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On a solution to the mean-variance portfolio selection via the mean-variance hedging in discrete time

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Abstract

In this paper, the multi-period mean-variance portfolio selection problem in discrete time is solved explicitly under the assumption which is called the deterministic mean-variance trade-off condition. We see that the optimal solution can be derived elementally through the dynamic programming, the optimal solution of the mean-variance hedging problem in discrete time and the ordinary Lagrange multiplier method.
Key words: Mean-variance portfolio selection, Dynamic programming, Mean--variance hedging, Lagrange multiplier method, Discrete time.
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1. Introduction

In this paper, the mean-variance portfolio selection problem in discrete time is solved explicitly under a certain assumption which simplifies the market model to some extent. Mean-variance portfolio selection is a problem of the allocation of wealth among various securities so as to attain the optimal trade-off between the expected return of the portfolio and its risk measured by the variance of the portfolio. This problem was first proposed and solved in the single-period setting by Markowitz [5]. Markowitz formulated the problem of minimizing a portfolio's variance subject to the constraint that its expected return equals a constant level. This analysis has long been recognized as the basis of modern portfolio theory.

Being widely used in both academia and industry, this mean-variance paradigm has also inspired the development of the multi-period mean-variance portfolio selections. As examples of studies in discrete-time multi-period mean-variance portfolio selections, we have Hakansson [2], Pliska [7] and Li and Ng [3]. Hakansson [2] has investigated relations between the optimal growth portfolio (i.e., the portfolio which is chosen so as to maximize its expected logarithmic utility) and the mean-variance efficiency. In the

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textbook Pliska [7], a multi-period mean-variance portfolio selection in a finite model (i.e., a security model where the probability space is a finite set) is treated. The most related article to our study is Li and Ng [3] and they have solved a mean-variance portfolio selection in a general discrete-time model where the growth rates of the security prices at each period are assumed to be independent random variables. They used the framework of multi-objective optimization and introduced an embedding technique which embeds the original problem in quadratic utility optimization problems so that dynamic programming can be used to obtain explicit solutions.

In the present paper, we solve a multi-period mean-variance portfolio selection in a discrete-time model. Li and Ng [3] referred above is the most related study to ours and our model is a little more general in the sense that the growth rates of the security price in our model are not independent random variables but only assumed that they satisfy the deterministic mean-variance trade-off condition explained below (see Remark 3.3). In our solution, we employ the ordinary Lagrange multiplier method and see that the problem of minimizing the Lagrangian with respect to the investment strategies can be regarded as a simple mean-variance hedging. Then we can construct the explicit solution to the mean-variance hedging by applying the result given by Gugushvili [1], which investigated the mean-variance hedging in a general discrete time model by dynamic programming. Although the assumption on the asset price process is milder than Li and Ng [3] as has been mentioned above, we consider a single risky asset model in our study in order to adapt the model to Gugushvili [1].

The rest of the paper is organized as follows. In Section 2, results of the mean-variance hedging problem in discrete time are recalled. The mean-variance portfolio selection is solved explicitly in Section 3. Section 4 concludes the paper.

2. Mean-variance hedging in discrete time

In this section, results of the mean-variance hedging problem in discrete time are recalled.

Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t=0,1,2,\ldots,T})$ be a filtered probability space where the filtration is assumed to satisfy $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$.

We set a discrete time market model with a finite terminal time $T > 0$. In the market, there is a risk-free asset. We assume that the growth rate of this risk-free asset is permanently zero. There is also a risky asset whose (discounted) price at time t is denoted by $S_t, t = 0, 1, 2, \ldots, T$ where $S_0 > 0$ is a constant and each S_t is $\mathcal{F}_t$-measurable. We use a notation $\Delta S_t := S_t - S_{t-1}, t = 1, 2, \ldots, T$.

We summarize results of the mean-variance hedging problem in discrete time. A mean-variance hedging is the problem to determine a value of a financial option by minimizing the expected value of the quadratic hedging error. Let an $\mathcal{F}_T$-measurable random variable H be the payoff of a financial option of which we want to know the price at initial time $t = 0$. A constant $c > 0$ denotes the agent's initial wealth. The whole set of his strategy is defined by

$$\Theta := \left\{ z = (z_t)_{t=0,1,2,\ldots,T-1} \middle| z \text{ is an adapted process} \right.$$
$$\left. \text{such that } \sum_{s=0}^{T-1} z_s \Delta S_{s+1} \text{ is square integrable} \right\}.$$

The terminal value of the agent's self-financed portfolio corresponding to a strategy z can be written as $c + \sum_{s=0}^{T-1} z_s \Delta S_{s+1}$. Then the mean-variance hedging problem is defined by

$$\inf_{z \in \Theta} E\left[\left(c + \sum_{s=0}^{T-1} z_s \Delta S_{s+1} - H \right)^2 \right]. \tag{1}$$

This problem is solved in Gugushvili [1] by the dynamic programming in discrete time and the optimal solution and the optimal strategy are given as follows.

Theorem 2.1 (Theorem 1 in Gugushvili [1]) *Define the value function $v(t,x)$ corresponding to the mean-variance hedging problem (1) as*

$$\begin{aligned} v(t,x) &= \inf_{z \in \Theta} E\left[\left(x + \sum_{s=t}^{T-1} z_s \Delta S_{s+1} - H \right)^2 \middle| \mathcal{F}_t \right], \\ v(T,x) &= (x-H)^2. \end{aligned}$$

Then $v(t,x)$ is a square trinomial in x,

$$v(t,x) = a_t x^2 + 2 b_t x + c_t$$

where a, b and c are adapted processes which are determined recurrently by

$$\begin{aligned} a_t &= E[a_{t+1}|\mathcal{F}_t] - \frac{(E[a_{t+1}\Delta S_{t+1}|\mathcal{F}_t])^2}{E[a_{t+1}(\Delta S_{t+1})^2|\mathcal{F}_t]}, \\ b_t &= E[b_{t+1}|\mathcal{F}_t] - \frac{E[a_{t+1}\Delta S_{t+1}|\mathcal{F}_t]E[b_{t+1}\Delta S_{t+1}|\mathcal{F}_t]}{E[a_{t+1}(\Delta S_{t+1})^2|\mathcal{F}_t]}, \\ c_t &= E[c_{t+1}|\mathcal{F}_t] - \frac{(E[b_{t+1}\Delta S_{t+1}|\mathcal{F}_t])^2}{E[a_{t+1}(\Delta S_{t+1})^2|\mathcal{F}_t]}, \\ a_T &= 1, \;\; b_T = -H, \;\; c_T = H. \end{aligned} \tag{2}$$

Moreover, the optimal strategy $\tilde{z}$ is given by

$$\tilde{z}_t = -\frac{E[b_{t+1}\Delta S_{t+1}|\mathcal{F}_t]}{E[a_{t+1}(\Delta S_{t+1})^2|\mathcal{F}_t]} - \left(c + \sum_{s=0}^{t-1} \tilde{z}_s \Delta S_{s+1} \right) \frac{E[a_{t+1}\Delta S_{t+1}|\mathcal{F}_t]}{E[a_{t+1}(\Delta S_{t+1})^2|\mathcal{F}_t]}. \tag{3}$$

Proof. See Theorem 1 in Gugushvili [1]. ■

3. Mean-variance portfolio selection in discrete time

In this section, the mean-variance portfolio selection problem is solved explicitly under a certain assumption.
A mean-variance portfolio selection seeks a strategy which minimizes the variance of the terminal value of a portfolio while keeping the expectation of the terminal value of the portfolio at a constant level. Imposing an assumption which simplifies the model on the growth rate of S, we derive an explicit solution to the mean-variance portfolio selection problem. In particular, the form of the optimal strategy is fully explicit and not even recurrent.
We consider the following problem:

$$\begin{aligned} &\inf_{z\in\Theta} Var\left(c+\sum_{s=0}^{T-1} z_s\Delta S_{s+1}\right), \\ &\text{subject to } \; E\left[c+\sum_{s=0}^{T-1} z_s\Delta S_{s+1}\right] \geq A \end{aligned} \tag{4}$$

where Var means the variance of random variables and A is a sufficiently large constant such that $A > c$. Here, we examine a specific case. Put

$$R_{t+1} := \frac{S_{t+1}-S_t}{S_t} \text{ and } M_t := \frac{(E[R_{t+1}|\mathcal{F}_t])^2}{E[R_{t+1}^2|\mathcal{F}_t]}, \;\; 0 \leq t \leq T-1.$$

We assume here that M_t are deterministic for all t. In this case, we obtain the next result.

Theorem 3.1 *Put $R_{t+1} := \Delta S_{t+1}/S_t$ and $M_t := (E[R_{t+1}|\mathcal{F}_t])^2/E[R_{t+1}^2|\mathcal{F}_t]$, $0 \leq t \leq T-1$. If M_t are deterministic, then the optimal strategy z^* to the problem (4) is given by*

$$z_t^* S_t = \left\{ \frac{A - c\prod_{s=0}^{T-1}(1-M_s)}{1-\prod_{s=0}^{T-1}(1-M_s)} - \left(c + \sum_{s=0}^{t-1} z_s^*\Delta S_{s+1}\right)\right\} \frac{E[R_{t+1}|\mathcal{F}_t]}{E[R_{t+1}^2|\mathcal{F}_t]} \tag{5}$$

or

$$z_t^* S_t = \frac{A-c}{1-\prod_{s=0}^{T-1}(1-M_s)} \prod_{s=0}^{t-1}\left(1 - \frac{R_{s+1}E[R_{s+1}|\mathcal{F}_s]}{E[R_{s+1}^2|\mathcal{F}_s]}\right) \frac{E[R_{t+1}|\mathcal{F}_t]}{E[R_{t+1}^2|\mathcal{F}_t]} \tag{6}$$

and the optimal solution is obtained by

$$Var\left(c+\sum_{s=0}^{T-1} z_s^*\Delta S_{s+1}\right) = \frac{(A-c)^2\prod_{s=0}^{T-1}(1-M_s)}{1-\prod_{s=0}^{T-1}(1-M_s)}. \tag{7}$$

We begin with solving the following problem:

$$\inf_{z\in\Theta} Var\left(c+\sum_{s=0}^{T-1} z_s\Delta S_{s+1}\right),$$
$$\text{subject to } E\left[c+\sum_{s=0}^{T-1} z_s\Delta S_{s+1}\right]=B \tag{8}$$

where B is a constant such that $B \geq A$. It is obvious that the solution of the original problem (4) can be obtained by minimizing the solution of the above problem (8) in terms of B. The Lagrangian corresponding to this problem (8) is obtained by

$$\mathcal{L}(z,\lambda)=E\left[\left(c+\sum_{s=0}^{T-1} z_s\Delta S_{s+1}-B\right)^2+\lambda\left(B-E\left[c+\sum_{s=0}^{T-1} z_s\Delta S_{s+1}\right]\right)\right]$$

where $\lambda \in \mathbb{R}$ is a Lagrange multiplier. By Theorem 2 in Section 8.4 of Luenberger [4], the optimal solution $z^B \in \Theta$ and $\lambda^B \in \mathbb{R}$ to the problem (8) is given by a saddle point of $\mathcal{L}$, i.e., z^B and λ^B which satisfy

$$\mathcal{L}(z^B,\lambda) \leq \mathcal{L}(z^B,\lambda^B) \leq \mathcal{L}(z,\lambda^B)$$

for all $z \in \Theta$ and $\lambda \in \mathbb{R}$. We start with minimizing $\mathcal{L}(z,\lambda)$ in terms of z for given $\lambda \in \mathbb{R}$.
As the example described in Section 3.3 of Øksendal and Sulem [6], since

$$\begin{aligned}\mathcal{L}(z,\lambda)&=E\left[\left(c+\sum_{s=0}^{T-1} z_s\Delta S_{s+1}-B\right)^2+\lambda\left(B-E\left[c+\sum_{s=0}^{T-1} z_s\Delta S_{s+1}\right]\right)\right]\\&=E\left[\left(c+\sum_{s=0}^{T-1} z_s\Delta S_{s+1}-\left(B+\frac{1}{2}\lambda\right)\right)^2\right]-\frac{1}{4}\lambda^2,\end{aligned}$$

we should solve the following problem in advance:

$$\inf_{z\in\Theta} E\left[\left(c+\sum_{s=0}^{T-1} z_s\Delta S_{s+1}-\left(B+\frac{1}{2}\lambda\right)\right)^2\right]. \tag{9}$$

This is a mean-variance hedging and can be solved by Theorem 2.1 above. We define the value function of this problem (9) by

$$v^{B+\frac{1}{2}\lambda}(t,x):=\operatorname*{essinf}_{z\in\Theta} E\left[\left(x+\sum_{s=t}^{T-1} z_s\Delta S_{s+1}-\left(B+\frac{1}{2}\lambda\right)\right)^2\middle|\mathcal{F}_t\right]$$

Then, by Theorem 2.1, $v^{B+\frac{1}{2}\lambda}$ can be written by

$$v^{B+\frac{1}{2}\lambda}(t,x) = a_t x^2 + 2b_t x + c_t$$

where a, b and c are adapted processes which are determined recurrently by

$$\begin{aligned} a_t &= E[a_{t+1}|\mathcal{F}_t] - \frac{(E[a_{t+1}\Delta S_{t+1}|\mathcal{F}_t])^2}{E[a_{t+1}(\Delta S_{t+1})^2|\mathcal{F}_t]}, \\ b_t &= E[b_{t+1}|\mathcal{F}_t] - \frac{E[a_{t+1}\Delta S_{t+1}|\mathcal{F}_t]E[b_{t+1}\Delta S_{t+1}|\mathcal{F}_t]}{E[a_{t+1}(\Delta S_{t+1})^2|\mathcal{F}_t]}, \\ c_t &= E[c_{t+1}|\mathcal{F}_t] - \frac{(E[b_{t+1}\Delta S_{t+1}|\mathcal{F}_t])^2}{E[a_{t+1}(\Delta S_{t+1})^2|\mathcal{F}_t]}, \\ a_T &= 1, \ \ b_T = -\left(B + \frac{1}{2}\lambda\right), \ \ c_T = \left(B + \frac{1}{2}\lambda\right)^2. \end{aligned} \tag{10}$$

Moreover, the optimal strategy z^B is determined by

$$z_t^B = -\frac{E[b_{t+1}\Delta S_{t+1}|\mathcal{F}_t]}{E[a_{t+1}(\Delta S_{t+1})^2|\mathcal{F}_t]} - \left(c + \sum_{s=0}^{t-1} z_s^B \Delta S_{s+1}\right) \frac{E[a_{t+1}\Delta S_{t+1}|\mathcal{F}_t]}{E[a_{t+1}(\Delta S_{t+1})^2|\mathcal{F}_t]}. \tag{11}$$

In the present setting, since $M_t = (E[R_{t+1}|\mathcal{F}_t])^2/E[R_{t+1}^2|\mathcal{F}_t]$, $0 \leq t \leq T-1$ are deterministic where $R_{t+1} = \Delta S_{t+1}/S_t$, (10) is, inductively, partially rewritten by

$$\begin{aligned} a_t &= a_{t+1}(1 - M_t), \ \ b_t = b_{t+1}(1 - M_t), \\ a_T &= 1, \ \ b_T = -\left(B + \frac{1}{2}\lambda\right), \end{aligned}$$

so that

$$a_t = \prod_{s=t}^{T-1}(1 - M_s), \ \ b_t = -\left(B + \frac{1}{2}\lambda\right)\prod_{s=t}^{T-1}(1 - M_s). \tag{12}$$

We can also deduce from (10) that

$$c_t = \left(B + \frac{1}{2}\lambda\right)^2 \prod_{s=t}^{T-1}(1 - M_s).$$

Then the value function becomes

$$v^{B+\frac{1}{2}\lambda}(t,x) = \left(x - \left(B + \frac{1}{2}\lambda\right)\right)^2 \prod_{s=t}^{T-1}(1 - M_s).$$

Now we can calculate λ^B. Substitute $z = z^B$ in the Lagrangian. Then the Lagrangian can be written as

$$\begin{aligned} \mathcal{L}(z^B, \lambda) &= v^{B+\frac{1}{2}\lambda}(0,c) - \frac{1}{4}\lambda^2 \\ &= \left(c - \left(B + \frac{1}{2}\lambda\right)\right)^2 \prod_{s=0}^{T-1}(1 - M_s) - \frac{1}{4}\lambda^2. \end{aligned}$$

Therefore $\lambda^B \in \mathbb{R}$ which maximizes this $\mathcal{L}(z^B, \lambda)$ is given by

$$\lambda^B = \frac{2(B-c)\prod_{s=0}^{T-1}(1-M_s)}{1-\prod_{s=0}^{T-1}(1-M_s)}.$$

Substituting (12) and $\lambda = \lambda^B$ into (11), we get

$$\begin{aligned} z_t^B S_t &= \left\{ B + \frac{1}{2}\lambda^B - \left(c + \sum_{s=0}^{t-1} z_s^B \Delta S_{s+1} \right) \right\} \frac{E[R_{t+1}|\mathcal{F}_t]}{E[R_{t+1}^2|\mathcal{F}_t]} \\ &= \left\{ \frac{B - c\prod_{s=0}^{T-1}(1-M_s)}{1-\prod_{s=0}^{T-1}(1-M_s)} - \left(c + \sum_{s=0}^{t-1} z_s^B \Delta S_{s+1} \right) \right\} \frac{E[R_{t+1}|\mathcal{F}_t]}{E[R_{t+1}^2|\mathcal{F}_t]} \end{aligned} \tag{13}$$

This can be also written by

$$z_t^B S_t = \left(\frac{B-c}{1-\prod_{s=0}^{T-1}(1-M_s)} - \sum_{s=0}^{t-1} z_s^B \Delta S_{s+1} \right) \frac{E[R_{t+1}|\mathcal{F}_t]}{E[R_{t+1}^2|\mathcal{F}_t]}$$

or

$$z_t^B S_t R_{t+1} = \left(\frac{B-c}{1-\prod_{s=0}^{T-1}(1-M_s)} - \sum_{s=0}^{t-1} z_s^B S_s R_{s+1} \right) \frac{R_{t+1} E[R_{t+1}|\mathcal{F}_t]}{E[R_{t+1}^2|\mathcal{F}_t]}.$$

By solving this equation, we obtain

$$z_t^B S_t = \frac{B-c}{1-\prod_{s=0}^{T-1}(1-M_s)} \prod_{s=0}^{t-1} \left(1 - \frac{R_{s+1} E[R_{s+1}|\mathcal{F}_s]}{E[R_{s+1}^2|\mathcal{F}_s]} \right) \frac{E[R_{t+1}|\mathcal{F}_t]}{E[R_{t+1}^2|\mathcal{F}_t]}. \tag{14}$$

Finally, the solution of (4) is given as follows. The solution of (8) is obtained as

$$Var\left(c + \sum_{s=0}^{T-1} z_s^B \Delta S_{s+1} \right) = \mathcal{L}(z^B, \lambda^B) = \frac{(B-c)^2 \prod_{s=0}^{T-1}(1-M_s)}{1-\prod_{s=0}^{T-1}(1-M_s)} \tag{15}$$

for each $B \geq A$. Obviously, this is minimized when $B = A$. Therefore the solution of the original problem (4) can be obtained by substituting $B = A$ into (13), (14) and (15) and it yields (5), (6) and (7), respectively. This concludes the proof. ■

Remark 3.2 We can directly check that $E[c + \sum_{s=0}^{T-1} z_s^* \Delta S_{s+1}] = A$. Indeed, from (6), we have

$$\begin{aligned} E[z_t^* S_t R_{t+1}] &= \frac{A-c}{1-\prod_{s=0}^{T-1}(1-M_s)} \prod_{s=0}^{t-1}(1-M_s) M_t \\ &= \frac{A-c}{1-\prod_{s=0}^{T-1}(1-M_s)} \left(\prod_{s=0}^{t-1}(1-M_s) - \prod_{s=0}^{t}(1-M_s) \right) \end{aligned}$$

for $t \geq 1$ and

$$E[z_0^* S_0 R_1] = \frac{A-c}{1-\prod_{s=0}^{T-1}(1-M_s)} M_0 = \frac{A-c}{1-\prod_{s=0}^{T-1}(1-M_s)}(1-(1-M_0)).$$

Therefore we obtain that

$$\begin{aligned}
&E\left[\sum_{s=0}^{T-1} z_s^* \Delta S_{s+1}\right] = \sum_{s=0}^{T-1} E[z_s^* S_s R_{s+1}] \\
&= \frac{A-c}{1-\prod_{s=0}^{T-1}(1-M_s)} \left\{1-(1-M_0)+\sum_{t=1}^{T-1}\left(\prod_{s=0}^{t-1}(1-M_s)-\prod_{s=0}^{t}(1-M_s)\right)\right\} \\
&= \frac{A-c}{1-\prod_{s=0}^{T-1}(1-M_s)}\left(1-\prod_{s=0}^{T-1}(1-M_s)\right) \\
&= A-c.
\end{aligned}$$

Remark 3.3 The condition that M_t are deterministic is called the *deterministic mean-variance trade-off* condition which is provided by Schweizer [8]. It is known that when this condition is satisfied, the variance-optimal martingale measure coincides with the minimal martingale measure, which can be obtained explicitly with ease (Corollary 4.2 in Schweizer [8]). This is one reason why we can get an explicit optimal solution to the mean-variance portfolio selection. We also note that it can be easily checked that $1/\prod_{s=0}^{T-1}(1-M_s)$ is equal to the square mean of the density of the variance-optimal martingale measure (or the minimal martingale measure) in the model (see (2.21) in Schweizer [8] for the expression of the density of the minimal martingale measure).

4. Conclusion

In this paper, the mean-variance portfolio selection problem in discrete time has been investigated using the result of the mean-variance hedging problem. Imposing the assumption that the security price process satisfies the deterministic mean-variance trade-off condition, we have derived the fully explicit optimal solution. However, the assumption was somewhat strong and we have only dealt with the single risky asset model. It is left for the future to get explicit solutions to the mean-variance selection in more general discrete-time models under milder assumptions.

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