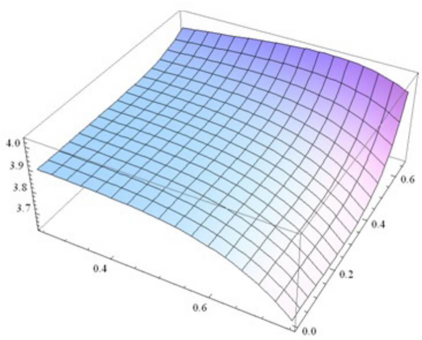

Figure 9: Radial concentration profile for $\omega = 0.01$, $\gamma = 0.001$, $k = 2$, $\epsilon = 0.4$

4. Conclusions

The presence of catheter affects concentration in the lumen region. As eccentricity increases, concentration decreases. As absorption and reaction coefficient at the wall increases, the concentration diffuses towards the wall. The present study can be improved by considering flow in the lumen region and variable diffusivity.

Appendix

$$\psi_i = -0.5772 + \sum_{i=1}^{k-1} \frac{1}{i} + log \frac{\alpha}{2},$$

$$\beta(i, j, k) = \frac{(-1)^i \alpha^{2i} c^{2i}}{2^{2i} (i!)^2} \beta_0 (i, j, k),$$

$$\beta_0 (i, j, k) = \frac{(i + k - 1)!}{k! (i - 1)!} \frac{(i + k + j - 1)!}{(k + j)! (i - 1)!},$$

$$A_1 = -\frac{i}{2} \left[\frac{c_T F_{12}}{F_{11} F_{22} - F_{12} F_{21}} \right],$$

$$A_2 = \frac{i}{2} \left[\frac{c_T F_{11}}{F_{11} F_{22} - F_{12} F_{21}} \right],$$

$$A_3 = \frac{i}{2} \left[\frac{G_{22} - c_T G_{12}}{G_{22} G_{11} - G_{12} G_{21}} \right],$$

$$A_4 = \frac{i}{2} \left[\frac{G_{21} - c_T G_{11}}{G_{12} G_{21} - G_{22} G_{11}} \right],$$

$$G_{11} = \sum \phi_1 (i, k, 0, l, m)\, 2 (k + j)\, \rho_1^{2(k+j)-1},$$

$$G_{12} = \sum \phi_2 (i, k, 0, l, m)\, (k + j)\, \rho_2^{2(k+j)-1},$$

$$G_{21} = \sum \phi_1 (i, k, 0, l, m)\, 2 (k + j)\, \rho_2^{2(k+j)-1} + i \sum \phi_1 (i, k, 0, l, m)\, \rho_2^{2(k+j)},$$

$$G_{22} = \sum \phi_2 (i, k, 0, l, m)\, 2 (k + j)\, \rho_2^{2(k+j)-1} + i \sum \phi_2 (i, k, 0, l, m)\, \rho_2^{2(k+j)},$$

$$\phi_1(i, j, k, l, m) = \beta(i, j, k)\left[\frac{[2(l-i)+j-1]!}{j!\,(2l-2i-1)!} + \rho^{-2j}\right] + \beta(i, m, k)\frac{[2(l-i)+j+m-1]!}{j!\,(2l-2i-1)!} + \beta(i, j+m, k)\frac{[2(l-i)+j-1]!}{j!\,(2l-2i-1)!},$$

$$\phi_2(i, j, k, l, m) = \beta(i, j, k)\left[\rho^{-2j} + \frac{[2(l-i)+j-2]!}{j!\,(2l-2i-2)!}\right] + \beta(i, m, k)\frac{[2(l-i)+j+m-2]!}{j!\,(2l-2i-2)!} + \beta(i, j+m, k)\frac{[2(l-i)+j-2]!}{j!\,(2l-2i-2)!},$$

$$F_{11} = \sum \beta(i, k, 0)\,2k\rho_1^{2k-1},$$

$$F_{21} = \sum \beta(i, k, 0)\,2k\rho_2^{2k-1} + i\sum \beta(i, k, 0)\,\rho_2^{2k},$$

$$F_{12} = \sum \beta(i, k, 0)\,2k\psi_i\rho_1^{2k-1} - 2\sum \beta(i, k, j)\frac{2(k+j)}{j}\rho_1^{2(k+j)} + \sum \beta(i, k, 0)\frac{2(k+j)}{m+j}\rho_1^{2(m+j)-1},$$

$$F_{22} = \sum \beta(i, k, 0)\,2k\psi_i\rho_2^{2k-1} + \sum \beta(i, k, 0)\frac{2(k+j)}{m+j}\rho_2^{2(k+j)-1} + \sum \beta(i, k, 0)\frac{\rho_2^{2k}}{m} - 2\sum \beta(i, k, j)\frac{2(k+j)}{j}\rho_2^{2(k+j)-1} + i\sum \beta(i, k, 0)\,\rho_2^{2k} - \sum \beta(i, k, j)\frac{\rho_2^{2(k+j)}}{j}.$$

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