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Implicative Filters of Lattice Pseudo Wajsberg Algebras

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Abstract

In this paper, we introduce the notion of implicative filter of lattice pseudo Wajsberg algebra. Further, we define Type-1implicative filter of lattice pseudo-Wajsberg algebra. Finally, we give Type-2 implicative filter of lattice pseudo-Wajsberg algebra and we obtain some of their related properties.

Keywords: Wajsberg algebra; Pseudo-Wajsberg algebra; Implicative filter; Type-1 implicative filter; Type-2 implicative filter.

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1. INTRODUCTION

A lattice is a mathematical structure with two binary operations, '$\vee$' and '$\wedge$' whereas a partially ordered set $(A, \leq)$. Where ' $\leq$ ' represents a partial order relation on a set A. Xu [11] proposed the concept of lattice implication algebras in 1993, which combines lattice with implication algebras and discussed some of their properties in [10] and [11]. Xu and Qin [9] introduced the notion of filter in a lattice implication algebras and investigated their properties.
Mordchaj Wajsbreg introduced the concept of Wajsberg algebras in 1935 and studied by Font, Rodriguez and Torrens in [3]. They defined lattice structure of Wajsberg

algebras. Also, they introduced the notion of an implicative filter of lattice Wajsberg algebras and discussed some of their properties. Pseudo-Wajsberg algebras are generalizations of Wajsberg algebras. Pseudo-Wajsberg algebras were introduced by Rodica Ceterchi [4] with the explicit purpose of providing a concept categorically equivalent to that of pseudo-MV algebras and which will have the same relationship with Wajsberg algebras as pseudo-MV algebras.

In the present paper, we introduce the notion of implicative filters of lattice pseudo-Wajsberg algebras, and investigate some of their properties with illustrations. Also, we obtain some equivalent conditions and we get the relation between them. Further, we investigate Type-1 and Type-2 implicative filters of lattice pseudo-Wajsberg algebras, and we discuss some of their necessary and sufficient conditions.

2. PRELIMINARIES

In this section, we recall some basic definitions and their properties which are helpful to develop the main results.

Definition 2.1[2]. An algebra $(A, \rightarrow, ^{-}, 1)$ with a binary operation " $\rightarrow$ " and a quasi-complement " $^{-}$ " is called a Wajsberg algebra if it satisfies the following axioms for all $x, y, z \in A$,

(i) $1 \rightarrow x = x$

(ii) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$

(iii) $(x \rightarrow y) \rightarrow \big((y \rightarrow z) \rightarrow (x \rightarrow z)\big) = 1$

(iv) $(x^{-} \rightarrow y^{-}) \rightarrow (y \rightarrow x) = 1$.

Definition 2.2[2]. A Wajsberg algebra A is called a lattice Wajsberg algebra if it satisfies the following axioms for all $x, y \in A$,

(i) The partial ordering " $\leq$ " on a lattice Wajsberg algebra A, such that $x \leq y$ if and only if $x \rightarrow y = 1$

(ii) $(x \vee y) = (x \rightarrow y) \rightarrow y$

(iii) $(x \wedge y) = \big((x^{-} \rightarrow y^{-}) \rightarrow y^{-}\big)^{-}$.

Note. From the above definition of an algebra $(A, \vee, \wedge, ^{-}, 0, 1)$ is a lattice Wajsberg algebra with lower bound 0 and upper bound 1.

Definition 2.3[2]. Let A be Wajsberg algebra. A non-empty subset F of A is called an implicative filter of A if it satisfies the following axioms for all $x, y \in A$,

(i) $1 \in F$

(ii) $x \in F$ and $x \rightarrow y \in F$ imply $y \in F$.

Definition 2.4[4]. An algebra $(A, \to, \rightsquigarrow, ^{-}, ^{\sim}, 1)$ with a binary operations $" \to ", " \rightsquigarrow "$ and quasi complements $" ^{-} "$, $" ^{\sim} "$ is called a pseudo-Wajsberg algebra if it satisfies the following axioms for all $x, y, z \in A$,

(i) (a) $1 \to x = x$

(b) $1 \rightsquigarrow x = x$

(ii) $(x \rightsquigarrow y) \to y = (y \rightsquigarrow x) \to x = (y \to x) \rightsquigarrow x = (x \to y) \rightsquigarrow y$

(iii) (a) $(x \to y) \to \big((y \to z) \rightsquigarrow (x \to z)\big) = 1$

(b) $(x \rightsquigarrow y) \rightsquigarrow \big((y \rightsquigarrow z) \to (x \rightsquigarrow z)\big) = 1$

(iv) $1^{-} = 1^{\sim} = 0$

(v) (a) $(x^{-} \rightsquigarrow y^{-}) \to (y \to x) = 1$

(b) $(x^{\sim} \to y^{\sim}) \rightsquigarrow (y \rightsquigarrow x) = 1$

(vi) $(x \to y^{-})^{\sim} = (y \rightsquigarrow x^{\sim})^{-}$

Definition 2.5[4]. An algebra $(A, \to, \rightsquigarrow, ^{-}, ^{\sim}, 1)$ is called a lattice pseudo-Wajsberg algebras if it satisfies the following axioms for all $x, y \in A$,

(i) A partial ordering " $\leq$ " on a lattice pseudo-Wajsberg algebra *A*, such that x ≤ y if and only if $x \to y = 1 \Leftrightarrow x \rightsquigarrow y = 1$.

(ii) $x \vee y = (x \to y) \rightsquigarrow y = (y \to x) \rightsquigarrow x = (x \rightsquigarrow y) \to y = (y \to x) \to x$

(iii) $x \wedge y = (x \rightsquigarrow (x \to y)^{\sim})^{-} = \big((x \to y) \to x^{-}\big)^{\sim}$

$= (y \rightsquigarrow (y \to x)^{\sim})^{-} = \big((y \to x) \to y^{-}\big)^{\sim}$

$= (y \to (y \rightsquigarrow x)^{-})^{\sim} = \big((y \rightsquigarrow x) \rightsquigarrow y^{\sim}\big)^{-}$

$= (x \to (x \rightsquigarrow y)^{-})^{\sim} = \big((x \rightsquigarrow y) \rightsquigarrow x^{\sim}\big)^{-}$

Proposition 2.6[3]. A Wajsberg algebra $(A, \to, ^{-}, 1)$ satisfies the following axioms for all $x, y \in A$,

(i) $x \to x = 1$

(ii) If $x \to y = y \to x = 1$, then $x = y$

(iii) $x \to 1 = 1$

(iv) $x \to (y \to x) = 1$

(v) If $x \to y = y \to z = 1$, then $x \to z = 1$

(vi) $(x \to y) \to \big((z \to x) \to (z \to y)\big) = 1$

(vii) $x \to (y \to z) = y \to (x \to z)$.

Proposition 2.7[4]. A lattice pseudo-Wajsberg algebra $(A, \to, \rightsquigarrow, ^{-}, ^{\sim}, 1)$ satisfies the following axioms for all $x, y \in A$,

(i) $x \to x = 1,\ x \rightsquigarrow x = 1$

(ii) $x \to (y \rightsquigarrow x) = 1,\ x \rightsquigarrow (y \to x) = 1$

(iii) $x \leq y \Rightarrow z \to x \leq z \to y\,;\ z \rightsquigarrow x \leq z \rightsquigarrow y$

(iv) $x \leq y \Rightarrow y \to z \leq x \to z\,;\ y \rightsquigarrow z \leq x \rightsquigarrow z$

(v) $x \leq y \to x\,;\ x \leq y \rightsquigarrow x$

(vi) $x \to y \leq (y \to z) \rightsquigarrow (x \to z)\,;\ x \rightsquigarrow y \leq (y \rightsquigarrow z) \to (x \rightsquigarrow z)$

(vii) $x \to y \leq (z \to x) \to (z \to y)\,;\ x \rightsquigarrow y \leq (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y)$

(viii) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$.

3. MAIN RESULTS

3.1. Implicative filter of lattice pseudo -Wajsberg algebra

In this section, we introduce implicative filter of lattice pseudo-Wajsberg algebra, and investigate some properties with illustrations.

Definition 3.1.1. Let A be lattice pseudo-Wajsberg algebra. A non-empty subset F of A is called an implicative filter of A if it satisfies the following axioms for all $x, y \in A$,

(i) $1 \in F$

(ii) $x \in F$ and $x \to y \in F$ imply $y \in F$

(iii) $x \in F$ and $x \rightsquigarrow y \in F$ imply $y \in F$.

Note: In this paper, we denote $\mathcal{F}(A)$ as set of all implicative filters, in a lattice pseduo-Wajsberg algebra.

Example 3.1.2. Consider a set $A = \{0, a, b, c, 1\}$. Define a partial ordering "≤" on A, such that $0 < a < b,\ c < 1$ and the binary operations " $\to$ " , " $\rightsquigarrow$ " and quasi complements " $^-$ " , " $^\sim$ " given by the following tables (1), (2), (3) and (4).

x	x^-
0	1
a	c
b	c
c	0
1	0

Table (1)

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	c	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

Table (2)

x	$x^\sim$
0	1
a	b
b	0
c	b
1	0

Table (3)

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	0	c	1	c	1
c	b	b	b	1	1
1	0	a	b	c	1

Table (4)

Now consider the set $F_1 = \{a, b, c, 1\}$. It can easily verified that F_1 is an implicative filter of A. But $F_2 = \{0, 1, a, b\}$ is not an implicative filter of A. Since, $(b \to a) = c \notin F_2$.

Remark 3.1.3. Let F be any implicative filter of lattice pseudo-Wajsberg algebra of A satisfying the axiom $x \rightsquigarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \rightsquigarrow z)$ for all $x, y \in A$.

Proposition 3.1.4. Let F be any implicative filter of lattice pseudo-Wajsberg algebra of A satisfying the following axioms for all $x, y \in A$,
(i) $(x \to 1) \to (y \to 1) = (x \to y) \to 1$ and
(ii) $(x \rightsquigarrow 1) \rightsquigarrow (y \rightsquigarrow 1) = (x \rightsquigarrow y) \rightsquigarrow 1$.

Proof. **(i)** From (i) and (vii) of proposition 2.6, we have

$$\begin{aligned}(x \to 1) \to (y \to 1) &= (x \to 1) \to \big(y \to ((x \to y) \to (x \to y))\big)\\ &= (x \to 1) \to \big((x \to y) \to (y \to (x \to y))\big)\\ &= (x \to 1) \to \big((x \to y) \to (x \to (y \to y))\big)\\ &= (x \to 1) \to \big((x \to y) \to (x \to 1)\big)\\ &= (x \to y) \to \big((x \to 1) \to (x \to 1)\big)\\ &= (x \to y) \to 1.\end{aligned}$$

(ii) From (i) of proposition 2.7 and remark 3.1.3, we have

$$\begin{aligned}(x \rightsquigarrow 1) \rightsquigarrow (y \rightsquigarrow 1) &= (x \rightsquigarrow 1) \rightsquigarrow \big(y \rightsquigarrow ((x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow y))\big)\\ &= (x \rightsquigarrow 1) \rightsquigarrow \big((x \rightsquigarrow y) \rightsquigarrow (y \rightsquigarrow (x \rightsquigarrow y))\big)\\ &= (x \rightsquigarrow 1) \rightsquigarrow \big((x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow (y \rightsquigarrow y))\big)\\ &= (x \rightsquigarrow 1) \rightsquigarrow \big((x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow 1)\big)\\ &= (x \rightsquigarrow y) \rightsquigarrow \big((x \rightsquigarrow 1) \rightsquigarrow (x \rightsquigarrow 1)\big)\\ &= (x \rightsquigarrow y) \rightsquigarrow 1.\end{aligned}$$

Proposition 3.1.5. Let F be any implicative filter of lattice pseudo-Wajsberg algebra of A satisfying the following axioms for all $x, y \in A$,
(i) $x \to ((x \to y) \to y) = 1$
(ii) $x \rightsquigarrow ((x \rightsquigarrow y) \rightsquigarrow y) = 1$.

Proof. **(i)** From (vii) and (i) of proposition 2.6, we have
$x \to ((x \to y) \to y) = (x \to y) \to (x \to y) = 1$
(ii) From the remark 3.1.3 and (i) of proposition 2.7, we have
$x \rightsquigarrow ((x \rightsquigarrow y) \rightsquigarrow y) = (x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow y) = 1$.

Proposition 3.1.6. Let F be any implicative filter of lattice pseudo-Wajsberg algebra of A satisfying the following axioms for all $x, y \in A$,
(i) $(x \to y) \to x \in F$ implies $x \in F$
(ii) $(x \rightsquigarrow y) \rightsquigarrow x \in F$ implies $x \in F$.

Proof. **(i)** Let F be any implicative filter of lattice pseudo-Wajsberg algebra of A and $(x \to y) \to x \in F$. From (i) (a) of definition 2.4, we have

$$1 \to ((x \to y) \to x) = (x \to y) \to x \in F.$$

Since $1 \in F$ and F is an implicative filter. Therefore, $x \in F$.

(ii) Similarly, we show that $(x \rightsquigarrow y) \rightsquigarrow x \in F$ implies $x \in F$.

Proposition 3.1.7. Every implicative filter F of A has the following property $x \leq y$ and $x \in F$ imply $y \in F$.

Proof. Let $x \in F$ and $x \leq y$

From (i) of definition 2.5, we have $x \to y = 1$ and $x \rightsquigarrow y = 1$

Since $1 \to x \in F,\ x \to y \in F$ and $1 \rightsquigarrow x \in F,\ x \rightsquigarrow y \in F$

From (ii) and (iii) of definition 3.1.1, we have $y \in F$.

Proposition 3.1.8. A non-empty subset F of A is an implicative filter of lattice pseudo-Wajsberg algebra if and only if it satisfies for all $x, y, z \in A$,

(i) $1 \in F$

(ii) $z \to ((x \to y) \to x) \in F$ and $z \in F$ imply $x \in F$

(iii) $z \rightsquigarrow ((x \rightsquigarrow y) \rightsquigarrow x) \in F$ and $z \in F$ imply $x \in F$.

Proof. If F satisfies (i), (ii) and (iii).

Let $x, z \in A$ be such that $z \to x \in F$ and $z \in F$

From (ii), we take $y = x$ then $z \to ((x \to x) \to x) = z \to (1 \to x) = z \to x \in F$

From (ii), we have $x \in F$

Similarly, we show that (iii)

Therefore, F is an implicative filter of A

Conversely, let F be an implicative filter of A

From $x, y, z \in A$, we have $z \to ((x \to y) \to x) \in F$ and $z \in F$

Then by proposition 3.1.7, we have $(x \to y) \to x \in F$

From (i) of proposition 3.1.6, we have $x \in F$

Similarly, we show that (iii).

Proposition 3.1.9. Let F be any implicative filter of lattice pseudo-Wajsberg algebra of A satisfying the following axioms for all $x, y, z \in A$,

(i) $x \to (y \to z) \in F$ and $x \to y \in F$ imply $x \to z \in F$

(ii) $x \rightsquigarrow (y \rightsquigarrow z) \in F$ and $x \rightsquigarrow y \in F$ imply $x \rightsquigarrow z \in F$.

***Proof.*(i)** Let $x, y, z \in A$ be such that $x \to (y \to z) \in F$ and $x \to y \in F$

From (vii) of proposition 2.6, and from (vii) of proposition 2.7,

$$x \to (y \to z) = y \to (x \to z) \leq (x \to y) \to (x \to (x \to z))$$

Consider, $((x \to z) \to z) \to (x \to z) = x \to (((x \to z) \to z) \to z)$

[from (vii) of proposition 2.6]

$$= x \to (x \to z) \in F$$

It follows that, $1 \to (((x \to z) \to z) \to (x \to z)) \in F$

From (i) (a) of definition 2.4, from (ii) of definition 3.1.1, $x \to z \in F$.

(ii) For any $x, y, z \in A$, such that $x \rightsquigarrow (y \rightsquigarrow z) \in F$ and $x \rightsquigarrow y \in F$

From remark 3.1.3, and from (vii) of proposition (2.7), we have
$x \rightsquigarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \rightsquigarrow z) \leq (x \rightsquigarrow y) \rightsquigarrow \big(x \rightsquigarrow (x \rightsquigarrow z)\big)$,
Consider $\big((x \rightsquigarrow z) \rightsquigarrow z\big) \rightsquigarrow (x \rightsquigarrow z) = x \rightsquigarrow \Big(\big((x \rightsquigarrow z) \rightsquigarrow z\big) \rightsquigarrow z\Big)$
[From remark 3.1.3]
$= x \rightsquigarrow (x \rightsquigarrow z) \in F$
It follows that, $1 \rightsquigarrow \Big(\big((x \rightsquigarrow z) \rightsquigarrow z\big) \rightsquigarrow (x \rightsquigarrow z)\Big) \in F$
From (i) (b) of definition 2.4, and from (iii) of definition 3.1.1, we have $x \rightsquigarrow z \in F$.

Proposition 3.1.10. Let *A* be a lattice pseudo-Wajsberg algebra and *F* be a non-empty subset of A. Then F is an implicative filter of A if and only if F satisfies for all $x, y, z \in F$,
(i) $x \to (y \to z) = 1$ implies $z \in F$
(ii) $x \rightsquigarrow (y \rightsquigarrow z) = 1$ implies $z \in F$.

Proof. **(i) and (ii)** Let$x \in F$. Since,$x \to (x \to 1) = 1$, it follows that 1$\in F$
From (i) of definition 3.1.1 holds for *F*.
Let $x \to z \in F$ and $x \in F$. Because $x \to \big((x \to z) \to z\big) = 1$, and so $z \in F$
From (ii) of definition 3.1.1 is true.
Similarly, from (iii) of definition 3.1.1 is true. Therefore, *F* is an implicative filter of *A*
Conversely, Let *F* be an implicative filter of *A*
Let $x, y \in F$ and $z \in A$, such that $x \to (y \to z) = 1$ and $x \rightsquigarrow (y \rightsquigarrow z) = 1$
From (i) of definition 3.1.1, we have $x \to (y \to z) \in F$ and $x \rightsquigarrow (y \rightsquigarrow z) \in F$
From (ii), (iii) of definition 3.1.1 twice, we have $z \in F$.

Proposition 3.1.11. Let *F* be any implicative filter of lattice pseudo-Wajsberg algebra of *A.* Then if it satisfies the following axioms for all $x, y, z \in A$.
(i) $x \to \big(y \to (y \to z)\big) \in F$ and $x \in F$ imply $y \to z \in F$
(ii) $x \rightsquigarrow \big(y \rightsquigarrow (y \rightsquigarrow z)\big) \in F$ and $x \in F$ imply $y \rightsquigarrow z \in F$. Then, $F \in \mathcal{F}(A)$.

Proof. **(i)** Let $x \to (y \to z) \in F$ and $x \to y \in F$ for all $x, y, z \in A$
From (vii) of proposition 2.6 and (vii) of proposition 2.7, we have

$$x \to (y \to z) = y \to (x \to z) \leq (x \to y) \to \big(x \to (x \to z)\big)$$

From proposition 3.1.7, $(x \to y) \to \big(x \to (x \to z)\big) \in F$
Since $x \to y \in F$, we have $x \to z \in F$
Therefore, $F \in \mathcal{F}(A)$.

(ii) Let $x \rightsquigarrow (y \rightsquigarrow z) \in F$ and $x \rightsquigarrow y \in F$ for all $x, y, z \in A$
From remark 3.1.3 and (vii) of proposition 2.7,
we have $x \rightsquigarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \rightsquigarrow z) \leq (x \rightsquigarrow y) \rightsquigarrow \big(x \rightsquigarrow (x \rightsquigarrow z)\big)$
From proposition 3.1.7, $(x \rightsquigarrow y) \rightsquigarrow \big(x \rightsquigarrow (x \rightsquigarrow z)\big) \in F$
Since $x \rightsquigarrow y \in F$, we have $x \rightsquigarrow z \in F$. Therefore, $F \in \mathcal{F}(A)$.

Proposition 3.1.12. Let $F \in \mathcal{F}(A)$. Then for all $x, y \in A$ the following axioms are equivalent.

(i) $F \in \mathcal{F}(A)$

(ii) (a) $x \to (x \to y) \in F$ implies $x \to y \in F$

(b) $x \to (x \rightsquigarrow y) \in F$ implies $x \rightsquigarrow y \in F$

(iii) (a) $x \to (y \to z) \in F$ implies $(x \to y) \to (x \to z) \in F$

(b) $x \rightsquigarrow (y \rightsquigarrow z) \in F$ implies $(x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z) \in F$.

Proof.**(i) ⇒ (ii)(a)**

Let $F \in \mathcal{F}(A)$ and let $x \to (x \to y) \in F$.

Since, $x \to x = 1 \in F$, from (i) of proposition 3.1.6, we have $x \to y \in F$.

Similarly, we show that (ii) (b).

(ii)(a) ⇒ (iii)(a)

If (ii) (a) holds and let $x \to (y \to z) \in F$, from (vii) of proposition 2.7, we have $x \to (y \to z) \leq x \to ((x \to y) \to (x \to z))$

From proposition 3.1.7 and from (vii) of proposition 2.6, we have $x \to \Big(x \to \big((x \to y) \to z\big)\Big) = \Big(x \to \big((x \to y) \to (x \to z)\big)\Big) \in F$

From (ii) (a), we have $x \to \big((x \to y) \to z\big) = (x \to y) \to (x \to z) \in F$

Similarly, we show that (iii) (b)

(iii) $(\boldsymbol{a})$ ⇒ (i)

If (iii) holds and let $x \to (y \to z) \in F$ and $x \to y \in F$.

From (iii), we have $(x \to y) \to (x \to z) \in F$ and $x \to y \in F$

It follows from (ii) of definition 3.1.1, $x \to z \in F$. Thus $F \in \mathcal{F}(A)$

Similarly, we show that (iii) (b) ⇒(i).

Proposition 3.1.13. If *F* be any implicative filter of lattice pseudo-Wajsberg algebra of *A*. Then $F \in \mathcal{F}(A)$ if and only if it satisfies $x \leq y \to z$ and $x \leq y \rightsquigarrow z$ implies $z \in F$ for all $x, y, z \in A$.

Proof. Let *F* be a non-empty subset of *A* and $F \in \mathcal{F}(A)$

From proposition 3.1.7, and from (ii), (iii) of definition 3.1.1, then $z \in F$

Conversely**,** if *F* satisfies $x \leq y \to z$ and $x \leq y \rightsquigarrow z$ implies $z \in F$

Since, $x \leq x \to 1$ and $x \leq x \rightsquigarrow 1$ for all $x \in F$, we have $1 \in F$

Let $x \to y \in F$ and $x \rightsquigarrow y \in F$ and $x \in F$

From proposition 3.1.7, and $x \leq y \to z$ and $x \leq y \rightsquigarrow z$ implies $z \in F$, then $y \in F$,

Hence $F \in \mathcal{F}(A)$.

Proposition 3.1.14. Let *A* be bounded and let *F* be any implicative filter of lattice pseudo-Wajsberg algebra of *A*. Then, for all $x, y \in A$ the following axioms are equivalent

(i) If $x \in F, x \to y \in F$ and $x \rightsquigarrow y \in F$ imply $y \in F$

(ii) If $(x \to y) \rightsquigarrow x \in F$, then $x \in F$ and if $(x \rightsquigarrow y) \to x \in F$, then $x \in F$

(iii) $\big((x \rightsquigarrow y) \to x\big) \to x \in F$ and $\big((x \to y) \rightsquigarrow x\big) \rightsquigarrow x \in F$

(iv) $(x^{-} \rightsquigarrow x) \rightsquigarrow x \in F$ and $(x^{\sim} \to x) \to x \in F$.

Proof.**(i) ⇒ (ii)**
If (i) holds, let $(x \to y) \rightsquigarrow x \in F,$ from (i) (a) of definition 2.4,
we have $1 \to \big((x \to y) \rightsquigarrow x\big)$=$(x \to y) \rightsquigarrow x \in F$
Since 1∈ F and F is an implicative filter. Hence, from (i), we have $x \in F$
Similarly to prove, if $(x \rightsquigarrow y) \to x \in F,$ then $x \in F.$
(ii) ⇒ (i)
If (ii) holds, let $x \to \big((y \to z) \rightsquigarrow y\big) \in F$ and $x \in F$
From (ii) of definition 3.1.1, we have $\big((y \to z) \rightsquigarrow y\big) \in F$
Hence, from (ii), we have $y \in F$
Similarly to prove $x \in F,\ x \rightsquigarrow y \in F$ imply $y \in F.$
(iii) ⇒ (iv)
If A is bounded, let y = 0 in (iii), then $\big((x \rightsquigarrow 0) \to x\big) \to x \in F$
Hence, $(x^{\sim} \to x) \to x \in F$ for all $x \in F$
Similarly to prove $(x^{-} \rightsquigarrow x) \rightsquigarrow x \in F$.
(iv) ⇒(iii)
Since A is bounded for any $y \in A,\ 0 \leq y$
For any $x \in A,\ x \to 0 \leq x \to y$ and so $x^{-} \leq x \to y$ [From (iii) of proposition 2.7]
$(x \to y) \rightsquigarrow x \leq x^{-} \rightsquigarrow x$ [From (iv) of proposition 2.7]
Then, we have $(x^{-} \rightsquigarrow x) \rightsquigarrow x \leq ((x \to y) \rightsquigarrow x) \rightsquigarrow x.$ [From (iv) of proposition 2.7]
Since F is an implicative filter and let $(x^{-} \rightsquigarrow x) \rightsquigarrow x \in F$
we have $\big((x \to y) \rightsquigarrow x\big) \rightsquigarrow x \in F$ for all $x, y \in A$ [From proposition 3.1.7]
Similarly, we prove the other cases.
(iii) ⇒ (i)
Let $x \to \big((y \to z) \rightsquigarrow y\big) \in F$ and $x \in F$
From (ii) of definition 3.1.1, $\big((y \to z) \rightsquigarrow y\big) \in F$ and from (iii),
we have $\big((y \to z) \rightsquigarrow y\big) \rightsquigarrow y \in F$
From (iii) of definition 3.1.1, we have $y \in F$
Similarly, we prove the other cases.

3.2. Type-1 Implicative filters of lattice pseudo-Wajsberg algebra

In this section, we define Type-1 implicative filter of lattice pseudo-Wajsberg algebra and obtain some useful results with illustrations.

Definition 3.2.1. A pseudo-Wajsberg algebra $(A, \to, \rightsquigarrow, {}^{-}, {}^{\sim}, 1)$ is called a Type-1 implicative filter of lattice pseudo-Wajsberg algebra if it satisfies the following axioms for all $x, y \in A,$
(i) $1 \in F$
(ii) $x \in F$ and $x \to y \in F$ imply $y \in F$
(iii) $x \in F$ and $x \rightsquigarrow y \in F$ imply $y \in F$
(iv) $(x \to y) \to x = x = (x \rightsquigarrow y) \rightsquigarrow x.$

Example 3.2.2. Consider a set $A = \{0, a, b, c, 1\}$. Define a partial ordering "≤" on A, such that $0 < a < b,\ c < 1$ and the binary operations " → " , " ↝ " and quasi complements " ⁻ " , " ˜ " given by the following tables (5), (6), (7) and (8).

x	x^-
0	1
a	c
b	b
c	b
1	0

Table (5)

→	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	c	1	1	1
c	b	c	c	1	1
1	0	a	b	c	1

Table (6)

x	$x^\sim$
0	1
a	c
b	c
c	a
1	0

Table (7)

↝	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	c	c	1	1	1
c	a	c	c	1	1
1	0	a	b	c	1

Table (8)

Now consider the set $F_1 = \{c, 1\}$. It can easily verified that F_1 is a Type-1 implicative filter of A. But $F_2 = \{0, b, 1\}$ is not a Type-1 implicative filter of A. Since, $(b \rightsquigarrow 0) \rightsquigarrow b = c \rightsquigarrow b,\ c \notin F_2$.

Proposition 3.2.3. Let $(A, \rightarrow, \rightsquigarrow, {}^-, {}^\sim, 1)$ be a Type-1 implicative filter of lattice pseudo-Wajsberg algebra. Then if it satisfies the following axioms for all $x, y \in A$,

(i) $x \rightarrow (x \rightarrow y) = x \rightarrow y$

(ii) $x \rightsquigarrow (x \rightsquigarrow y) = x \rightsquigarrow y$

(iii) $x \rightarrow y = \big((y \rightarrow x) \rightarrow x\big) \rightarrow y$

(iv) $x \rightsquigarrow y = \big((y \rightsquigarrow x) \rightsquigarrow x\big) \rightsquigarrow y$

(v) $\big((y \rightarrow x) \rightarrow x\big) \rightarrow x \leq y \rightarrow x$

(vi) $\big((y \rightsquigarrow x) \rightsquigarrow x\big) \rightsquigarrow x \leq y \rightsquigarrow x$

(vii) $\big((y \rightsquigarrow x) \rightarrow x\big) = \Big(\big((y \rightsquigarrow x) \rightarrow x\big) \rightarrow x\Big) \rightarrow x$

(viii) $\big((y \rightarrow x) \rightsquigarrow x\big) = \Big(\big((y \rightarrow x) \rightsquigarrow x\big) \rightsquigarrow x\Big) \rightsquigarrow x$

(ix) $y \to x = \big(\big((y \rightsquigarrow x) \to x\big) \to (y \to x)\big)$

(x) $y \rightsquigarrow x = \big(\big((y \to x) \rightsquigarrow x\big) \rightsquigarrow (y \rightsquigarrow x)\big)$

Proof. **(i) and (ii)** From (iv) of definition 3.2.1,
we have $x \to (x \to y) = \big((x \to y) \to x\big) \to (x \to y)$
$= x \to y$ for all $x, y \in A$.
Similarly, to prove $x \rightsquigarrow (x \rightsquigarrow y) = x \rightsquigarrow y$ for all $x, y \in A$.
(iii) and (iv) From (iv) of definition 3.2.1 and from (vii) of proposition 2.7,
we have $x \to y \leq \big((y \to x) \to x\big) \to \big((y \to x) \to y\big) = \big((y \to x) \to x\big) \to y$
From $x \leq (y \to x) \to x$, we have $\big((y \to x) \to x\big) \to y \leq x \to y$
[From (iv) of proposition 2.7]
Therefore, $x \to y = \big((y \to x) \to x\big) \to y$ for all $x, y \in A$.
Similarly, to prove $x \rightsquigarrow y = \big((y \rightsquigarrow x) \rightsquigarrow x\big) \rightsquigarrow y$ for all $x, y \in A$.
(v) and (vi) From (iii), we have

$$1 = x \to (y \to x) = \big(\big((y \to x) \to x\big) \to x\big) \to (y \to x),$$

Hence, $\big((y \to x) \to x\big) \to x \leq y \to x$ for all $x, y \in A$. [From (i) of definition 2.5]
Similarly, to prove $\big((y \rightsquigarrow x) \rightsquigarrow x\big) \rightsquigarrow x \leq y \rightsquigarrow x$ for all $x, y \in A$.
(vii) and (viii) From (ii) of definition 2.7 and (iii),
we have $1 = x \to (y \rightsquigarrow x) = \big(\big((y \rightsquigarrow x) \to x\big) \to x\big) \to (y \rightsquigarrow x)$,
Then, $\big(\big((y \rightsquigarrow x) \to x\big) \to x\big) \leq y \rightsquigarrow x$ [From (i) of definition 2.5]
$(y \rightsquigarrow x) \to x \leq \big(\big((y \rightsquigarrow x) \to x\big) \to x\big) \to x$ [From (iv) of definition 2.7]
From (v), we have $\big((y \rightsquigarrow x) \to x\big) = \big(\big((y \rightsquigarrow x) \to x\big) \to x\big) \to x$
Therefore, $\big((y \rightsquigarrow x) \to x\big) = \big(\big((y \rightsquigarrow x) \to x\big) \to x\big) \to x$ for all $x, y \in A$.
Similarly, to prove $\big((y \to x) \rightsquigarrow x\big) = \big(\big((y \to x) \rightsquigarrow x\big) \rightsquigarrow x\big) \rightsquigarrow x$ for all $x, y \in A$.
(ix) and (x) From y $\leq (y \rightsquigarrow x) \to x$,
$y \to x = \big((y \rightsquigarrow x) \to x\big) \to (y \to x)$ for all $x, y \in A$.[From (i), and(v) proposition 2.7]
Similarly, to prove $y \rightsquigarrow x = \big(\big((y \to x) \rightsquigarrow x\big) \rightsquigarrow (y \rightsquigarrow x)\big)$ for all $x, y \in A$.

Proposition 3.2.4. Let $(A, \to, \rightsquigarrow, {}^{-}, {}^{\sim}, 1)$ be a pseudo-Wajsberg algebra. Then F is called a Type-1 implicative filter of lattice pseudo-Wajsberg algebra if and only if F is an implicative filter of lattice pseudo-Wajsberg algebra.

Proof. Let F is an implicative filter of lattice pseudo-Wajsberg algebra.
To prove: $(x \to y) \to x = x = (x \rightsquigarrow y) \rightsquigarrow x,$ for all $x, y \in A$
Let $y = 1, (x \to y) \to x = (x \to 1) \to x = 1 \to x = x$
Similarly, $(x \rightsquigarrow y) \rightsquigarrow x = (x \rightsquigarrow 1) \rightsquigarrow x = 1 \rightsquigarrow x = x$
Therefore, F is Type-1 implicative filter of lattice pseudo -Wajsberg algebra.
The converse part is straight forward.

3.3. Type-2 Implicative filters of lattice pseudo-Wajsberg algebra

In this section, we define Type-2 implicative filter of lattice pseudo-Wajsberg algebra and obtain some useful results with illustrations.

Definition 3.3.1. A pseudo-Wajsberg algebra $(A, \to, \rightsquigarrow, ^-, ^\sim, 1)$ is called a Type-2 implicative filter of lattice pseudo-Wajsberg algebra if it satisfies the following axioms for all $x, y \in A$,

(i) $1 \in F$
(ii) $x \in F$ and $x \to y \in F$ imply $y \in F$
(iii) $x \in F$ and $x \rightsquigarrow y \in F$ imply $y \in F$
(iv) $(x \rightsquigarrow y) \to x = x = (x \to y) \rightsquigarrow x$.

Example 3.3.2. Consider a set $A = \{0, a, b, c, 1\}$. Define a partial ordering "$\leq$" on A, such that $0 < a < b,\ c < 1$ and the binary operations "$\to$", "$\rightsquigarrow$" and quasi complements "$^-$", "$^\sim$" given by the following tables (9), (10), (11) and (12)

x	x^-
0	1
a	b
b	b
c	0
1	0

Table (9)

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	b	c	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Table (10)

x	$x^{\sim}$
0	1
a	b
b	b
c	0
1	0

Table (11)

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	b	b	1	1	1
c	0	b	b	1	1
1	0	a	b	c	1

Table (12)

Now consider the set $F_1 = \{c, 1\}$, it can easily verified that F_1 is a Type-2 implicative filter of A. But $F_2 = \{a, b, 1\}$ is not a Type-2 implicative filter of A. Since, $(b \to a) \rightsquigarrow b = c \rightsquigarrow b, c \notin F_2$.

Proposition 3.3.3. Let $(A, \to, \rightsquigarrow, {}^{-}, {}^{\sim}, 1)$ be a Type-2 Implicative filter of lattice pseudo-Wajsberg algebra. Then if it satisfies the following axioms for all $x, y \in A$,

(i) $x \to (x \to y) = x \to y$

(ii) $x \rightsquigarrow (x \rightsquigarrow y) = x \rightsquigarrow y$

(iii) $x \rightsquigarrow (x \to y) = x \to y$

(iv) $x \to (x \rightsquigarrow y) = x \rightsquigarrow y$

(v) $x \to y = x \rightsquigarrow y$.

Proof.**(i)** From (iv) of definition 3.3.1,

we have $x \to (x \to y) = \big((x \to y) \rightsquigarrow x\big) \to (x \to y) = x \to y$

(ii) From (iv) of definition 3.3.1,

we have $x \rightsquigarrow (x \rightsquigarrow y) = \big((x \rightsquigarrow y) \to x\big) \rightsquigarrow (x \rightsquigarrow y) = x \rightsquigarrow y$

(iii) From (1) and (ii) of definition 2.4,

we have $\big(x \rightsquigarrow (x \to y)\big) \to (x \to y) = \big(x \to (x \to y)\big) \rightsquigarrow (x \to y)$

$= (x \to y) \rightsquigarrow (x \to y) = 1.$

Then, $x \rightsquigarrow (x \to y) \leq x \to y$. [From (i) of definition 2.5]

From (v) of proposition 2.7, we have $x \to y \leq x \rightsquigarrow (x \to y)$.

Therefore, $x \rightsquigarrow (x \to y) = x \to y$ for all $x, y \in A$.

(iv) From (2) and (ii) of definition 2.4,

we have $\big(x \to (x \rightsquigarrow y)\big) \rightsquigarrow (x \rightsquigarrow y) = \big(x \rightsquigarrow (x \rightsquigarrow y)\big) \to (x \rightsquigarrow y)$

$= (x \rightsquigarrow y) \to (x \rightsquigarrow y) = 1.$

Then, $x \to (x \rightsquigarrow y) \leq x \rightsquigarrow y$. [From (i) of definition 2.5]

From (v) of proposition 2.7, we have $x \rightsquigarrow y \leq x \to (x \rightsquigarrow y)$.

Therefore, $x \to (x \rightsquigarrow y) = x \rightsquigarrow y$ for all $x, y \in A$.

(v) From (3), (4) and (iv) of definition 3.3.1,
we have $x \to y = x \rightsquigarrow (x \to y)$

$= x \to (x \rightsquigarrow y)$ [From (viii) of definition 2.7]

$= x \rightsquigarrow y$

Therefore, $x \to y = x \rightsquigarrow y$ for all $x, y \in A$.

Proposition 3.3.4. Let $(A, \to, \rightsquigarrow, {}^{-}, {}^{\sim}, 1)$ be a pseudo-Wajsberg algebra. Then F is called a Type-2 implicative filter of lattice pseudo-Wajsberg algebra if and only if F is an implicative filter of lattice pseudo-Wajsberg algebra.

Proof. Let F is an implicative filter of lattice pseudo-Wajsberg algebra.
To Prove: $(x \rightsquigarrow y) \to x = x = (x \to y) \rightsquigarrow x$ for all $x, y \in A$.
Consider, $x = 1 \to x = (x \rightsquigarrow y) \to x$ and $x = 1 \rightsquigarrow x = (x \to y) \rightsquigarrow x$
Therefore, $(x \rightsquigarrow y) \to x = x = (x \to y) \rightsquigarrow x$ for all $x, y \in A$.
Hence, F is Type-2 implicative filter of lattice pseudo -Wajsberg algebra.
The converse part is straight forward.

4. CONCLUSION

In this paper, we have introduced the notion of an implicative filter of lattice pseudo-Wajsberg algebra. The properties and equivalent conditions of implicative filter of lattice pseudo-Wajsberg algebra are discussed. In addition, we have introduced the notions of Type-1 and Type-2 implicative filters of lattice pseudo-Wajsberg algebra, and also investigated properties and some necessary and sufficient conditions of Type-1 and Type-2 implicative filters of lattice pseudo-Wajsberg algebra. Further we extend this idea as fuzzyfications of the implicative filter of lattice pseudo-Wajsberg algebra.

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