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Stability Results of Non Linear Differential Equations using Lyapunov Methods

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Abstract

This paper studies stability and asymptotic stability of Non linear differential equations by Lyapunov direct method. Sufficient conditions for stability and asymptotic stability are given in terms of non differentiable Lyapunov-like functions. An application to a class of non linear control systems with feedback control is also given.

AMS subject classification:
Keywords: Lynapunor-like functions, asymptotic stability, stability, stabilizability.

1. Introduction

Consider a non linear differential equation of the form

$$\dot{x}(t) = f(t, x(t)), t \geqslant 0 \tag{1.1}$$

It is well known that there are two major approaches to the Lyapunov stability analysis of system (1.1); the direct method and the linearization method.

In general, the most powerful technique to investigate the stability of (1.1) is by the use of direct method, but can also be done through the linearization method. For this direct method, one usually assumes the existence of the so called Lyapunov function, a positive definite function with negative derivative along the solution path of the system. In last twenty years, the Lyapunov direct method has been a fruitful technique in stability analysis of non linear differential equations [1]. We know that there are great number of authors studying Lyapunov-like functions extension and generalization [2, 3, 4]. To reduce a given complicated system into a relatively simpler system and provide a useful application to control systems, we recognize that the Lyapunov-like functions serve as a main instrument [5].

Unlike in the papers [6, 7] where stability results on non-Lyaunov function were given, we are hereby investigating the asymptotic stability of class of non linear differential equations in terms of non-smooth Lyapunov-like functions.

Here, the Lyapunov-like functions we are going to use in this paper need not be differentiable or even continuous. As an application based on the stability results obtained for system (1.1), we obtain sufficient conditions for stabilizability of non linear control systems by non-linear feedback controls.

In this paper, we shall give main notations and definitions of Lyapunov-like functions, stability and asymptotic stability with the proposed Lyapunov-like functions in section 2. In section 3, we present main Theorems on sufficient conditions for stability and asymptotic stability with the proposed Lyapunov-like functions. An application to stabilizability of non linear control systems is also given. We then conclude the paper in section 4.

2. Notations and Definitions

The following notations and definitions will be used throughout this paper: $X = E^n$ denotes the n-dimensional Euclidean space with the corresponding norm $||.||$, E denotes the real line; E^+ denotes the set of non-negative real numbers,Z^+ denotes the set of non-negative integers; B_ϵ denotes an open ball with radius ϵ and centered at zero. Now, consider the following non-linear differential equation with initial conditions.

$$\begin{cases} \dot{x} = f(t, x(t)), t \geqslant t_0 \in E^+ \\ x(t_0) = x_0 \end{cases} \tag{2.1}$$

where the states x(t) takes values in X, $f(t, x(t)) : E^+ \times X \to X$ is a given non-linear function and $f(t, 0) = 0$ for all $t \in E^+$. We are assuming that the conditions on the system (2.1) are such that the existence of its solution is guaranteed.

Definition 2.1. The zero solution $x(t) = 0$ is stable for (2.1) if for all $\varepsilon > 0$, and all $t_0 \geq 0$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that if $|x_0| < \delta$, then $f(t, t_0, x_0)$ exist on

$[t_0, \infty)$and satisfies
$|f(t, t_0, x_0)| < \varepsilon$ for all $t \geq t_0$.

Definition 2.2. The zero solution $x(t_0) = 0$ is asymptotically stable for (2.1) if for all $t_0 \geq 0$, there exist a $\delta_0 > 0$ such that if $|x_0| < \delta$ then $|f(t, t_0, x_0)| \to 0$ as $t \to \infty$.

From Definitions 2.1 and 2.2, we can see that asymptotic stability implies stability. So, any zero solution which is asymptotically stable is stable. It has to be noted that in the above definitions 2.1 and 2.2, if the number $\delta > 0$ is independent of t_0, then the zero solution x = 0 of the system is said to be uniformly stable and uniformly asymptotically stable respectively.

Definition 2.3. A function $f(t, x) : E^+ \times X \to X$ is said to be Lipschitz in x uniformly with respect to $(u.w.r.t)$ $t \in E^+$ if there is a number $L > 0$ such that

$$||f(t, x_1) - f(t, x_2)|| \leq L||x_1 - x_2||, \forall (t, x_1, x_2) \in E^+ \times X \times X.$$

Definition 2.4. Let $V(t, x) : E^+ \times W \to E$ be a given function, where $W \subset X$ is some open neighborhood of the origin. Then we define

$$D^+V(t, x) = \limsup_{h \to 0^+} \frac{V(t+h, x(t+h)) - V(t, x)}{h}$$

and

$$D_-V(t, x) = \liminf_{y \to 0, h \to 0^+} \Delta_{h,y} V(t, x)$$

where

$$\Delta_{h,y} = \frac{V(t+h, x(t+h)) - V(t, x)}{h}$$

Definition 2.5. A positive definite function $V(t, x) : E^+ \times W \to E$ is a weak Lyapunov function of system (1.1) if it is continuous in $(t, x) \in E^+ \times W$ and Lipschitz in $x \in W$ and there is a non-decreasing continuous function $\gamma(.) : E^+ \to E \setminus_{(0)}$. such that

$$D^+V(t, x) \leq -\gamma(||x||) < 0, \forall t \in E^+, x \in W \setminus_{(0)} .[8] \tag{2.2}$$

Definition 2.6. A function $V(t, x) : E^+ \times W \to E$ is a Lyapunov-like function of the system (1.1) if it satisfies the following conditions:

i. There exist non-decreasing function $a(t) : E^+ \to E^+ \setminus_{(0)}$, a non-increasing function $b(t) : E^+ \to E^+ \setminus_{(0)}$,
$a > 0, b > 0$ such that

$$a(t)||x||^a \leq V(t, x) \leq b(t)||x||^b, \ \forall (t, x) \in E^+ \times W \tag{2.3}$$

ii. There are non-decreasing functions $\gamma(.) : E^+ \to E \setminus_{(0)}$ and $C(.) : E^+ \to E \setminus_{(0)}$ such that

$$D_-V(t, x) \leq c(t)\gamma(V(t, x)), \ \forall t \in E^+, x \in W \setminus_{(0)} \tag{2.4}$$

Definition 2.7. A function $V(t,x) : E^+ \times W \to X$ is said to be a generalized Lyapunov-like function of system (1.1) if it satisfies the following conditions:

(i) There exist functions $a(t,h),\ b(t,h) : E^+ \to E\setminus_{(0)}$ $a(t,0) = b(t,0) = 0.$ which are continuously strictly increasing in $h \in E^+$, $a(t,h)$ is non-decreasing in t, $b(t,h)$ is non-increasing in t such that

$$a(t,||x||) < V(t,x) \leq b(t,||x||),\ \forall (t,x) \in E^+ \times W \tag{2.5}$$

(ii) For every $T > 0$, there are sequences of positive numbers $\{t_n\}$ going to zero and a function $\gamma(t,h) : E^+ \to E\setminus_{(0)}$ which is integrable, non-decreasing in (t,h), such that

$$\lim_{n\to\infty, y\to 0} \Delta_{h,y} V(t,x) \leq -\int_t^{t+T} \gamma(s,||x||)ds, \forall\, t \in E^+, x \in W\setminus_{(0)} \tag{2.6}$$

Note:
If the function $V(t,x)$ is continuous in $t \in E^+$ and Lipschitzian in x, and satisfies condition (2.2), then the Lyapunov function $V(t,x(t))$ is non-increasing in t. Note that the Lyapunov-like function in terms of Definitions 2.6. and 2.7. is not necessarily continuous in t, Lipschitzian in x and then is not, in general, non-increasing in t, since the functions $a(t)$, $b(t)$, are not assumed to be conditions in t. Note also that condition (2.4) or (2.5) means that the Lyapunov-like function $V(t,x(t))$ is non-increasing along the solution path of the system on the sequence.

3. Stability Results

Here we give the stability criterions using Lyapunor-like functions. Let us start with the following Theorem from [3] which gives a sufficient condition for the asymptotic stability of system (1.1) with the weak Lyapunov function:

Theorem 3.1. [3] Assume that $||f(t,x)|| \leq M, \forall (t,x) \in E^+ \times W$. If the system (1.1) admits a weak Lyapunov function, then the zero solution is uniformly asymptotically stable. Here we need the following Lemma in our work.

Lemma 3.2. Let $G(t,x), V(t,x) : E^+ \times W \to E$ be functions, where $V(t,x)$ is continuous in $x \in W \subset X$ and satisfies the condition;

$$D_- V(t,x) \leq G(t,x), \forall (t,x) \in E^+ \times W,$$

then for every solution $x(t)$ of the system (1.1)

$$\liminf_{h\to 0^+} \frac{V(t+h, x(t+h)) - V(t,x)}{h} \leq G(t,x) u.w.r.t.t \in E^+ \tag{+}$$

Proof. We assume for contradiction that for every sequence $\{t_n\}$ going to 0, then, there exists a solution $\bar{x}(t)$ of system (1.1) such that

$$\liminf_{n\to\infty} \frac{V(T+t_n, \bar{x}(T+t_n)) - V(T, \bar{x}(T))}{t_n} > G(T, \bar{x}(T))$$

for some $T > 0$. Then there exists an integer $N_1 > 0$, a positive number ϵ_0 small enough such that for all $n > N_1$

$$\frac{V(T+t_n, \bar{x}(T+t_n)) - V(T, \bar{x}(T))}{t_n} > G(T, \bar{x}(T)) + \epsilon_0$$

Nothing that $V(T, \bar{x}(T)) = \bar{x}(T) + t_n f(T, \bar{x}(T)) + 0(t_n)$ for some function

$$0(h) : E^+ \to E^+; \lim_{h\to 0} \frac{0(h)}{h} = 0$$

we have;

$$\frac{V(T+t_n, t_n f(T, \bar{x}(T)) + 0(t_n)) - V(T, \bar{x}(T))}{t_n} > G(T, \bar{x}(T)) + \epsilon_0 \qquad (*)$$

On the other hand, by the assumption, there is a sequence $\{t_n\}$ which goes to 0^+, a number $N_2 > 0$, and for $t = T, \bar{x}(T) = \bar{x}, \epsilon > 0$, there is a number $\delta > 0$ such that

$$V(t + {t_n}^1, x + {t_n}^1 f(t, x) + {t_n}^1 y) - V(t, x) \leq G(t, x) + \epsilon_0 \qquad (**)$$

for all $y \in \beta_\delta, n > N_2$. Let us put in the inequality $(*)$ $t_n = {t_n}^1$ Then taking a number $N > Max\{N_1, N_2\}$ and $n > N$ large enough so that $\frac{0(t_n^1)}{t_n}^1 \in B_\delta$, from $(*)$ it follows that

$$\frac{V(T + {t_n}^1, \bar{x} + {t_n}^1 f(T, \bar{x}) + t^1 y) - V(T, \bar{x})}{t_n^1} > G(T, \bar{x}) + \epsilon_0$$

which contradicts the condition$(**)$, so $(+)$ is true. ∎

Theorem 3.3. Assume that

$$||f(t, x||) \leq M(t), \forall (t, x) \in E^+ \times W \qquad (3.1)$$

where $M(t) : E^+ \to E^+$ is an integerable function satisfying the condition;

$$\lim_{h\to 0} \int_t^{t+h} M(s)ds = 0 \; u \; w.r.t. \; t \in E^+ \qquad (3.2)$$

If the system (1.1) admits a Lyapunov-like function, then the zero solution is asymptotically stable.

Proof. *(a) First, for Stability.* Let $\delta_1 > 0$ be chosen so that $\beta_{\delta 1} \in W$. From the condition (2.4) and above Lemma 3.1, it follows that there is a sequence $\{t_n\} > 0$ such that for all solution $x(t)$ of the system $t \in E^+$,

$$\lim_{h\to\infty} \frac{V(t+t_n, x(t+t_n)) - V(t+t_n)}{t_n} \leq -C\gamma V(t, x(t)) \leq 0 \tag{3.3}$$

Let us take $\epsilon > 0$ as an arbitrary number satisfying $\epsilon < \delta_1$ such that $\beta_\epsilon \subset \beta_\delta \subset W$. Let $a(t), b(t), a > 0, b > 0$ be the functions and numbers in the assumption (2.3). For any $t_0 \in E^+$, we set

$$\delta_2 = [\frac{a(t_0)}{b(t_0)}\varepsilon^a]^{\frac{1}{b}} > 0, 0 < \delta_3 \leq min\{\delta_1, \delta_2, \varepsilon\}$$

Suppose $x(t)$ is an arbitrary solution of system (1.1) with $||x_n|| < \delta_3$. We shall show that $||x(t)|| < \delta$ for all $t > t_0$. For this, using condition (9), we have a positive number $N \in Z^+$ such that for all $n > N, t > t_0$, we have:

$$\int_t^{t+t_n} M(s)ds < Min\{\delta_1 - \varepsilon, \delta_2 - \delta_3, \delta_1 - \delta_3\} \tag{3.4}$$

For any fixed number $n_0 > N$, and setting $t_{n_0} = h$, it follows from (3.3) that

$$V(t+h, x(t+h)) - V(t, x(t)) \leq 0 \, \forall \, t \in E^+ \tag{3.5}$$

Let us check the solution of system (1.1) evaluated at $x_0 + h$, and using (3.4) we have the estimate

$$||(x(t_0+h)|| \leq ||x_0|| + \lim_{h\to 0}\int_{t_0}^{t_0+h} M(s)ds \leq \delta_3 + \int_{t_0}^{t_0+h} M(s)ds < \delta_1$$

which gives $x(t_0 + h) \in \beta_{\delta 1}$.

Using (2.3) and (3.5), we get

$$a(t_0)||x(t_0+h)||^a \leq V(t_0+h, x(t_0+h)) \leq V(t_0, x_0)$$

$$\leq b(t_0)||x_0||^b < b(t_0)\delta_2^b = a(t_0)\varepsilon^a.$$

which gives

$$||x(t)+h)|| < \varepsilon.$$

We now consider the solution $x(t)$ evaluated at time $(t_0 + 2h)$. By the same argument, we can show that $||x(t_0 + 2h)|| < \delta$, and then applying (2.3) and (3.5) again, we get

$$a(t_0)||x(t_0+h)||^a \leq V(t_0+h, x(t_0+h)) \leq V(t_0, x_0).$$

$$\leq b(t_0)\delta_2^b = a(t_0)\varepsilon^a.$$

which gives

$$||x(t_0 + h)|| < \varepsilon.$$

Repeating the same arguments, we obtain.

$$||x(t_0 + Kh)|| < \varepsilon \ \forall K \in E^+ \qquad (3.6).$$

Let $t \geq t_0$ be an arbitrary number. For $n_0 > N, t_{n_0} = h$, there are number $K_0 \in Z^+\backslash_{\{0\}}$ and $\tau \in [0, h)$ such that $t - t_0 = K_0 h + \tau$. Consider now the solution $x(t)$ evaluated at $t_0 + K_0 h + \tau$. From (3.2) and (3.6) we can see that $x(t_0 + K_0 h + \tau) \in \beta_{\delta 1}$. Considering conditions (2.3) and (3.5) we obtain

$$a(t_0)||x(t)||^a \leq V(t_0 + K_0 h + \tau, x(t_0 + K_0 h + \tau)).$$

$$\leq V(t_0 + (K_0 - 1)h + \tau), x(t_0 + (K_{0-1})h + \tau)).$$

$$< b(t_0), ||x(t_0 + \tau)||^b.$$

Estimating the solution $x(t)$ evaluated at $t + \tau$ by using (3.4), we get

$$||x(t_0 + \tau)|| \leq \delta_2.$$

So we get that

$$a(t_0)||x(t)||^a < b(t_0)\delta_2^b = a(t_0)\varepsilon,$$

as desired.

(b) For asymptotic Stability. We have to show that there is a number $\delta > 0$ such that for any solution $x(t)$ of (1.1), with $||x(t_0)|| < \delta$, for every $\varepsilon > 0$ there exists a number $N > 0$ such that $||x(t)|| < \varepsilon$ for all $t > t_0 + N$.

We note first that the system is stable, so for $\delta_1 > 0$, where δ_1 is chosen so that $B_\delta \subset W$, we can easily find a numbers δ_2 such that any solution $x(t)$ of the system with $||x(t_0)|| < \delta_2$ we get

$$||x(t)|| < \delta_2, \forall \ t > t_0$$

Consider any solution $x(t)$ of (1.1) with $||x_0|| < \delta = Min\{\delta_1, \delta_2\}$, we have $x(t) \in W, \forall \ t > t_0$. Let $\varepsilon > 0$ be an arbitrary given number. Let us define;

$$\delta_3 = \left[\frac{a(t_0)^{\varepsilon a}}{a(t_0)}\right]^{\frac{1}{b}}$$

and $\delta_4 \in (0, \delta_3)$. In view of (2.4), there is a sequence $\{t_n\} > 0$ going to zero and number $N_1 > 0$ such that for all $n > N_1$ the condition (3.3) holds. Due to (3.2), there is a number $N_1 > 0$ such that for all nN_1 , the condition (3.3) holds. Due to (3.2), there is a number $N_2 > 0$ such that for all $n > N_2$,

$$\int_t^{t+t_n} M(s)ds < \delta_3 - \delta_4 \qquad (3.7).$$

We shall show that for any fixed number $n_0 > N_2 = Max\{N_1, N_2\}$, there is an integer $K > 0$ such that

$$||x(t_0 + Kt_{n_0})|| < \delta_4 \tag{3.8}$$

If (3.8) is not true, then $||x(t_0 + Kt_{n_0})|| < \delta_4$, for all $K \in Z^+$, which will lead to contradiction as $K \to \infty$. Thus (3.8) is true. From (3.7) and (3.8) we get

$$||x(t_0 + Kt_{n_0} + \tau)|| \leq ||x\bar{t_0}|| + \int_t^{t+\tau_0} M(s)ds$$

$$< \delta_4 + \int_t^{t+\tau_0} M(s)ds$$

where $\bar{t_0} := t_0 + Kt_0$.
Then it follows that

$$a(t_0)||x(t)||^a < V(t_0 + Kt_0 + \tau_0, x(t_0 + Kt_{n_0} + \tau_0))$$

$$\leq b(t_0)||x(t_0 + Kt_{n_0} + \tau_0)||^b < b(t_0)\delta_3^b$$

$$= a(t_0)\varepsilon^a.$$

This gives

$$||x(t)|| < \varepsilon, \forall\, t > t_0 + Kt_{n_0}$$

completing the proof. ■

3.1. Application to Stabilizability

We conclude this section with an application to some stabilizability problem of a class of non linear control systems with feedback controls. Consider a non linear control system of the form,

$$\dot{x}(t) = f(t, x(t), u(t)), t \geq 0 \tag{3.9}$$

where the state $x(t) \in X$; the controls $u(t)$ takes values in E^m; $f(t, x, u)$ is a given non linear function with $f(t, 0, 0) = 0, t \geq 0$. We recall that the system (3.9) is stabilizable by a feedback control $u(t) = g(x(t))$, where $g(x) : X \to U, g(0) = 0$ is a given function, if the zero solution of the following system,

$$\dot{x}(t) = f(t, x(t), g(x(t))) : F(t, x(t), x(t_0)), x_0, t_0 \geq 0 \tag{3.10}$$

is asymptotically stable [3, 5]. Some sufficient conditions, which are stated below are for the stabilizability using Lyapunov functions, were established for a class of non linear autonomous system.

$$\dot{x}(t) = f(x(t), u(t)), u(t) \geq 0 \tag{3.11}$$

We put in these two Theorems, without proofs to conclude these results.

Theorem 3.4. [5] Consider autonomous system (3.11). If there is a function $g(x)$: $E^n \to U, g(0) = 0, g(x) \in C^1(t, x)$ and a positive definite function $V(x) : E^n \to E^+$, $V(x) \in C^1(x)$ and $V(x)$ is proper ($i.e.$ $\lim_{(|x|) \to \infty} V(x) = +\infty$) such that;

$$\frac{\partial V}{\partial x_i} f^i(x, g(x)) < 0 \,\forall\, i = 1, 2, \ldots, n, \ \forall x \neq 0$$

then the system is stabilizable by feedback control $u(t) = g(x(t))$.

Theorem 3.5. Assume that there exists a function $g(x) : E^n \to U$ such that the system (3.10) satisfies the condition (3.1). Assume also that there exists a function $V(t, x)$: $E^+ \times W \to E$, where $W \subset X$ is an open neighbourhood of zero, and functions $a(t, h), b(t, h) : E^+ \times E^+ \backslash_{\{0\}}$ which are continuously strictly increasing in h; $a(t, h)$ is non-decreasing in t, $b(t, h)$ is non-increasing in t, such that

(i)
$$a(t, ||x||) \leq V(t, x) \leq b(t, ||x||) \ \forall (t, x) \in E^+ \times W \backslash_{\{0\}} \tag{3.12}$$

(ii) For every $T > 0$, there is a sequence of positive numbers t_n going to zero, a positive function $\gamma(t, h) : E^+ \times E^+ \to W$, which is integrable and strictly increasing in (t, h) such that

$$\lim_{n \to \infty, y \to 0} \triangle_{t_n, y} V(t, x) \leq \int_0^{t+\tau} \gamma(s, ||x||) ds, \forall (t, x) \in E^1 \times W \backslash_{\{0\}}$$

then the system (3.9) is stabilizable by feedback control $u(t) = g(x(t))$.

4. Conclusions

Stability and asymptotic stability of non linear differential equations by Lyapunov direct method has been investigated. This result can be extended to any other spaces like Banach Space or Hilbert Space. However, the stability results obtained can be applied to some stabilization problems on non linear control systems by feedback controls, which can be taken as an added results to the existing ones.

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