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Signed Knot Algebras

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Abstract

In this paper, we present another class of algebras SK_n, which we call signed knot algebras. The explanation behind this name is that these new algebras have a premise comprising of signed knot graph. The connection ~ of SK_n fulfilled proportionality connection furthermore the increase of SK_n into associative algebra.

Keywords: Signed knot diagram, Knot graphs and Brauer diagram

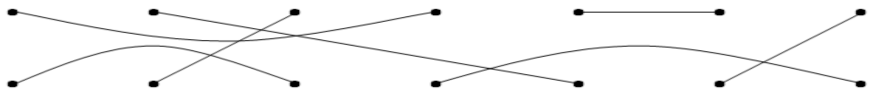
INTRODUCTION:

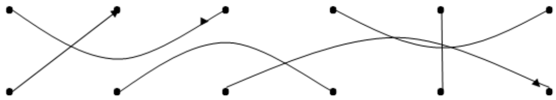
Brauer[1] introduced algebras, known as Brauer's algebras, in connection with the problem of decomposition of a tensor product representation into irreducible ones. These algebras have a basis consisting of undirected graphs. Wenzl[2] obtained the structure of these algebras D_{n+1} by making use of conditional expectations and by an inductive procedure from the structure of D_{n-1} and D_n. Parvathi and Kamaraj[3] introduced signed Brauer's algebra, which has a basis consisting of signed diagrams $\vec{D}_n$. Kamaraj and Mangayarkarasi[4] introduced knot diagrams K_n using Brauer graphs without horizontal edges and also used two types of knots only. We are spurred by these to present knot in signed brauer diagram, which prompts another class of algebras called as signed knot algebras.

1. Preliminaries:

In this section we recall the basic definition needed for our purpose.

1.1 Definition [1] [2]: A Brauer diagram is a graph on 2n vertices with n edges, with the vertices being arranged in two rows and each row consisting of n vertices, and every vertex is the vertex of only one edge.





1.2 Definition:[3] A signed diagram is a Brauer graph in which every edge is labeled by a positive or a negative sign.

1.3Knot graphs [4]

Let S_n be the symmetric group of order n and $\pi \in S_n$. A knot graph of order n is a special graph which is defined from π as follows:

1.3.1 Definition

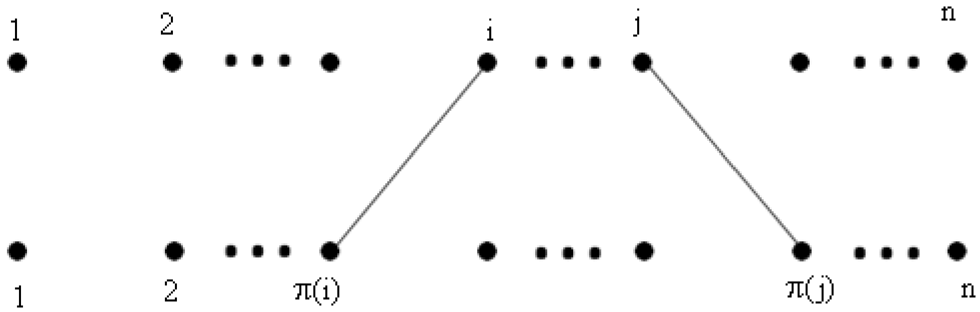
Let $\pi \in S_n$. π can be represented by a graph, which is called the Brauer diagram. Consider two edges $e_i = (i, \pi(i))$ and $e_j = (j, \pi(j))$, where vertices i and j are in the upper row and $\pi(i)$ and $\pi(j)$ are in the lower row.

Now we define $f_\pi : A(\pi) \to \{1,-1\}$ as follows.

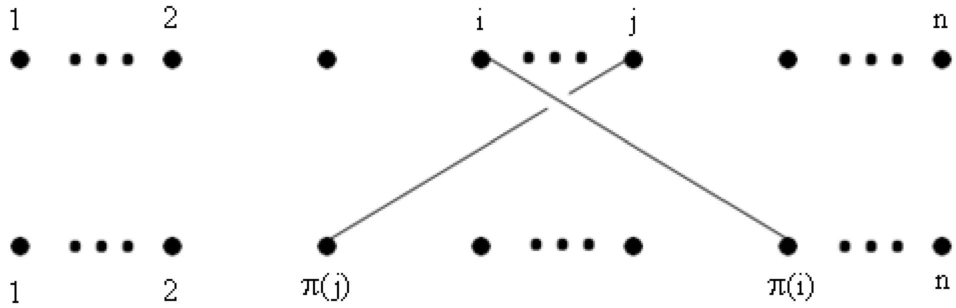
Where $A(\pi) = \{(e_i, e_j) : i < j,\ i, j = 1,2,...,n\}$

(i) If and $\pi(i) < \pi(j)$, then the edges are drawn in usual Brauer diagram.

(i.e.,) $f(e_i, e_j) = 0; \quad i < j$

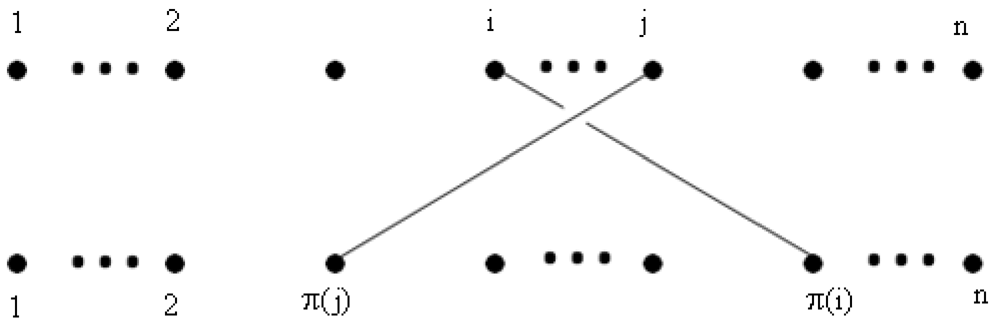


(ii). Positive Knot: If $i < j$ and $\pi(j) > \pi(i)$ then the edges are drawn in two cases as shown below. (i.e.,) $f(e_i, e_j) = +1$



In case 1, $(i, \pi(i))$ is the upper edge and $(j, \pi(j))$ is the lower edge. It can also be said that the edge $(j, \pi(j))$ is lower than the edge $(i, \pi(i))$.

Negative knot : (i.e.,) $f(e_i, e_j) = -1$



In case 2, the edge $(j, \pi(j))$ is higher than $(i, \pi(i))$ or else $(i, \pi(i))$ is lower than $(j, \pi(j))$. The above graph is called a knot graph of order n.

2. SIGNED KNOT ALGEBRAS

We are motivated by the above types of knots to introduce the signed knot diagram as follows.

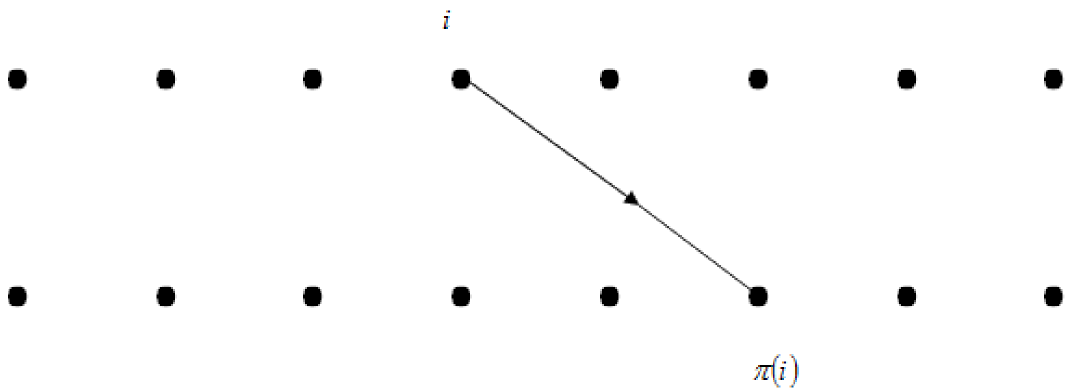
2.1.1 Signed Knot Diagram.

We introduce sign (orientation) in edges in a knot diagram.

Define a map $S_\pi : E(\pi) \to \{1,-1\}$ which is called as signed map. A knot diagram with signed map S_π is called as signed knot diagram.

Case1: Let $(\pi, f_\pi) \in K_n$, then positive sign is introduce as follows.

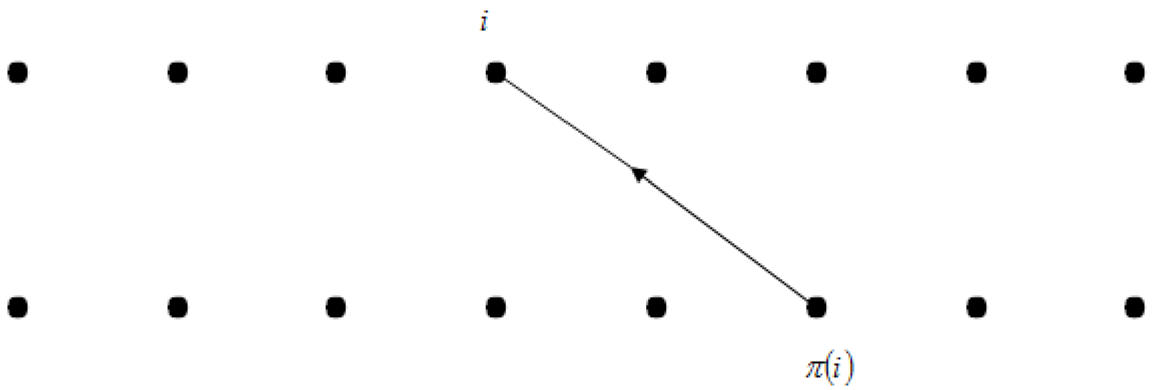
In this case, we say the edge $e = (i, \pi(i))$ is said to be positively oriented, if it has downward arrow as follows.



Here we define $S_\pi(e_i) = 1$

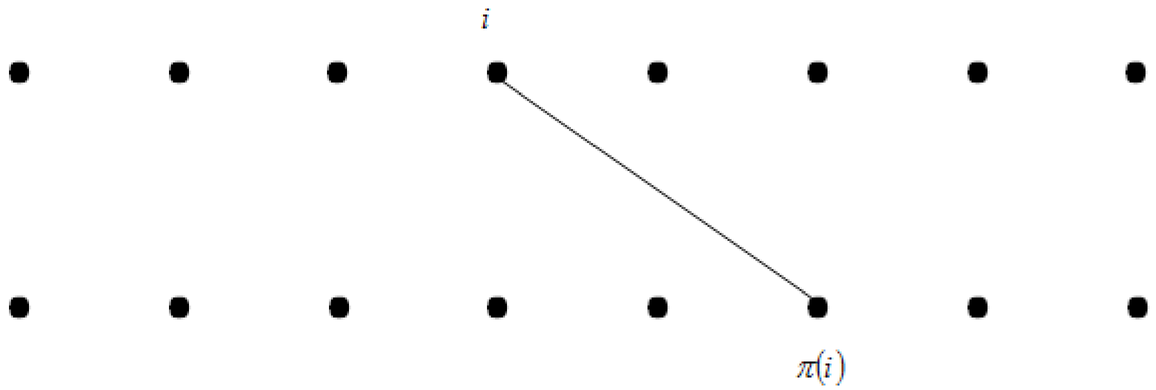
Case2: Let $(\pi, f_\pi) \in K_n$, then negative sign is introduce as follows.

In this case, we say the edge $e = (i, \pi(i))$ is said to be negatively oriented, if it has upper arrow as follows.



Here we define $S_\pi(e_i) = -1$

2.1.2 Notation : A positively oriented edge is also represented as (i.e.,) without any orientation, means as positive sign.

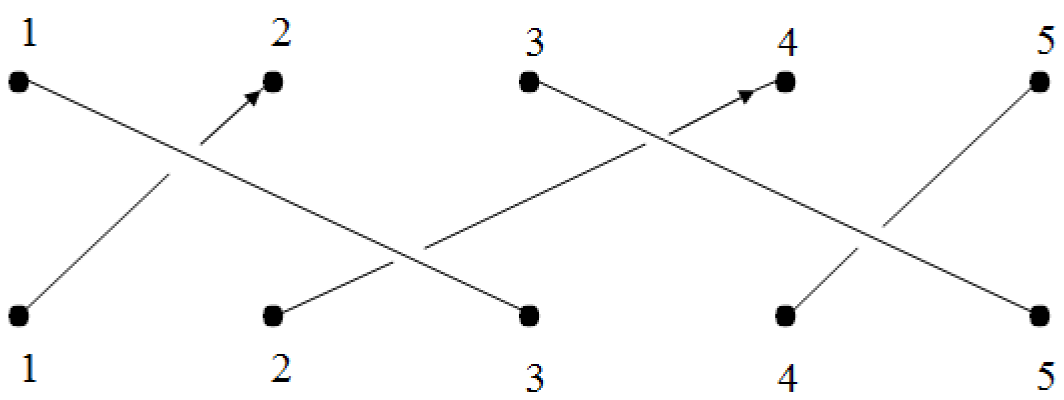


2.1.3: Signed knot diagram is represented as (π, f_π, S_π)

2.1.4 Notation: For all fixed $\pi \in S_n$ **,** $SK_\pi = \left\{ \begin{matrix} f_\pi \text{ is a knot mapping} \\ S_\pi \text{ is a signed mapping} \end{matrix} \right\}$

2.1.5Notation: $SK_n = \bigcup_{\pi \in S_n} SK_\pi$

2.1.6 Example: $\boldsymbol{SK_5}$



$e_1 = (1, 3);\ e_2 = (2, 1);\ e_3 = (3, 5);\ e_4 = (4, 2);\ e_5 = (5, 4)$

$S(e_1) = 1;\ S(e_2) = -1;\ S(e_3) = 1;\ S(e_4) = -1;\ S(e_5) = 1$

2.1.7 Proposition:

If $f_\pi(e_i, e_j) \neq g_\pi(e_i, e_j);\ \pi \in S_n$, then $f_\pi(e_i, e_j) \neq 0$.

Proof: Suppose $f_\pi(e_i, e_j) \neq 0$ this implies that $g_\pi(e_i, e_j) \neq 0$

(i.e.,) $f_\pi(e_i, e_j) \neq 0$

Which is contradiction our result. Hence $f_\pi(e_i, e_j) \neq 0$

2.1.8 Remark

$f_\pi(e_i, e_j) \neq 0 \Leftrightarrow g_\pi(e_i, e_j) \neq 0$ for any knot mapping SK_n.

2.1.9 Definition:

Let $\pi \in S_n$, We define a relation ~in SK_π as follows

Let $a, b \in SK_\pi$ an we say a ~b if any one of the following cases are true.

$a = (\pi, f_\pi, S_\pi)$ and $b = (\sigma, f'_\sigma, S'_\sigma)$

Case1: $f_\pi = f'_\sigma$ and $S_\pi = S'_\sigma$ (or)

Case2 : $f_\pi(e_i, e_j) \neq f'_\sigma(e_i, e_j)$ (or)

Precisely one of the accompanying is valid

(i)If $S_\pi(e_i) \neq S'_\sigma(e_i)$ then $S_\pi(e_j) \neq S'_\sigma(e_j)$

(ii)If $S_\pi(e_i) = S'_\sigma(e_i)$ then $S_\pi(e_j) \neq S'_\sigma(e_j)$

Case3: $f_\pi(e_i, e_j) = f'_\sigma(e_i, e_j) \neq 0$ (or)

(i)If $S_\pi(e_i) \neq S'_\sigma(e_i)$ then $S_\pi(e_j) \neq S'_\sigma(e_j)$ (or)

(ii) If $S_\pi(e_i) = S'_\sigma(e_i)$ then $S_\pi(e_j) = S'_\sigma(e_j)$

Case4: $f_\pi(e_i, e_j) = f'_\sigma(e_i, e_j) = 0$

(i) If $S_\pi(e_i) = S'_\sigma(e_i)$ then $S_\pi(e_j) = S'_\sigma(e_j)$(or)

(ii) If $S_\pi(e_i) \neq S'_\sigma(e_i)$ then $S_\pi(e_j) \neq S'_\sigma(e_j)$

2.1.10 Remark:

If $a, b, c = \{-1, \quad 1\}$ and $a \neq b \neq c$ then $a = c$

2.1.11 Theorem: The relation ~ is an Equivalence relation.

Proof:

Reflexive: Proof is obvious by definition

Symmetric:

Let $\left(\pi, f_{\pi}^{'}, S_{\pi}^{'}\right) \sim \left(\pi, f_{\pi}^{''}, S_{\pi}^{''}\right)$

Interchanging $'$ and $''$ in the definition, we get $\left(\pi, f_{\pi}^{''}, S_{\pi}^{''}\right) \sim \left(\pi, f_{\pi}^{'}, S_{\pi}^{'}\right)$

Hence it is ~ satisfies symmetric

Transitive: Let $(\pi, f_{\pi}, S_{\pi}) \sim \left(\pi, f_{\pi}^{'}, S_{\pi}^{'}\right)$ and $\left(\pi, f_{\pi}^{'}, S_{\pi}^{'}\right) \sim \left(\pi, f_{\pi}^{''}, S_{\pi}^{''}\right)$

Claim: $(\pi, f_{\pi}, S_{\pi}) \sim \left(\pi, f_{\pi}^{''}, S_{\pi}^{''}\right)$

Case 1: $f_{\pi}(e_i, e_j) \neq 0$ by 2.1.9 $f'_{\pi}(e_i, e_j) \neq 0$ and $f''_{\pi}(e_i, e_j) \neq 0$

Subcase1:

If $f_{\pi}(e_i, e_j) \neq f'_{\pi}(e_i, e_j)$ and $f_{\pi}'(e_i, e_j) \neq f_{\pi}''(e_i, e_j)$ then $f_{\pi}(e_i, e_j) = f_{\pi}''(e_i, e_j) \neq 0$

Let $f_{\pi}(e_i, e_j) \neq f'_{\pi}(e_i, e_j)$, by 2.1.9

Precisely one of the accompanying is valid

(i) If $S_{\pi}(e_i) \neq S_{\pi}^{'}(e_i)$ then $S_{\pi}(e_j) \neq S_{\pi}^{'}(e_j)$

(ii) If $S_{\pi}(e_i) = S_{\pi}^{'}(e_i)$ then $S_{\pi}(e_j) = S_{\pi}^{'}(e_j)$

Let $f'_{\pi}(e_i, e_j) \neq f''_{\pi}(e_i, e_j)$ by 2.1.9

Precisely one of the accompanying is valid

(iii) If $S'_{\pi}(e_i) \neq S_{\pi}^{''}(e_i)$ then $S'_{\pi}(e_j) \neq S_{\pi}^{''}(e_j)$

(iv) If $S'_{\pi}(e_i) = S_{\pi}^{''}(e_i)$ then $S'_{\pi}(e_j) = S_{\pi}^{''}(e_j)$

By [(i) and (iii)] ; we get the result , If $S_{\pi}(e_i) = S_{\pi}^{''}(e_i)$ then $S_{\pi}(e_j) = S_{\pi}^{''}(e_j)$

Hence $f_{\pi}(e_i, e_j) = f''_{\pi}(e_i, e_j) \neq 0$

By [(ii) and (iv)]; we get the result , If $S_{\pi}(e_i) = S_{\pi}^{''}(e_i)$ then $S_{\pi}(e_j) = S_{\pi}^{''}(e_j)$

Hence $f_{\pi}(e_i, e_j) = f''_{\pi}(e_i, e_j) \neq 0$

By [(i) and (iv)] ;we get the result, If $S_{\pi}(e_i) \neq S_{\pi}^{''}(e_i)$ then $S_{\pi}(e_j) \neq S_{\pi}^{''}(e_j)$

Hence $f_{\pi}(e_i, e_j) = f''_{\pi}(e_i, e_j) \neq 0$

By [(ii) and (iii)] ;we get the result, If $S_\pi(e_i) \neq S''_\pi(e_i)$ then $S_\pi(e_j) \neq S''_\pi(e_j)$

Hence $f_\pi(e_i, e_j) = f''_\pi(e_i, e_j) \neq 0$

Therefore, if $S_\pi(e_i) \neq S''_\pi(e_i)$ then $S_\pi(e_j) \neq S''_\pi(e_j)$ (or)

If $S_\pi(e_i) = S''_\pi(e_i)$ then $S_\pi(e_j) = S''_\pi(e_j)$

Hence $f_\pi(e_i, e_j) = f''_\pi(e_i, e_j) \neq 0$

Subcase 2: $f_\pi(e_i, e_j) \neq f'_\pi(e_i, e_j)$ and $f'_\pi(e_i, e_j) = f''_\pi(e_i, e_j)$

Let $f_\pi(e_i, e_j) \neq f'_\pi(e_i, e_j)$, by 2.1.9

Precisely one of the accompanying is valid

(v) If $S_\pi(e_i) \neq S'_\pi(e_i)$ then $S_\pi(e_j) \neq S'_\pi(e_j)$

(vi) If $S_\pi(e_i) = S'_\pi(e_i)$ then $S_\pi(e_j) = S'_\pi(e_j)$

Let $f'_\pi(e_i, e_j) = f''_\pi(e_i, e_j)$, by 2.1.9

(vii) If $S'_\pi(e_i) \neq S''_\sigma(e_i)$ then $S'_\pi(e_j) \neq S''_\pi(e_j)$ (or)

(viii) If $S'_\pi(e_i) = S''_\pi(e_i)$ then $S'_\pi(e_j) = S''_\pi(e_j)$

By [(v) and (vii)] ; we get the result , If $S_\pi(e_i) = S''_\pi(e_i)$ then $S_\pi(e_j) = S''_\pi(e_j)$

Hence $f_\pi(e_i, e_j) = f''_\pi(e_i, e_j)$

By [(v) and (viii)] ; we get the result , If $S_\pi(e_i) \neq S''_\pi(e_i)$ then $S_\pi(e_j) \neq S''_\pi(e_j)$

Hence $f_\pi(e_i, e_j) = f''_\pi(e_i, e_j)$

By [(vi) and (vii)] ; we get the result , If $S_\pi(e_i) \neq S''_\pi(e_i)$ then $S_\pi(e_j) \neq S''_\pi(e_j)$

Hence $f_\pi(e_i, e_j) = f''_\pi(e_i, e_j)$

By [(vi) and (viii)] ; we get the result , If $S_\pi(e_i) = S''_\pi(e_i)$ then $S_\pi(e_j) = S''_\pi(e_j)$

Hence $f_\pi(e_i, e_j) = f''_\pi(e_i, e_j)$

Therefore, if $S_\pi(e_i) \neq S''_\pi(e_i)$ then $S_\pi(e_j) \neq S''_\pi(e_j)$ (or)

If $S_\pi(e_i) = S''_\pi(e_i)$ then $S_\pi(e_j) = S''_\pi(e_j)$

Hence $f_\pi(e_i, e_j) \neq f''_\pi(e_i, e_j)$

Subcase 3: $f_\pi(e_i, e_j) = f'_\pi(e_i, e_j)$ and $f_\pi(e_i, e_j) \neq f''_\pi(e_i, e_j)$

It is obvious

Subcase4: $f_\pi(e_i, e_j) = f'_\pi(e_i, e_j) \neq 0$ and $f'_\pi(e_i, e_j) = f''_\pi(e_i, e_j) \neq 0$

Let $f_\pi(e_i, e_j) = f'_\pi(e_i, e_j) \neq 0$, by 2.1.9

(ix) If $S_\pi(e_i) \neq S'_\pi(e_i)$ then $S_\pi(e_j) \neq S'_\pi(e_j)$ (or)

(x) If $S_\pi(e_i) = S'_\pi(e_i)$ then $S_\pi(e_j) = S'_\pi(e_j)$

Let $f'_\pi(e_i, e_j) = f''_\pi(e_i, e_j) \neq 0$, by 2.1.9

(xi) If $S'_\pi(e_i) \neq S''_\sigma(e_i)$ then $S'_\pi(e_j) \neq S''_\pi(e_j)$ (or)

(xii) If $S'_\pi(e_i) = S''_\pi(e_i)$ then $S'_\pi(e_j) = S''_\pi(e_j)$

By [(ix) and (xi)] ;we get the result, If $S_\pi(e_i) = S''_\pi(e_i)$ then $S_\pi(e_j) = S''_\pi(e_j)$

Hence $f_\pi(e_i, e_j) = f''_\pi(e_i, e_j) \neq 0$

By [(ix) and (xii)] ; we get the result, If $S_\pi(e_i) \neq S''_\pi(e_i)$ then $S_\pi(e_j) \neq S''_\pi(e_j)$

Hence $f_\pi(e_i, e_j) = f''_\pi(e_i, e_j) \neq 0$

By conditions [(x) and (xi)] ; we get the result, If $S_\pi(e_i) \neq S''_\pi(e_i)$ then $S_\pi(e_j) \neq S''_\pi(e_j)$

Hence $f_\pi(e_i, e_j) = f''_\pi(e_i, e_j) \neq 0$

By [(x) and (xii)] ; we get the result, If $S_\pi(e_i) = S''_\pi(e_i)$ then $S_\pi(e_j) = S''_\pi(e_j)$

Hence $f_\pi(e_i, e_j) = f''_\pi(e_i, e_j) \neq 0$

Therefore, if $S_\pi(e_i) \neq S''_\pi(e_i)$ then $S_\pi(e_j) \neq S''_\pi(e_j)$ (or)

If $S_\pi(e_i) = S''_\pi(e_i)$ then $S_\pi(e_j) = S''_\pi(e_j)$

Hence $f_\pi(e_i, e_j) \neq f''_\pi(e_i, e_j) \neq 0$

Case2: $f_\pi(e_i, e_j) = 0$ by 2.1.8 $f'_\pi(e_i, e_j) = 0$ and $f''_\pi(e_i, e_j) = 0$

If $f_\pi(e_i, e_j) = f'_\pi(e_i, e_j) = 0$ and $f'_\pi(e_i, e_j) = f''_\pi(e_i, e_j) = 0$

Then $f_\pi(e_i, e_j) = f''_\pi(e_i, e_j) = 0$

Proof:

Let $f_\pi(e_i, e_j) = f'_\pi(e_i, e_j) = 0$, by 2.1.9

(xiii) If $S_\pi(e_i) \neq S'_\pi(e_i)$ then $S_\pi(e_j) \neq S'_\pi(e_j)$ (or)

(xiv) If $S_\pi(e_i) = S'_\pi(e_i)$ then $S_\pi(e_j) = S'_\pi(e_j)$

Let $f'_\pi(e_i, e_j) = f''_\pi(e_i, e_j) = 0$, by 2.1.8

(xv) If $S'_\pi(e_i) \neq S''_\sigma(e_i)$ then $S'_\pi(e_j) \neq S''_\pi(e_j)$ (or)

(xvi) If $S'_\pi(e_i) = S''_\pi(e_i)$ then $S'_\pi(e_j) = S''_\pi(e_j)$

By [(xiii) and (xv)] ; we get the result, If $S_\pi(e_i) = S''_\pi(e_i)$ then $S_\pi(e_j) = S''_\pi(e_j)$

Hence $f_\pi(e_i, e_j) = f''_\pi(e_i, e_j) = 0$

By [(xiii) and (xvi)]; we get the result, If $S_\pi(e_i) \neq S''_\pi(e_i)$ then $S_\pi(e_j) \neq S''_\pi(e_j)$

Hence $f_\pi(e_i, e_j) = f''_\pi(e_i, e_j) = 0$

By [(xiv) and (xv)] ; we get the result, If $S_\pi(e_i) \neq S''_\pi(e_i)$ then $S_\pi(e_j) \neq S''_\pi(e_j)$

Hence $f_\pi(e_i, e_j) = f''_\pi(e_i, e_j) = 0$

By [(xiv) and (xvi)] ; we get the result, If $S_\pi(e_i) = S''_\pi(e_i)$ then $S_\pi(e_j) = S''_\pi(e_j)$

Hence $f_\pi(e_i, e_j) = f''_\pi(e_i, e_j) = 0$

Therefore, if $S_\pi(e_i) \neq S''_\pi(e_i)$ then $S_\pi(e_j) \neq S''_\pi(e_j)$ (or)

If $S_\pi(e_i) = S''_\pi(e_i)$ then $S_\pi(e_j) = S''_\pi(e_j)$

Hence $f_\pi(e_i, e_j) = f''_\pi(e_i, e_j) = 0$

Hence $(\pi, f_\pi, S_\pi) \sim (\pi, f''_\pi, S''_\pi)$

2.1.12 : Remark : Now consider $SK_\pi / \sim$,we can also use the notation SK_π instead of $SK_\pi / \sim$

2.1.13Notation: $(f_\pi, S_\pi) = \overline{f_\pi}$

2.1.14 Definition:

Let $(\pi, f_\pi, S_\pi) \in SK_n$

we define $\overline{f_\pi}(e_i, e_j)$ as follows:

Case1:

$$\overline{f_\pi}(e_i, e_j) = \begin{cases} 1^+ \text{ if } f_\pi(e_i, e_j) = 1 \text{ and } S(e_i) = S(e_j) = 1 \\ 1^- \text{ if } f_\pi(e_i, e_j) = 1 \text{ and } S(e_i) \neq S(e_j) \end{cases}$$

Case2:

$$\overline{f_\pi}(e_i, e_j) = \begin{cases} (-1)^+ \text{ if } f_\pi(e_i, e_j) = -1 \text{ and } S(e_i) = S(e_j) = 1 \\ (-1)^- \text{ if } f_\pi(e_i, e_j) = -1 \text{ and } S(e_i) \neq S(e_j) \end{cases}$$

Case3:

$$\overline{f_\pi}(e_i, e_j) = \begin{cases} 0^+ \text{ if } f_\pi(e_i, e_j) = 0 \text{ and } S(e_i) = S(e_j) = 1 \\ 0^- \text{ if } f_\pi(e_i, e_j) = 0 \text{ and } S(e_i) \neq S(e_j) \end{cases}$$

2.1.15: Multiplication of SK_n

We define multiplication among the signed knot diagram as follows.

Let $\bar{a}, \bar{b} \in SK_n$ To define $\bar{a}\bar{b}$, consider $ab \in D_n$

If $\bar{a} = (\pi, \overline{f_\pi}), \bar{b} = (\sigma, \overline{f_\sigma})$ then $\bar{a} \bullet \bar{b} = (\pi \bullet \sigma, \overline{f_{\pi\bullet\sigma}})$; where $\overline{f_{\pi\bullet\sigma}} = \overline{f_\pi} * \overline{f_\sigma}$ and is defined as follows.

*	0^+	0^-	1^+	1^-	$(-1)^+$	$(-1)^-$
0^+	0^+	0^-	1^+	1^-	$(-1)^+$	$(-1)^-$
0^-	0^-	0^+	1^-	1^+	$(-1)^-$	$(-1)^+$
1^+	1^+	1^-	0^-	0^+	0^+	0^-
1^-	1^-	1^+	0^+	0^-	0^-	0^+
$(-1)^+$	$(-1)^+$	$(-1)^-$	0^+	0^-	0^-	0^+
$(-1)^-$	$(-1)^-$	$(-1)^+$	0^-	0^+	0^+	0^-

2.1.16 Proposition:

The multiplication * is well defined.

Proof:

Let $\bar{a}, \bar{b} \in SK_n$ and $\bar{a} \sim \overline{a_1}$ and $\bar{b} \sim \overline{b_1}$

Claim: $\bar{a}\bar{b} \sim \overline{a_1}\,\overline{b_1}$

It is enough to prove that for $\bar{a}, \bar{b} \in \{1^+, 1^-, 0^+, 0^-, (-1)^+, (-1)^-\}$

Case 1: Let $\bar{a} = 1^+, \bar{b} = 1^+$ and $\overline{a_1} = (-1)^-, \overline{b_1} = (-1)^-$

$\bar{a}\bar{b} = 1^+ * 1^+ = 0^-$ and $\overline{a_1}\,\overline{b_1} = (-1)^- * (-1)^- = 0^-$

i.e., $0^- \sim 0^-$

Similarly, let $\bar{a} = 1^-, \bar{b} = 1^-$ and $\overline{a_1} = (-1)^+, \overline{b_1} = (-1)^+$

Subcase 1: Let $\bar{a} = 1^+, \bar{b} = 1^+$ and $\overline{a_1} = 1^+, \overline{b_1} = (-1)^-$

$\bar{a}\bar{b} = 1^+ * 1^+ = 0^-$ and $\overline{a_1}\,\overline{b_1} = 1^+ * (-1)^- = 0^-$

i.e., $0^- \sim 0^-$

Similarly, let $\bar{a} = 1^+, \bar{b} = 1^+$ and $\overline{a_1} = (-1)^-, \overline{b_1} = 1^+$ &

Let $\bar{a} = 1^-$, $\bar{b} = 1^-$ and $\overline{a_1} = (-1)^+, \overline{b_1} = 1^-$

Case 2: Let $\bar{a} = 1^{+}$, $\bar{b} = 1^{-}$ and $\overline{a_1} = (-1)^{-}$, $\overline{b_1} = (-1)^{+}$

$\bar{a}\bar{b} = 1^{+} * 1^{-} = 0^{+}$ and $\overline{a_1}\,\overline{b_1} = (-1)^{-} * (-1)^{+} = 0^{+}$

i.e., $0^{+} \sim 0^{+}$

Subcase 1: Let $\bar{a} = 1^{+}$, $\bar{b} = 1^{-}$ and $\overline{a_1} = 1^{+}$, $\overline{b_1} = (-1)^{+}$

$\bar{a}\bar{b} = 1^{+} * 1^{-} = 0^{+}$ and $\overline{a_1}\,\overline{b_1} = 1^{+} * (-1)^{+} = 0^{+}$

i.e., $0^{+} \sim 0^{+}$

Similarly, let $\bar{a} = 1^{+}$, $\bar{b} = 1^{-}$ and $\overline{a_1} = (-1)^{-}$, $\overline{b_1} = 1^{-}$

Case 3: Let $\bar{a} = 1^{+}$, $\bar{b} = 0^{+}$ and $\overline{a_1} = (-1)^{-}$, $\overline{b_1} = 0^{+}$

$\bar{a}\bar{b} = 1^{+} * 0^{+} = 1^{+}$ and $\overline{a_1}\,\overline{b_1} = (-1)^{-} * 0^{+} = (-1)^{-}$

i.e., $1^{+} \sim (-1)^{-}$

Similarly, let $\bar{a} = 1^{-}$, $\bar{b} = 0^{+}$ and $\overline{a_1} = (-1)^{+}$, $\overline{b_1} = 0^{+}$

Case 4: Let $\bar{a} = 1^{+}$, $\bar{b} = 0^{-}$ and $\overline{a_1} = (-1)^{-}$, $\overline{b_1} = 0^{-}$

$\bar{a}\bar{b} = 1^{+} * 0^{-} = 1^{-}$ and $\overline{a_1}\,\overline{b_1} = (-1)^{-} * 0^{-} = (-1)^{+}$

i.e., $1^{-} \sim (-1)^{+}$

Similarly, let $\bar{a} = 1^{-}$, $\bar{b} = 0^{-}$ and $\overline{a_1} = (-1)^{+}$, $\overline{b_1} = 0^{-}$

Proceeding in this way, we get the multiplication * is well defined.

2.1.17 Theorem: If $\bar{a}$, $\bar{b}$, $\bar{c} \in SK_n$, then $\left(\bar{a}\bar{b}\right)\bar{c} = \bar{a}\left(\bar{b}\bar{c}\right)$

Proof:

It is enough to prove that for $\bar{a}, \bar{b} \in \{1^{+}, 1^{-}, 0^{+}, 0^{-}\}$

Case 1: Let $\bar{a} = 1^{+}$, $\bar{b} = 1^{-}$, $c = 0^{+}$

$\bar{a}\bar{b} = 1^{+} * 1^{-} = 0^{+} \Rightarrow \left(\bar{a}\bar{b}\right)\bar{c} = 0^{+} * 0^{+} = 0^{+}$

$\bar{b}\bar{c} = 1^{-} * 0^{+} = 1^{-} \Rightarrow \bar{a}\left(\bar{b}\bar{c}\right) = 1^{+} * 1^{-} = 0^{+}$

Therefore $\left(\bar{a}\bar{b}\right)\bar{c} = \bar{a}\left(\bar{b}\bar{c}\right)$

Case 2: Let $\bar{a}=1^{+}$, $\bar{b}=1^{-}$, $c=0^{-}$

$\bar{a}\bar{b}=1^{+}*1^{-}=0^{+}\Rightarrow(\bar{a}\bar{b})\bar{c}=0^{+}*0^{-}=0^{-}$

$\bar{b}\bar{c}=1^{-}*0^{-}=1^{+}\Rightarrow\bar{a}(\bar{b}\bar{c})=1^{+}*1^{+}=0^{-}$

Therefore $(\bar{a}\bar{b})\,\bar{c}=\bar{a}(\bar{b}\bar{c})$

Case 3: Let $\bar{a}=1^{+}$, $\bar{b}=0^{+}$, $c=0^{-}$

$\bar{a}\bar{b}=1^{+}*0^{+}=1^{+}\Rightarrow(\bar{a}\bar{b})\bar{c}=1^{+}*0^{-}=1^{-}$

$\bar{b}\bar{c}=0^{+}*0^{-}=0^{-}\Rightarrow\bar{a}(\bar{b}\bar{c})=1^{+}*0^{-}=1^{-}$

Therefore $(\bar{a}\bar{b})\,\bar{c}=\bar{a}(\bar{b}\bar{c})$

Case 4: Let $\bar{a}=1^{-}$, $\bar{b}=0^{+}$, $c=0^{-}$

$\bar{a}\bar{b}=1^{-}*0^{+}=1^{-}\Rightarrow(\bar{a}\bar{b})\bar{c}=1^{-}*0^{-}=1^{+}$

$\bar{b}\bar{c}=0^{+}*0^{-}=0^{-}\Rightarrow\bar{a}(\bar{b}\bar{c})=1^{-}*0^{-}=1^{+}$

Therefore $(\bar{a}\bar{b})\,\bar{c}=\bar{a}(\bar{b}\bar{c})$

Case 5: Let $\bar{a}=1^{+}$, $\bar{b}=1^{+}$, $c=1^{-}$

$\bar{a}\bar{b}=1^{+}*1^{+}=0^{-}\Rightarrow(\bar{a}\bar{b})\bar{c}=0^{-}*1^{-}=1^{+}$

$\bar{b}\bar{c}=1^{+}*1^{-}=0^{+}\Rightarrow\bar{a}(\bar{b}\bar{c})=1^{+}*0^{+}=1^{+}$

Therefore $(\bar{a}\bar{b})\,\bar{c}=\bar{a}(\bar{b}\bar{c})$

Similarly, let $\bar{a}=1^{+}$, $\bar{b}=1^{+}$, $c=0^{+}$ and $\bar{a}=1^{+}$, $\bar{b}=1^{+}$, $c=0^{-}$

Case 6: Let $\bar{a}=1^{-}$, $\bar{b}=1^{-}$, $c=1^{+}$

$\bar{a}\bar{b}=1^{-}*1^{-}=0^{-}\Rightarrow(\bar{a}\bar{b})\bar{c}=0^{-}*1^{+}=1^{-}$

$\bar{b}\bar{c}=1^{-}*1^{+}=0^{+}\Rightarrow\bar{a}(\bar{b}\bar{c})=1^{-}*0^{+}=1^{-}$

Therefore $(\bar{a}\bar{b})\,\bar{c}=\bar{a}(\bar{b}\bar{c})$

Similarly, let $\bar{a}=1^{-}$, $\bar{b}=1^{-}$, $c=0^{+}$ and $\bar{a}=1^{-}$, $\bar{b}=1^{-}$, $c=0^{+}$

Case 7: Let $\bar{a}=1^{+}$, $\bar{b}=1^{+}$, $c=1^{+}$

$\bar{a}\bar{b} = 1^{+} * 1^{+} = 0^{+} \Rightarrow (\bar{a}\bar{b})\bar{c} = 0^{+} * 1^{+} = 1^{+}$

$\bar{b}\bar{c} = 1^{+} * 1^{+} = 0^{+} \Rightarrow \bar{a}(\bar{b}\bar{c}) = 1^{-} * 0^{+} = 1^{+}$

Therefore $(\bar{a}\bar{b})\,\bar{c} = \bar{a}(\bar{b}\bar{c})$

Case 8: Let $\bar{a} = 1^{-}$, $\bar{b} = 1^{-}$, $c = 1^{-}$

$\bar{a}\bar{b} = 1^{-} * 1^{-} = 0^{-} \Rightarrow (\bar{a}\bar{b})\bar{c} = 0^{-} * 1^{-} = 1^{+}$

$\bar{b}\bar{c} = 1^{-} * 1^{-} = 0^{-} \Rightarrow \bar{a}(\bar{b}\bar{c}) = 1^{-} * 0^{-} = 1^{+}$

Therefore $(\bar{a}\bar{b})\,\bar{c} = \bar{a}(\bar{b}\bar{c})$

Case 9: Let $\bar{a} = 0^{+}$, $\bar{b} = 0^{+}$, $c = 0^{+}$

It is trivial

Case 10: Let $\bar{a} = 0^{-}$, $\bar{b} = 0^{-}$, $c = 0^{-}$

$\bar{a}\bar{b} = 0^{-} * 0^{-} = 0^{+} \Rightarrow (\bar{a}\bar{b})\bar{c} = 0^{+} * 0^{-} = 0^{-}$

$\bar{b}\bar{c} = 0^{-} * 0^{-} = 0^{+} \Rightarrow \bar{a}(\bar{b}\bar{c}) = 0^{-} * 0^{+} = 0^{-}$

Therefore $(\bar{a}\bar{b})\,\bar{c} = \bar{a}(\bar{b}\bar{c})$

Hence $(\bar{a}\bar{b})\,\bar{c} = \bar{a}(\bar{b}\bar{c})$

2.1.18 Result: The free algebra generated by SK_n over F(x) is called signed knot algebras.

2.1.19 Result: The dimension of signed knot algebra is of the form $2 + 2^{1k_1} + 2^{2k_2} + \ldots + 2^{ik_i} + 2^{nk_n}$; $n \geq 3$

Where $i = \pi(i)$ = number of knots in π.

$k_i = |\{\pi \in S_n : i(\pi) = i\}|$

n = maximum number of knots.

2.1.20 Notation: F denotes a field and F(x) is the field of fractions, where x is indeterminate.

2.1.21 Result: The free algebra $F\,SK_n$ is called signed knot algebras.

REFERENCES

[1] R. Brauer, "Algebras which are connected with semi-simple continuous groups", *Ann. Math.*, **1937**, *38*, 854–872 .

[2] H. Wenzl, "The structure of Brauer's centralizer algebra", *Ann. Math.*, **1988**, *128*, 173–193.

[3] M. Parvathi and M. Kamaraj, "Signed Brauer's algebras", *Commun. Algebra*., **1998**, *26*(3), 839–855.

[4] M. Kamaraj and R. Mangayarkarasi, "Knot symmetric algebras", *Res. J. Pure Algebra*., **2011**, *1*(6), 141–151.