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Köthe-Toeplitz Duals of The Cesàro Sequence Spaces Defined on a Generalized Orlicz Space

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Abstract

In this paper, we present the generalized Köthe-Toeplitz duals of Cesàro sequence space with terms in a generalized Orlicz space. We modify the definition of the Cesàro sequence spaces using modular on the generalized Orlicz space.

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1. Introduction

Theory of sequence spaces are very important and useful topic in many areas. Therefore, this topic gains a lot of attention by some researchers. (See e.g. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]). Cesàro sequence spaces are spaces closely related to theory of matrix operators. Discussions about Cesàro sequence spaces can be found in ([1, 2, 10]). Research in the theory of sequence spaces is a very active activities in many areas of research. Because of some needs, both in theoretical and applied field, some researchers do some modification and generalization on scalar valued sequence spaces. Some authors generalized the scalar

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valued sequence spaces into Orlicz spaces ([2, 3, 8, 9, 10]) and some others generalize them into vector valued sequence spaces ([4, 5, 10]).

A dual space is a very closely related concept to sequence spaces. A kind of dual spaces, that is Köthe-Toeplitz dual, is discussed in [1, 4, 5, 7, 10]. [1] and [7] discuss Köthe-Toeplitz duals of various scalar valued Cesàro sequence spaces. Meanwhile, [4, 5] discuss generalized Köthe-Toeplitz duals of vector valued sequence spaces.

Let X be a Banach space and $0 < p < \infty$. [4, 5] defined the sequence space $\mathcal{W}_p$,

$$\mathcal{W}_p = \left\{ (u_k) : \frac{1}{N} \sum_{k=1}^{N} \|u_k - u_0\|^p \to 0 \ (N \to \infty), \text{ for some } u_0 \in X \right\} \tag{1}$$

and characterized a generalized Köthe-Toeplitz of the space. Furthermore, the results of [4] are extended by [10] to vector valued Orlicz sequence spaces. In [10], the Köthe-Toeplitz duals of the spaces are also observed.

We recall what so called an Orlicz function. A function $\phi : \mathbb{R} \to [0, \infty)$ is called an Orlicz function, if it is continuous, even, convex, $\phi(x) = 0 \Leftrightarrow x = 0$, and $\lim_{x\to\infty} \phi(x) = \infty$. For any Orlicz function ϕ, the function ψ defined by $\psi(y) = \sup\{|y|x - \phi(x) : x \geq 0\}$ is an Orlicz function as well. Further, both of them satisfy

$$|xy| \leq \phi(x) + \psi(y), \tag{2}$$

for every $x, y \in \mathbb{R}$. An Orlicz function ψ is said to be complementary to an Orlicz function ϕ if the inequality (2) holds. An Orlicz function ϕ is said to satisfy the Δ_2-condition if there is a $K > 0$ such that $\phi(2x) \leq K\phi(x)$ for each $x \geq 0$. If the Orlicz function ϕ satisfies the Δ_2-condition, then there exist $p > 1$ and $c > 0$ such that $\phi(x) \leq c|x|^p$ for all $x \geq 0$ (See e.g. [3]). For any Orlicz function $\phi : \mathbb{R} \to [0, \infty)$, we denote by ϕ^{-1} a function from $[0, \infty)$ into itself such that

$$\phi^{-1}(y) = x \Leftrightarrow y = \phi(x), \ x \geq 0$$

In this paper, we are going to construct an Orlicz space by using an Orlicz function ϕ, denoted $\mathcal{W}_\phi$, so that the space $\mathcal{W}_p$ given in (1) is its special case. Further, we will define and characterize a generalized Köthe-Toeplitz duals of $\mathcal{W}_\phi$ for an Orlicz function ϕ which satisfies a certain condition.

Let ϕ and ψ be Orlicz functions, which are complementary each other, and E a bounded closed subset of $\mathbb{R}$. We denote by $\mathcal{L}_\phi$ the space of all Lebesgue measurable real valued functions u on E such that $|\int_E u(x)v(x)dx| < \infty$ for every measurable real valued function v on E with $\int_E \psi(v(x))dx \leq 1$. It is easy to check that the functional ρ_ϕ,

$$\rho_\phi(u) = \int_E \phi(u(x))dx,$$

is a convex modular on $\mathcal{L}_\phi$. Further, with respect to the Orlicz norm

$$\|u\|_\phi = \sup\left\{\left|\int_E u(x)v(x)dx\right| : \rho_\psi(v) \le 1\right\},$$

$\mathcal{L}_\phi$ is a Banach space. In addition, it is true that: (i) $\|u\|_\phi \le \rho_\phi(u) + 1$ for every $u \in \mathcal{L}_\phi$ and (ii) $\rho_\phi(u) \le 1$ if and only if $\|u\|_\phi \le 1$ (See e.g. [3, 8]). Further, if the Orlicz function ϕ satisfies the Δ_2-condition, it can be proved that

$$\mathcal{L}_\phi = \{u : E \to \mathbb{R} : \ u \text{ measurable and } \int_E \phi(u(x))dx < \infty\} \tag{3}$$

By taking the Orlicz funtion $\phi = |.|^p$, $1 \le p < \infty$, then following (3) we have $\mathcal{L}_\phi = L_p$, that is the Lebesgue space. The Luxemburg on $\mathcal{L}_\phi$ is a norm $\|\cdot\|_{(\phi)}$ defined by

$$\|u\|_{(\phi)} = \inf\{t > 0 : \rho_\phi(u/t) \le 1\},\ u \in \mathcal{L}_\phi$$

We can prove that $\|u\|_{(\phi)} \le \|u\|_\phi \le 2\|u\|_{(\phi)}$ for every $u \in \mathcal{L}_\phi$. In other words, the Orlicz norm is equivalent to the Luxemburg norm. Furthermore, if the function ϕ satisfies the Δ_2-condition, then $\rho_\phi(u/\|u\|_{(\phi)}) = 1$ (See e.g. [3]). It can also be verified that there is a real number $c > 0$ such that

$$\int_E \phi(au(x))dx \le c\phi(a), \tag{4}$$

for every $a > 0$ and $u \in \mathcal{L}_\phi$ with $\|u\|_\phi \le 1$.

For any Banach space Y, the notation Y^* denotes the collection of all continuous linear functionals on Y. For any $f \in Y^*$ and $y \in Y$, the notation $\langle f, y\rangle$ means $f(y)$. Let $\mathcal{B}(\mathcal{L}_\phi, Y)$ be the collection of all bounded linear mappings from $\mathcal{L}_\phi$ into Y. The adjoint of $T \in \mathcal{B}(\mathcal{L}_\phi, Y)$ is an operator $T^* \in \mathcal{B}(Y^*, \mathcal{L}_\phi^*)$ such that $\langle f, Tu\rangle = \langle T^* f, u\rangle$ for all $f \in Y^*$ and $u \in \mathcal{L}_\phi$. The notations U and U^* represent $\{u \in \mathcal{L}_\phi : \|u\|_\phi \le 1\}$ and $\{f \in Y^* : \|f\| \le 1\}$, respectively.

2. Cesàro Sequence Spaces in $\mathcal{L}_\phi$ and Their Properties

Let X be a Banach space. Notations ω and $\omega(X)$ denote the collection of all sequences in $\mathbb{R}$ and in X, respectively. For any Orlicz function ϕ satisfying the Δ_2-condition, we define the following sequence spaces:

$$\mathcal{W}_{0,\phi} = \left\{(u_k) \in \omega(\mathcal{L}_\phi) : \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \int_E \phi(u_k(x))dx = 0\right\}$$

$$\mathcal{W}_{\phi} = \left\{(u_k) \in \omega(\mathcal{L}_\phi) : (u_k - u_0) \in \mathcal{W}_{0,\phi}, \text{ for some } u_0 \in \mathcal{L}_\phi\right\}$$

$$\mathcal{W}_{\infty,\phi} = \left\{(u_k) \in \omega(\mathcal{L}_\phi) : \sup_N \frac{1}{N} \sum_{k=1}^{N} \int_E \phi(u_k(x))dx < \infty\right\}$$

$$w_{0,\phi} = \left\{(a_k) \in \omega : \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \phi(a_k) = 0\right\},$$

$$w_{\phi} = \{(a_k) \in \omega : (a_k - a_0) \in w_{0,\phi}, \text{ for some } a_0 \in \mathbb{R}\}$$

$$w_{\infty,\phi} = \left\{(a_k) \in \omega : \sup_{N\in\mathbb{N}} \frac{1}{N} \sum_{k=1}^{N} \phi(a_k) < \infty\right\}.$$

As we mentioned before, the Lebesgue space L_p, $1 \le p < \infty$, is a special case of $\mathcal{L}_\phi$, where $\phi = |.|^p$. Notice that in the Lebesgue space L_p, $\phi(\|u\|_{(\phi)}) = \rho_\phi(u)$. However, in any Orlicz space, this property is not always true (see [9]). The following lemma is needed to discuss the main results.

Lemma 2.1. Let ϕ be an Orlicz function satisfies the Δ_2-condition. If $(u_k) \in \mathcal{W}_{\infty,\phi}$, there exists $c > 0$ such that

$$\phi(\|u_k\|_\phi) \le c\rho_\phi(u_k), \quad \forall k \in \mathbb{N}.$$

Proof. Let (u_k) be any member of $\mathcal{W}_{\infty,\phi}$. We may assume that $u_k \neq 0$ for every $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, we define a function v_k on E by

$$v_k(x) = \phi^{-1}\left(\frac{\rho_\phi(u_k)}{\mu(E)}\right) = \alpha_k,\ x \in E.$$

Since $(u_k) \in \mathcal{W}_\phi$, then there exists $M > 0$ such that $\rho_\phi(u_k) \le Mk$, and so $\dfrac{\rho_\phi(u_k)}{\mu(E)} < \infty$ for each $k \in \mathbb{N}$. This implies

$$\rho_\phi(v_k) = \int_E \phi(v_k(x))dx = \frac{\rho_\phi(u_k)}{\mu(E)}\mu(E) = \rho_\phi(u_k),$$

and hence $\|v_k\|_{(\phi)} = \|u_k\|_{(\phi)}$. Since ϕ satisfies the Δ_2-condition, we have

$$1 = \int_E \phi\left(\frac{v_k(x)}{\|v_k\|_{(\phi)}}\right)dx = \phi\left(\frac{\alpha_k}{\|v_k\|_{(\phi)}}\right) \cdot \mu(E),$$

so $\|v_k\|_{(\phi)} = \dfrac{\alpha_k}{\phi^{-1}(1/\mu(E))}$. If $0 < \phi^{-1}(1/\mu(E)) < 1$, then there exists $M > 0$ such that

$$\phi(\|v_k\|_{(\phi)}) \leq M\phi(\alpha_k) = \frac{M}{\mu(E)}\rho_\phi(u_k).$$

If $\phi^{-1}(1/\mu(E)) \geq 1$,

$$\phi(\|v_k\|_{(\phi)}) \leq \phi(\alpha_k) = \frac{1}{\mu(E)}\rho_\phi(u_k).$$

Since, $\|.\|_\phi$ and $\|.\|_{(\phi)}$ are equivalent, then the assertion follows. ■

Following the Lemma 2.1, for each $N \in \mathbb{N}$, we have

$$\frac{1}{N}\sum_{k=1}^{N}\phi(\|u_k\|_\phi) = \frac{c}{N}\sum_{k=1}^{N}\rho_\phi(u_k).$$

Hence, we have the following results.

Theorem 2.2. For any Orlicz function ϕ that satisfies the Δ_2-condition, then the following statements hold.

(i) If $(u_k) \in \mathcal{W}_\phi$, then $(\|u_k\|_\phi) \in w_\phi$.

(ii) If $(u_k) \in \mathcal{W}_{\infty,\phi}$, then $(\|u_k\|_\phi)_{k\in\mathbb{N}} \in w_{\infty,\phi}$.

By using the fact that ρ_ϕ is a modular, it is easy to show that the function $\rho : \omega(\mathcal{L}_\phi) \to [0, \infty]$ defined by

$$\rho(u) = \sup_{N\in\mathbb{N}}\left\{\frac{1}{N}\sum_{k=1}^{N}\int_E \phi(u_k(x))dx\right\}, \quad u = (u_k)$$

is a modular (See e.g. [2]). Note that the convexity of ϕ implies the convexity of ρ. For every $u \in \mathcal{W}_{\infty,\phi}$, we have $\rho(u) < \infty$. Further, each of the space $\mathcal{W}_{0,\phi}, \mathcal{W}_\phi$ and $\mathcal{W}_{\infty,\phi}$ is a Banach space with respect to the Luxemburg norm

$$\|u\|_\rho = \inf\left\{t > 0 : \rho\left(\frac{u}{t}\right) \leq 1\right\}.$$

3. Köthe-Toeplitz Duals

We recall that the α-dual and the β-dual of any subspace $\lambda \in \omega$ are defined as

$$\lambda^\alpha = \left\{(a_n) \in \omega : \sum_{k=1}^{\infty}|a_n b_n| < \infty \text{ for every } (b_k) \in \lambda\right\}.$$

$$\lambda^\beta = \left\{(a_n) \in \omega : \left|\sum_{k=1}^{\infty} a_n b_n\right| < \infty \text{ for every } (b_k) \in \lambda\right\}.$$

Let X and Y be Banach spaces, respectively. A collection of all sequences of operators in $\mathcal{B}(X,Y)$ will be denoted by $\omega(\mathcal{B}(X,Y))$. For any subspace $W \subset \omega(X)$, we define its α-dual and β-dual, both with respect to Y, as follow.

$$[W]_Y^\alpha = \left\{(A_k) \in \omega(\mathcal{B}(X,Y)) : \sum_{k=1}^{\infty} \|A_k u_k\|_Y < \infty \text{ for every } (u_k) \in W\right\}.$$

$$[W]_Y^\beta = \left\{(A_k) \in \omega(\mathcal{B}(X,Y)) : \sum_{k=1}^{\infty} A_k u_k \text{ converges in } Y \text{ for every } (u_k) \in W\right\}.$$

It is clear that for any subspaces $V, W \in \omega(X)$ with $V \subset W$, $[W]_Y^\alpha \subset [V]_Y^\alpha$ and $[W]_Y^\beta \subset [V]_Y^\beta$. Now we observe our main results.

Theorem 3.1. Let Y be a Banach space and ϕ an Orlicz function that satisfies the Δ_2-condition. Then $(A_k) \in [\mathcal{W}_{0,\phi}]_Y^\alpha$ if and only if $(\|A_k\|) \in [w_{0,\phi}]^\alpha$.

Proof. For necessity, let $(A_k) \in [\mathcal{W}_{0,\phi}]_Y^\alpha$. Since $A_k \in \mathcal{B}(\mathcal{L}_\phi, Y)$ for every $k \in \mathbb{N}$, then for each $k \in \mathbb{N}$ there exists $u_k \in U$ such that

$$\|A_k\| \leq 2\|A_k u_k\|_Y.$$

Let (a_k) be an arbitrary element of $w_{0,\phi}$. By (4), we can choose some $c > 0$ such that

$$\frac{1}{N}\sum_{k=1}^{N}\int_E \phi(a_k u_k(x))dx \leq \frac{c}{N}\sum_{k=1}^{N}\phi(a_k)\int_E \phi(u_k(x))dx \leq \frac{c}{N}\sum_{k=1}^{N}\phi(a_k),$$

which implies $(a_k u_k) \in \mathcal{W}_{0,\phi}$. Therefore,

$$\sum_{k=1}^{\infty} |a_k\|A_k\|| \leq 2\sum_{k=1}^{\infty} |a_k|\|A_k u_k\|_Y = 2\sum_{k=1}^{\infty} \|A_k(a_k u_k)\|_Y < \infty,$$

i.e. $(\|A_k\|) \in [w_{0,\phi}]^\alpha$.

For sufficiency, let $(\|A_k\|) \in [w_{0,\phi}]^\alpha$ and given any $(u_k) \in \mathcal{W}_{0,\phi}$. By Theorem 2.2, $(\|u_k\|_\phi) \in w_{0,\phi}$. Hence

$$\sum_{k=1}^{\infty} \|A_k u_k\|_Y \leq \sum_{k=1}^{\infty} \|A_k\|\|u_k\|_\phi < \infty,$$

which means $(A_k) \in [\mathcal{W}_{0,\phi}]_Y^\alpha$. ■

Theorem 3.2. Let Y be a Banach space. If the Orlicz function ϕ satisfies the Δ_2-condition then

$$[\mathcal{W}_\phi]_Y^\alpha = [\mathcal{W}_{0,\phi}]_Y^\alpha.$$

Proof. Since $\mathcal{W}_{0,\phi} \subseteq \mathcal{W}_\phi$, then $[\mathcal{W}_\phi]^\alpha_Y \subseteq [\mathcal{W}_{0,\phi}]^\alpha_Y$. Conversely, let $(A_k) \in [\mathcal{W}_{0,\phi}]^\alpha_Y$. Given any $(u_k) \in \mathcal{W}_\phi$, there exists $u_0 \in \mathcal{L}_\phi$ such that $(u_k - u_0) \in \mathcal{W}_{0,\phi}$. Hence, we have

$$\sum_{k=1}^{\infty} \|A_k(u_k - u_0)\|_Y < \infty.$$

Since ϕ satisfies the Δ_2-condition, there exist $c > 0$ and $p > 1$ such that $\phi(x) \leq c|x|^p$ for each $x \geq 0$. This implies $w_{0,p} \subset w_{0,\phi}$, and hence $[w_{0,\phi}]^\alpha \subset [w_{0,p}]^\alpha$. Since by Theorem $(\|A_k\|) \in [w_{0,\phi}]^\alpha$, then $(\|A_k\|) \in [w_{0,p}]^\alpha$, and so $\sum_{r=0}^{\infty} 2^{r/p} \left(\sum_{k=2^r}^{2^{r+1}-1} \|A_k\|^q \right)^{1/q} < \infty$, for some number q such that $1/p + 1/q = 1$ (See e.g. [6]). Hence,

$$\begin{aligned}
\sum_{k=1}^{\infty} \|A_k u_0\|_Y &\leq \sum_{k=1}^{\infty} \|A_k\| \|u_0\|_\phi \\
&\leq \sum_{r=0}^{\infty} 2^{r/p} \left(\sum_{k=2^r}^{2^{r+1}-1} \|A_k\|^q \right)^{1/q} \left(\frac{1}{2^r} \sum_{k=2^r}^{2^{r+1}-1} \|u_0\|_\phi^p \right)^{1/p} \\
&\leq \sum_{r=r_0}^{\infty} 2^{r/p} \left(\sum_{k=2^r}^{2^{r+1}-1} \|A_k\|^q \right)^{1/q} \|u_0\|_\phi \\
&< \infty.
\end{aligned}$$

Finally,

$$\sum_{k=1}^{\infty} \|A_k u_k\|_Y \leq \sum_{k=1}^{\infty} \|A_k(u_k - u_0)\|_Y + \sum_{k=1}^{\infty} \|A_k u_0\|_Y < \infty,$$

i.e., $(A_k) \in [\mathcal{W}_\phi]^\alpha_Y$. ■

Theorem 3.3. For any Banach space Y and any Orlicz function ϕ that satisfies the Δ_2-condition, $(A_k) \in [\mathcal{W}_{\infty,\phi}]^\alpha_Y$ if and only if $(\|A_k\|) \in [w_{\infty,\phi}]^\alpha$.

Proof. Given $(A_k) \in [\mathcal{W}_{\infty,\phi}]^\alpha_Y$. For each $k \in \mathbb{N}$, let $u_k \in U$ such that

$$\|A_k\| \leq 2\|A_k u_k\|_Y,$$

and let $(a_k) \in w_{\infty,\phi}$. Following (4), we have

$$\sup_{N \in \mathbb{N}} \frac{1}{N} \sum_{k=1}^{N} \int_E \phi(a_k u_k(x)) dx \leq \sup_{N \in \mathbb{N}} \frac{c}{N} \sum_{k=1}^{N} \phi(a_k) < \infty,$$

which shows that $(a_k u_k) \in \mathcal{W}_{\infty,\phi}$. Therefore,

$$\sum_{k=1}^{\infty} |a_k \| A_k \|| \le 2 \sum_{k=1}^{\infty} |a_k| \| A_k u_k \| = 2 \sum_{k=1}^{\infty} \| A_k (a_k u_k) \|_Y < \infty.$$

This means that $(\|A_k\|) \in [w_{\infty,\phi}]^{\alpha}$.

Conversely, let $(\|A_k\|) \in [w_{\infty,\phi}]^{\alpha}$ and $(u_k) \in \mathcal{W}_{\infty,\phi}$. By Theorem 2.2, we have $(\|u_k\|_\phi) \in w_{\infty,\phi}$. So,

$$\sum_{k=1}^{\infty} \| A_k u_k \|_Y \le \sum_{k=1}^{\infty} \| A_k \| \| u_k \|_\phi < \infty,$$

which means $(A_k) \in [\mathcal{W}_{\infty,\phi}]_Y^{\alpha}$. ■

Theorem 3.4. Let Y be a Banach space. If the Orlicz function ϕ satisfies the Δ_2-condition, then $(A_k) \in [\mathcal{W}_{0,\phi}]_Y^{\beta}$ if and only if $(\|A_k^* f\|)_{k\in\mathbb{N}} \in [w_{0,\phi}]^{\alpha}$ for every $f^* \in U^*$.

Proof. Let $(A_k) \in [\mathcal{W}_{0,\phi}]_Y^{\beta}$ and $f \in U^*$. Since $A_k^* f \in \mathcal{L}_\phi^*$, we can choose $u_k \in U$ such that

$$\| A_k^* f \| \le 2 | A_k^* f u_k |,$$

for every $k \in \mathbb{N}$. Given any element $(a_k) \in w_{0,\phi}$. For every $k \in \mathbb{N}$, define $v_k = |a_k| sgn(A_k^* f(u_k)) u_k$. By using (4), it is easy to show that $(v_k) \in \mathcal{W}_{0,\phi}$. Therefore $\sum_{k=1}^{\infty} A_k v_k$ converges, and hence $\sum_{k=1}^{\infty} A_k^* f v_k = \sum_{k=1}^{\infty} |a_k| |A_k^* f u_k|$ converges. Consequently, $\sum_{k=1}^{\infty} |a_k| \| A_k^* f \| \le 2 \sum_{k=1}^{\infty} |a_k| |A_k^* f u_k| < \infty$, i.e. $(\|A_k^* f\|)_{k\in\mathbb{N}} \in [w_{0,\phi}]^{\alpha}$.

Conversely, let $(\|A_k^* f\|)_{k\in\mathbb{N}} \in [w_{0,\phi}]^{\alpha}$ for every $f \in U^*$ and $(u_k) \in \mathcal{W}_{0,\phi}$. By Theorem 2.2 $(\|u_k\|_\phi) \in w_{0,\phi}$, so $\sum_{k=1}^{\infty} \| A_k^* f \| \| u_k \|_\phi < \infty$. Let $\varepsilon > 0$ be an arbitrary and take $k_0 \in \mathbb{N}$ such that

$$\sum_{k=m}^{n} \| A_k^* f \| \| u_k \|_\phi < \varepsilon, \quad \text{for all } m, n \ge k_0.$$

By the Hanh-Banach Theorem, there is an $f \in U^*$ such that

$$\begin{aligned}\|\sum_{k=m}^{n} A_k u_k\|_Y &= \left|\langle f, \sum_{k=m}^{n} A_k u_k\rangle\right| \\ &\le \sum_{k=m}^{n} |\langle f, A_k u_k\rangle| = \sum_{k=m}^{n} |A_k^* f u_k| \\ &\le \sum_{k=m}^{n} \|A_k^* f\| \|u_k\|_\phi < \varepsilon,\end{aligned}$$

which shows that $\left(\sum_{k=1}^{n} A_k u_k\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in Y and therefore $\sum_{k=1}^{\infty} A_k u_k$ converges. ■

Theorem 3.5. Let Y be a Banach space and ϕ an Orlicz function that satisfies the Δ_2-condition. Then $[\mathcal{W}_\phi]_Y^\beta = [\mathcal{W}_{0,\phi}]_Y^\beta$.

Proof. It is clear that $[\mathcal{W}_\phi]_Y^\beta \subseteq [\mathcal{W}_{0,\phi}]_Y^\beta$. Let $(A_k) \in [\mathcal{W}_{0,\phi}]_Y^\beta$ and $(u_k) \in \mathcal{W}_\phi$. There exists $u_0 \in \mathcal{L}_\phi$ such that $(u_k - u_0) \in \mathcal{W}_{0,\phi}$, so $\sum_{k=1}^{\infty} A_k(u_k - u_0)$ converges. By Theorem 3.4, $(\|A_k^* f\|) \in [w_{0,\phi}]^\alpha$ for every $f \in U^*$. Analogous to the proof of Theorem 3.2, there exist $p > 1$ such that $[w_{0,\phi}]^\alpha \subset [w_{0,p}]^\alpha$. Consequently $(\|A_k^* f\|)_{k\in\mathbb{N}} \in [w_{0,p}]^\alpha$, and so $\sum_{r=0}^{\infty} 2^{r/p}(\sum_{k=2^r}^{2^{r+1}-1} \|A_k^* f\|^q)^{1/q} < \infty$, for some real number q such that $1/p + 1/q = 1$. Given any $\varepsilon > 0$, there exists $r_0 \in \mathbb{N}$ such that

$$\sum_{r=r_0}^{\infty} 2^{r/p} \left(\sum_{k=2^r}^{2^{r+1}-1} \|A_k^* f\|^q\right)^{1/q} < \frac{\varepsilon}{\|u_0\|_\phi + 1}$$

for every $r \ge r_0$. Let $m, n \ge 2^{r_0}$, there exists $f \in U^*$ such that

$$\begin{aligned}\left\|\sum_{k=m}^{n} A_k u_0\right\|_Y &= \left|\left\langle f, \sum_{k=m}^{n} A_k u_0\right\rangle\right| \le \sum_{k=m}^{n} |A_k^* f(u_0)| \le \sum_{k=m}^{n} \|A_k^* f\| \|u_0\|_\phi \\ &\le \sum_{r=r_0}^{\infty} 2^{r/p} \left(\sum_{k=2^r}^{2^{r+1}-1} \|A_k^* f|^q\right)^{1/q} \left(\frac{1}{2^r} \sum_{k=2^r}^{2^{r+1}-1} \|u_0\|_\phi^p\right)^{1/p} \\ &\le \sum_{r=r_0}^{\infty} 2^{r/p} \left(\sum_{k=2^r}^{2^{r+1}-1} \|A_k^* f\|^q\right)^{1/q} \|u_0\|_\phi < \varepsilon,\end{aligned}$$

i.e. $\left(\sum_{k=1}^{n} A_k u_0\right)_n$ is a Cauchy sequence in Y. Hence, $\sum_{k=1}^{\infty} A_k u_0$ converges. Finally,

$$\sum_{k=1}^{\infty} A_k u_k = \sum_{k=1}^{\infty} A_k(u_k - u_0) + \sum_{k=1}^{\infty} A_k u_0$$

converges, which means $(A_k) \in [\mathcal{W}_\phi]_Y^\beta$. ■

Theorem 3.6. Let Y be a Banach space and ϕ an Orlicz function which satisfies the Δ_2-condition. Then $(A_k) \in [\mathcal{W}_{\infty,\phi}]_Y^\beta$ if and only if $(\|A_k^* f\|)_{k\in\mathbb{N}} \in [w_{\infty,\phi}]^\alpha$ for every $f \in U^*$.

Proof. Let $(A_k) \in [\mathcal{W}_{\infty,\phi}]_Y^\beta$ and $f \in U^*$. For each $k \in \mathbb{N}$, let $u_k \in U$ such that

$$\|A_k^* f\| \leq 2|A_k^* f(u_k)|,$$

and let $(a_k) \in w_{\infty,\phi}$. Define (v_k) with $v_k = |a_k| sgn(A_k^* f(u_k)) u_k$ for each $k \in \mathbb{N}$. It is clear that $(v_k) \in \mathcal{W}_{\infty,\phi}$, so $\sum_{k=1}^{\infty} A_k v_k$ converges, and this implies $\sum_{k=1}^{\infty} A_k^* f(v_k)$ converges. Since $A_k^* f(v_k) = |a_k||A_k^* f(u_k)|$ for each $k \in \mathbb{N}$, then

$$\sum_{k=1}^{\infty} |a_k||A_k^* f(u_k)| < \infty.$$

Hence,

$$\sum_{k=1}^{\infty} |a_k|\|A_k^* f\| \leq 2\sum_{k=1}^{\infty} |a_k||A_k^* f(u_k)| < \infty.$$

This means $(\|A_k^* f\|) \in [w_{\infty,\phi}]^\alpha$.

Conversely, let $(\|A_k^* f\|)_{k\in\mathbb{N}} \in [w_{\infty,\phi}]^\alpha$ and $(u_k) \in \mathcal{W}_{\infty,\phi}$. By Theorem 2.2, $(\|u_k\|_\phi)_{k\in\mathbb{N}} \in w_{\infty,\phi}$, and so $\sum_{k=1}^{\infty} \|A_k^* f\|\|u_k\|_\phi < \infty$. Further, analogous to the proof of Theorem 3.4, we have $\sum_{k=1}^{\infty} A_k u_k$ converges, i.e. $(A_k) \in [\mathcal{W}_{\infty,\phi}]_Y^\beta$. ■

4. Concluding Remarks

The definition of the Cesàro sequence spaces can be modified in a generalized Orlicz space by using a modular. Some characteristic of the spaces have been able to be formulated and observed. Moreover, the generalized Köthe-Toeplitz duals of Cesàro sequence spaces with terms in a generalized Orlicz space can be presented as well.

References

[1] V. K. Bhardwaj and S. Gupta, 2013, "Cesàro summable difference sequence space," Journal of Inequalities and Application, 315, pp. 1–9.

[2] Haryadi, Supama and Zulijanto, A., 2017, “Cesaro sequence space on a generalized Orlicz space,” Far East Journal of Mathematical Sciences, 101(9), pp. 1897–1911.

[3] M.A. Krasnosel’skii and Y.B. Rutickii, 1961, “Convex Functions and Orlicz Spaces,” Netherlands: P. Noordhoff Ltd.

[4] I.J. Maddox, 1980, “Generalized Köthe-Toeplitz Duals,” Int. J. Math. Math. Sci., 3, pp. 423–432.

[5] I.J. Maddox, 1974, “Kuttner’s theorem for operators,” Composition Mathematica, 29, pp. 35–42.

[6] Malkowsky E and V. Rakoevi, 2000, “An Introduction into The Theory of Sequence Spaces and Measures of Noncompactness,” Zb. Rad. Beogr., 9, pp. 143–234.

[7] Md. F. Rahman and A. B. M. R. Karim, 2015, “Weighted Cesàro sequence space and related matrix transformation,” Pure and Applied Mathematics Journal, 4(6), pp. 237–241.

[8] M.M. Rao and Z.D. Ren, 1991, “Theory of Orlicz Spaces,” New York: Marcel Dekker Inc.

[9] R. B. Darst and R. E. Zink, 1995, “A note on the definition of an Orlicz space,” Real Analysis Exchange, 21(1), pp. 356–362.

[10] Y. Yilmaz and M.K. Ozdemir, 2005, “Köthe-Toeplitz duals of some vector-valued Orlicz sequence space,” Soochow Journal of Mathematics, 31(3), pp. 389–402.