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The Quenching for a Nonlinear Diffusion Equation with Nonlinear Singular Boundary Condition

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Abstract

This paper study the quenching behavior for the nonlinear diffusion equation $\frac{\partial(\psi(u))}{\partial t} = u_{xx} + f(x)(1-u)^{-p}$, $0 < x < 1$, $t > 0$ with nonlinear singular boundary condition. It is proved that the solution of the equation will quench in a finite time and the unique quenching point is $x = 0$. At the same time, it has been proved that the u_t will blow up at $x = 0$.

Keyword: quenching; nonlinear singular boundary condition; quenching rate.

1. INTRODUCTION

In this paper this equation

$$\frac{\partial \psi(u)}{\partial t} = u_{xx} + f(x)(1-u)^{-p}, \qquad 0 < x < 1, \quad t > 0,$$

$$u_x(0,t) = 0, u_x(1,t) = -B(u(1,t)), \qquad t > 0, \qquad (1.1)$$

$$u(x,0) = u_0(x), \qquad 0 \le x \le 1,$$

is considered, where $\psi(s)$ is a properly smooth and strictly monotonically increasing function and $\psi(0)=0,\ \ \psi(1)=1, \psi''(s)\le 0$.

$B(s)$ satisfies the following conditions:

$B(s)>0, B'(s)<0,\ \ B''(s)\ge 0$, $\lim_{s\to 0^+} B(s)=+\infty, s>0$;

$f(x)$ is a continuous function and $f(x)>0, f'(x)<0$. The initial datum $u_0(x):[0,1]\to(0,1)$ is smooth enough and nonincreasing and p is a positive constant.

When $\psi(u)=u^m$, the model(1.1) known as the classical porous medium equation which shows a number of physical phenomenon in the nature such as the flow of an isentropic gas through a porous medium and heat transfer or diffusion([1][2]).

The earliest quenching problem was proposed by Kawarada[3], Kawarada study the quenching behavior of $u_t=u_{xx}+\dfrac{1}{(1-u)}$.

Selcuk and Ozalp have proved the quenching behavior of a semilinear heat equation with a singular boundary outflux[4].

$$u_t=u_{xx}+(1-u)^{-p}, 0<x<1, 0<t<T,$$
$$u_x(0,t)=0, u_x(1,t)=-u^{-q}(1,t), 0<t<T,$$
$$u(x,0)=u_0(x), 0\le x\le 1$$

They have proved a finite-time quenching for the solution. Further, they show that quenching occurs on the boundary under certain conditions and the time derivative blows up at a quenching point. They also get a quenching rate and a lower bound for the quenching time.

In recent years, more and more scholars pay attention to the quenching behaviour for nonlinear diffusion equations. The author has studied this equation[5]

$$\partial_t u-(|u_x|^{p-2}u_x)_x+\chi_{\{u>0\}}u^{-\beta}=0, \quad \text{in} \quad I\times(0,\infty)$$

$$u(L_1,t)=u(L_2,t)=0, \qquad t\in(0,\infty)$$

$$u(x,0)=u_0(x), \qquad \text{in} \quad I$$

They showed that the solutions occurs quenching in a finite time and get the upper and lower bounds at the quenching time.

Nie,Wang and Zhou showed the quenching for singular and degenerate quasilinear diffusion equations. They obtain the critical length for the quenching phenomenon occurs and got the location of the quenching points and the blowing up of the derivative of the solution with respect to the time at the quenching time[6].In [7] author introduced the quenching for p-Laplacian with singular boundary.

Ying.Y showed the quenching phenomenon of this equation[8]

$$\partial(A(u))_t = u_{xx} + (1-u)^{-\alpha}, 0 < x < 1, t > 0$$
$$u_x(0,t) = 0, u_x(1,t) = -B(u(1,t)), t > 0$$
$$u(x,0) = u_0(x), 0 \le x \le 1$$

In [9]-[11],more works of the quenching phenomenon for singular and degenerate parabolic equations has been studied. Some authors have studied the quenching behaviour of degenerate parabolic equations with two nonlinear heat sources, see([12]-[14]).

This article has mainly studied the quenching for a nonlinear diffusion equation with nonlinear singular boundary condition. Motivated by the work [8],we will study the quenching of the solution in the finite time and the time derivative blows up at a quenching point. This paper is divided into two parts. First, the quenching for the solution of equation (1.1) on the boundary is proved.，and prove that u_t blows up at $x=0$ .Then the rate of quenching and the lower bound of quenching time are calculated.

2. QUENCHING ON THE BOUNDARY

Definition1.1: u is a solution of (1.1),we say u quenches in the finite time ,if there exists a finite time $T(0 < T < \infty)$,such that

$$\lim_{t \to T^-} \max\{u(x,t) : 0 \le x \le 1\} \to 1 \qquad \text{or} \qquad \lim_{t \to T^-} \min\{u(x,t) : 0 \le x \le 1\} \to 0$$

The equation (1.1) is rewritten into the following form:

$$u_t = a(u)u_{xx} + a(u)f(x)(1-u)^{-p}, \qquad 0 < x < 1, \quad t > 0,$$

$$u_x(0,t) = 0, u_x(1,t) = -B(u(1,t)), \qquad t > 0, \qquad (2.1)$$

$$u(x,0)=u_0(x)\,, \qquad 0\le x\le 1\,,$$

where, $a(u)=\dfrac{1}{\psi'(u)}$.

Lemma2.1. Assume that the initial datum satisfies $u_o''(x)+f(x)(1-u_0(x))^{-p}>0$, for some $T_1>0$, the solution u of (2.1) exists in $(0,T_1)$, then $u_x(x,t)<0, u_t(x,t)\ge 0$ in $(0,1]\times(0,T_1)$.

Proof. let $g(x,t)=u_x(x,t)$, then,

$$g_t=a(u)g_{xx}+a'(u)u_{xx}g+a'(u)f(x)(1-u)^{-p}g+pa(u)f(x)(1-u)^{-p-1}g$$

$$+a(u)f'(x)(1-u)^{-p}, \qquad 0<x<1, 0<t<T_1$$

$$g(0,t)=0\,, \quad g(1,t)=-B(u(1,t))\,, \qquad 0<t<T_1$$

$$g(x,0)=u'_o(x)\,, \qquad 0\le x\le 1$$

We can get $g(x,t)<0$ by the maximum principle, so $u_x(x,t)<0$.

Let $s(x,t)=u_t(x,t)$, we have,

$$s_t=a(u)s_{xx}+a'(u)u_{xx}s+a'(u)f(x)(1-u)^{-p}s+pa(u)f(x)(1-u)^{-p-1}s\,, \quad 0<x<1, 0<t<T_1$$

$$s_x(0,t)=0, s_x(1,t)=-B'(u(1,t))s(1,t)\,, \qquad 0<t<T_1$$

$$s(x,0)=a(u_0(x))u_0''(x)+a(u_0(x))f(x)(1-u_0(x))^{-p}\,, \qquad 0\le x\le 1$$

we have $s(x,t)\ge 0$, by the maximum principle, thus $u_t(x,t)\ge 0$ in $(0,1]\times(0,T_1)$.

Theorem2.1: Suppose that $u_o''(x)+f(x)(1-u_0(x))^{-p}>0$ holds. There is a finite time T so that the solution u of the equation (1.1) is quenched in this time. And $x=0$ is the unique quenching point.

Proof: by the maximum principle, we have $0 < u(\bullet, t) < 1$, $t \in (0, T_1)$.

Since

$$u_0''(x) + f(x)(1-u_0(x))^{-p} > 0$$

Integrating the inequality with respect x from 0 to 1, we can get,

$$\eta = -B(u_0(1)) + \int_0^1 f(x)(1-u_0(x))^{-p}\, dx > 0$$

Set

$$A(t) = \int_0^1 (1-\psi(u(x,t)))dx$$

Thus,

$$A'(t) = B(u(1,t)) - \int_0^1 f(x)(1-u(x,t))^{-p}\, dx \le -\eta$$

Integrating the inequality with respect t from 0 to t, we obtain,

$A(t) \le A(0) - \eta t$, that is ,there is a $t_0 > 0$, such that $A(t_0) = 0$. There exists $T(0 < T < t_0)$

such that $\lim\limits_{t \to T^-} u(0,t) = 1$ by $u_x(x,t) < 0 (0 < x \le 1)$. So the quenching phenomenon is bound to occur at $x = 0$.

Let's prove quenching will not occur in $(0,1) \times (\theta, T)$ for some $\theta (0 < \theta < T)$.

Set

$$G(x,t) = u_x + \lambda(b_2 - x), \qquad \text{in } (b_1, b_2) \times [\theta, T)$$

Where $b_2 \in (0,1), b_1 \in (0, b_2)$ and λ is a positive constant.

Thus,

$$G_t = a(u)G_{xx} + \frac{a'(u)}{a(u)} u_x u_t + pa(u) f(x)(1-u)^{-p} u_x + a(u) f'(x)(1-u)^{-p}$$

Since $u_x(x,t) < 0, u_t(x,t) \ge 0$, $a'(u) = -\dfrac{\psi''(u)}{(\psi'(u))^2} \ge 0$, $f'(x) < 0$ in $(0,1] \times [0,T)$

We have $$G_t \le a(u)G_{xx}$$

That is

$G_t - a(u)G_{xx} \le 0$， in $(b_1, b_2) \times [\theta, T)$

If λ is small enough， then,

$G(b_1, t) = u_x(b_1, t) + \lambda(b_2 - b_1) < 0$
$G(b_2, t) = u_x(b_2, t) < 0$
$G(x, \theta) < 0$

in $(0,1] \times [\theta, T)$. We can know that $G(x,t)$ will not be able to reach a positive maximum within its area by the maximum principle.so $G(x,t) < 0$ can be obtained by the maximum principle.

That is

$u_x < -\lambda(b_2 - x)$

Integrating the inequality with respect x from b_1 to b_2,we obtain,

$$u(b_2, t) < u(b_1, t) - \frac{\lambda(b_2 - b_1)^2}{2} < 1 - \frac{\lambda(b_2 - b_1)^2}{2} < 1$$

Therefore, the quenching phenomenon of u will not occur in $(0,1]$.This theorem proves to be completed.

Theorem2.2：If $u_0''(x) + f(x)(1 - u_0(x))^{-p} > 0$ holds, and $p \ge 1$,then u_t blows up at $x = 0$.

Proof：**Suppose** that u_t is bounded in $(0,1] \times [0, T)$. Then there is a positive constant I, which makes $u_t < I$.

Hence, $$a(u)(u_{xx} + f(x)(1-u)^{-p}) < I$$

Because of $\psi''(s) < 0$, $\psi'(s)$ is nonincreasing. So there are σ and τ ,which make

$0 < \tau < u < 1$ in $[0, \sigma] \times [0, T)$,thus,

$$a(u)=\frac{1}{\psi'(u)}\geq a(\tau)$$

So

$$a(\tau)(u_{xx}+f(x)(1-u)^{-p})\leq a(u)(u_{xx}+f(x)(1-u)^{-p})<I\ ,$$

That is

$$u_{xx}+f(x)(1-u)^{-p}\leq\frac{I}{a(\tau)}$$

Upper in equation is on the left and right sides also multiplied by u_x, and integrating with respect to x from 0 to x.

Then we have,

$$\int_0^x f(x)(1-u)^{-p}u_x dx>-\frac{1}{2}u_x^{\ 2}+\frac{I}{a(\tau)}[u(x,t)-u(0,t)]$$

We get $0<m\leq f(x)\leq M$ because of $f(x)$ is a continuous function and $f(x)>0$.

Since $u_x(x,t)<0$

We have $f(x)(1-u)^{-p}u_x\leq m(1-u)^{-p}u_x$

Thus

$$\int_0^x m(1-u)^{-p}u_x dx>-\frac{1}{2}u_x^{\ 2}+\frac{I}{a(\tau)}[u(x,t)-u(0,t)]$$

When $p=1$, we have

$$\ln[1-u(0,t)]>-\frac{1}{2m}u_x^{\ 2}+\ln[1-u(x,t)]+\frac{I}{a(\tau)m}[u(x,t)-u(0,t)]$$

When $p\neq 1$, we get

$$\frac{(1-u(0,t))^{-p+1}}{-p+1}>-\frac{1}{2m}u_x^{\ 2}+\frac{(1-u(x,t))^{-p+1}}{-p+1}+\frac{I}{a(\tau)m}[u(x,t)-u(0,t)] \qquad \text{..........(2.1)}$$

When $t\to T^-$ and $p\geq 1$,the left of (2.1) tends to negative infinity, and the right is limited. So there is a contradiction.

3. THE QUENCHING RATE AND THE LOWER BOUND OF QUENCHING TIME

Theorem3.1： If $u_0''(x)+f(x)(1-u_0(x))^{-p}>0$ and $u_0'(x)\le -xB(u_0(x)),0\le x\le 1$ hold. Then there exists a positive constant C such that

$$u(0,t)\ge 1-C(T-t)^{\frac{1}{p+1}}$$

proof： Now give a function $J(x,t)=u_x(x,t)+xB(u(x,t))$ in $(0,1]\times[0,T)$.

According to $u_x<0,B'(s)<0,B''(s)\ge 0,f'(x)<0$， we have,

$$J_t-a(u)J_{xx}=a'(u)u_x(u_{xx}+(1-u)^{-p})+a(u)\left[pf(x)(1-u)^{-p-1}-2B'(u)\right]u_x$$
$$+a(u)(1-u)^{-p}\left[f'(x)+xf(x)B'(u)\right]-a(u)xB''(u)u_x^{\ 2}\le 0$$

By using of $u_0'(x)\le -xB(u_0(x)),0\le x\le 1$,we get

$$J(x,0)=u_x(x,0)+xB(u(x,0))\le 0$$

$$J(0,t)=0,J(1,t)=0$$

where $t\in(0,T)$.Using the maximum principle, we obtain $J(x,t)\le 0$ in $[0,1]\times[0,T)$.

Hence,

$$J_x(0,t)=\lim_{\rho\to 0^+}\frac{J(\rho,t)-J(0,t)}{\rho}=\lim_{\rho\to 0^+}\frac{J(\rho,t)}{\rho}\le 0$$

According to (2.1)， $u_{xx}=\dfrac{u_t}{a(u)}-f(x)(1-u)^{-p}$

Thus,

$$J_x(0,t)=u_{xx}(0,t)+B(u(0,t))=\frac{u_t}{a(u(0,t))}-f(0)(1-u(0,t))^{-p}+B(u(0,t))\le 0$$

That is，

$$a(u(0,t))J_x(0,t)=u_t(0,t)-a(u(0,t))f(0)(1-u(0,t))^{-p}+a(u)B(u(0,t))\le 0$$

so，

$$u_t(0,t)\le a(u(0,t))f(0)(1-u(0,t))^{-p}\le\frac{1}{\psi'(1)}f(0)(1-u(0,t))^{-p}\ ....(3.1)$$

Integrating with respect to t from t to T for (3.1),we get

$$u(0,t)\ge 1-C(T-t)^{\frac{1}{p+1}}$$

where， $C=\left(\frac{f(0)(p+1)}{\psi'(1)}\right)^{\frac{1}{p+1}}$.

Remark3.1： Integrating with respect to t from 0 to T in (3.1).The lower boundary of quenching time can be obtained. That is

$$T\ge\frac{\psi'(1)(1-u_0(0))^{p+1}}{f(0)(p+1)}.$$

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