Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 14, Number 5 (2018), pp. 625-634

http://www.ripublication.com

Implicative Filters of Residuated Lattice Wajsberg Algebras

A. Ibrahim * and V. Nirmala**

** P.G and Research Department of Mathematics,
H.H. Rajah's College, Pudukkotai, Tamilnadu, India.
Email: ibrahimaadhil@yahoo.com; dribrahimaadhil@gmail.com*

*****Department of Mathematics, Rathinam Technical campus, Coimbatore,
Tamilnadu, India.
Email: nirmalakutty9@gmail.com***

Abstract

In this paper, we introduce the notion of implicative filter of residuated lattice Wajsberg algebra. Further, we prove every implicative filter is a filter of residuated lattice Wajsberg algebra, and a subset of implicative filter is also an implicative filter. Finally, we obtained some properties of implicative filter with illustrations.

Keywords: Wajsberg algebra; Lattice Wajsberg algebra; Residuated lattice Wajsberg algebra; Implicative filter.

Mathematical Subject classification 2010: 03G10, 03E70, 06B99.

INTRODUCTION

Residuated lattices were introduced by Ward and Dilworth [8]. The algebraic counterparts of some non-classical logics satisfy residuation and those logics can be considered in a frame of residuated lattices. Much logical algebra, such as Boolean algebras, MV- algebras, MTL-algebras is particular residuated lattice. The filter theory of logical algebras plays an important role in the studying of these algebras and the completeness of the corresponding non-classical logical. The filter theory of residuated lattice has been widely studied, and some important results have been obtained. Young Bae Jun [9, 10] proposed the concept of implication filters and fantastic filters of implication algebras. Young Bae Jun,Yang Xu and Keyun Qin [11] proposed the concept of positive implicative and associative filters of lattice implication algebras [10]. Mordchaj Wajsberg [10] introduced the concept of Wajsberg algebras in 1935 and studied by Font, Rodriguez and Torrens in [5]. They

[5] defined lattice structure of Wajsberg algebras. Also, they introduced the notion of an implicative filter of lattice Wajsberg algebras and discussed some of their properties [5]. Basheer Ahamed and Ibrahim [2, 3] introduced the definitions of fuzzy implicative filter of lattice Wajsberg algebras and obtained some properties. All the above results motivate us to further investigation of the relations between implicative filter and residuated lattice Wajsberg algebras.

In this present paper, we give an equivalent condition of an implicative filter, and provide some equivalent conditions that an implicative filter is a filter of residuated lattice Wajsberg algebra and discuss some of their properties with illustrations.

PRELIMINARIES

In this section we recall some basic definitions and their properties which are helpful to develop our main results

Definition 2.1[5] Let $(A, \rightarrow, *, 1)$ be a algebra with quasi complement "*" and a binary operation "→" is called a Wajsberg algebra if and only if it satisfies the following axioms for all $x, y, z \in A$,

(i) $1 \rightarrow x = x$

(ii) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$

(iii) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$

(iv) $(x^* \rightarrow y^*) \rightarrow (y \rightarrow x) = 1$.

Proposition 2.2[5]. A Wajsberg algebra $(A, \rightarrow, *, 1)$ satisfies the following axioms for all $x, y, z \in A$,

(i) $x \rightarrow x = 1$

(ii) If $(x \rightarrow y) = (y \rightarrow x) = 1$ then $x = y$.

(iii) $x \rightarrow 1 = 1$

(iv) $(x \rightarrow (y \rightarrow x)) = 1$

(v) If $(x \rightarrow y) = (y \rightarrow z) = 1$ then $x \rightarrow z = 1$

(vi) $(x \rightarrow y) \rightarrow ((z \rightarrow x) \rightarrow (z \rightarrow y)) = 1$

(vii) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$

(viii) $x \rightarrow 0 = x \rightarrow 1^* = x^*$

(ix) $(x^*)^* = x$

(x) $(x^* \rightarrow y^*) = y \rightarrow x$.

Definition 2.3 [5]. A Wajsberg algebra A is called a lattice Wajsberg algebra if it satisfies the following conditions for all $x, y \in A$,

(i) The partial ordering " $\leq$ " on a lattice Wajsberg algebra A such that $x \leq y$ if and only if $x \to y = 1$

(ii) $(x \vee y) = (x \to y) \to y$

(iii) $(x \wedge y) = (x^* \to y^*) \to y^*)^*$.

Thus $(A, \vee, \wedge, *, \to, 0, 1)$ is a lattice Wajsberg algebra with lower bound 0 and upper bound 1.

Proposition 2.4[5]. A Wajsberg algebra $(A, \to, *, 1)$ satisfies the following axioms for all $x, y, z \in A$,

(i) If $x \leq y$ then $x \to z \geq y \to z$ and $z \to x \leq z \to y$

(ii) $x \leq y \to z$ if and only if $y \leq x \to z$

(iii) $(x \vee y)^* == (x^* \wedge y^*)$

(iv) $(x \wedge y)^* = (x^* \vee y^*)$

(v) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$

(vi) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$

(vii) $(x \to y) \vee (y \to x) = 1$

(viii) $x \to (y \vee z) = (x \to y) \vee (x \to z)$

(ix) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$

(x) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$

(xi) $(x \wedge y) \to z = (x \to y) \to (x \to z)$.

Definition 2.5 [8]. A residuated lattice is a algebra $(A, \vee, \wedge, *, \to, 0, 1)$ of type (2, 2, 2, 2, 0, 0) satisfying the following conditions,

(i) $(A, \vee, \wedge, 0, 1)$is a bounded lattice

(ii) $(A, \odot, 1)$ is a commutative monoid

(iii) $x \odot y \leq z$ if and only if $x \leq y \to z$ for all $x, y, z \in A$.

Definition 2.6[5]. Let $(A, \vee, \wedge, *, \to)$be a lattice Wajsberg algebra. If a binary operation "$\odot$"on A satisfies $x \odot y = (x \to y^*)^*$ for all $x, y \in A$. Then, $(A, \vee, \wedge, *, \to, \odot, 0, 1)$ is called a residuated lattice Wajsberg algebra.

Definition 2.7[5]. Let A be a Wajsberg algebra, a subset F of A is called an implicative filter of A, if it satisfies the following axioms for all $x, y, z \in A$,

(i) $1 \in F$

(ii) $x \in F$ and $x \to y \in F$imply $y \in F$.

IMPLICATIVE FILTERS OF RESIDUATED LATTICE WAJSBERG ALGEBRA

In this section, we define implicative filter of residuated lattice Wajsberg algebra and obtain some useful results with illustrations.

Definition 3.1. Let A be a residuated lattice Wajsberg algebra. A non-empty subset F of A is called a implicative filter of A if for all $x, y \in A$,

(i) $1 \in F$

(ii) If $x, y \in F$ then $x \odot y \in F$

(iii) $x \in F$ and $x \to y \in F$ imply $y \in F$.

Example 3.2 Let $A = \{0, a, b, 1\}$ be a set with Figure (1) as a partial ordering. Define a quasi complement "*" and a binary operation "→" on A as in tables (1) and (2)

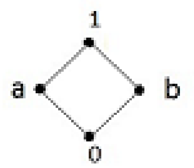

Figure (1)

x	x^*
0	1
a	b
b	a
1	0

Table (1)

→	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	b	a	1	1
1	0	a	b	1

Table (2)

Define ∨ and ∧ operations on A as follow:

$(x \vee y) = (x \to y) \to y$

$(x \wedge y) = ((x^* \to y^*) \to y^*)^*$ for all $x, y \in A$.

Then $(A, \vee, \wedge, 0, 1)$ is a residuated lattice Wajsberg algebra. It is easy to verify that $F_1 = \{b, 1\}$ is an implicative filter of Wajsberg algebra of A. But, $F_2 = \{a, 1\}$ is not an implicative filter of residuated lattice Wajsberg algebra of A, since $a \odot 1 = (a \to 1^*)^* = b \notin F_2$.

Example 3. 3 Let $A = \{0, a, b, c, 1\}$ be a set with Figure (2) as a partial ordering. Define a quasi-complement "*", a binary operation" $\odot$ "and "→" on A as in tables (3) and (4).

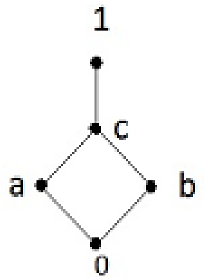

Figure (2)

x	x^*
0	1
a	b
b	a
c	c
1	0

Table (3)

→	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	a	b	1	b	1
c	c	c	c	1	1
1	0	a	b	c	1

Table (4)

Define ∨ and ∧ operations on A as follow:

$(x \vee y) = (x \to y) \to y$

$(x \wedge y)$= $((x^* \to y^*) \to y^*)^*$ for all $x, y \in A$.

Then $(A, \vee, \wedge, 0, 1)$ is a residuated lattice Wajsberg algebra. It is easy to verify that $F_3 = \{0, a, c, 1\}$ is an implicative filter of residuated lattice Wajsberg algebra. But, $F_4 = \{0, b, c, 1\}$ is not an implicative filter of residuated lattice Wajsberg algebra, since $(b \odot \mathrm{c}) = (b \to c^*)^* = a \notin F_4$.

Proposition 3.4 Let F be an implicative filter in residuated lattice Wajsberg algebra and let $x \in F$ and $x \leq y$, then $y \in F$.

Proof. Let $x \in F$ and $x \leq y$. From (i) of definition 2.4, we have $x \to y = 1$. Since $(x \odot y) \in F$ and $1 \to x \in F, x \to y \in F$. From (ii) and (iii) of 3.1, we have $y \in F$. □

Proposition 3.5 In a residuated lattice Wajsberg algebra every implicative filter is a filter

Proof. Let F be an implicative filter of residuated lattice Wajsberg algebra. If $x \in F$ and $x \to y \in F$, then $1 \to x \in F$ and $1 \to (x \to y) \in F$ it follows that $y = 1 \to y \in F$. □

Note. But the converse of the above proposition is not true.

Proposition 3.6 Let A be a residuated lattice Wajsberg algebra. Then $F \subseteq A$ is an implicative filter of A if and only if it satisfies for all $x, y \in F$ and $z \in A$ and $x \leq y \to z$ implies $z \in F$.

Proof. Let F be an implicative filter of A. By proposition 3.4, $x \leq y$and $x \in F$ imply $y \in F$. Let $z \in F$, we have $x \leq y \to z$. Suppose $x \leq x \to 1$ for all$x \in F$. By definition 2.7, we have 1$\in F$. Let $x \to y \in F$ and$x \in F$, $(x \to (y \to y) = 1$, $x \leq x \to y$ implies $y \in F$. Hence F is an implicative filter. □

Proposition 3.7 Let A be a residuated lattice Wajsberg algebra. Then $F \subseteq A$ is an implicative filter of A if and only if it satisfies the following

(i) 1$\in F$

(ii) If $x, y \in F$ then $x \odot y \in F$

(iii) For any $x, y, z \in A, x \to y \in F, y \to z \in F$ imply $x \to z \in F$.

Proof. Let F be an implicative filter of A. Then, by the definition 3.1 we have (i) and (ii). Now, $x \to y \in F$, $y \to z \in F$. By,$(x \to y) \to ((y \to z) \to (x \to z)) = 1 \in F$. It follows that $x \to z \in F$. Conversely, if $F \subseteq A$ satisfies (i), (ii) and (iii) we have x and $x \to y \in F$. It follows that$1 \to x \in F, x \to y \in F$, and hence, we get $y = 1 \to y \in F$. □

Proposition 3.8 Let F be a non empty subset of residuated lattice Wajsberg algebra A. F is an implicative filter of A if and only if it satisfies the following

(i) 1$\in F$

(ii) If $x, y \in F$ then $x \odot y \in F$

(ii) For any $x, y, z \in A, (z \to y) \to x \in F, y \in F$ imply $z \to x \in F$.

Proof. Suppose that (i), (ii) and (iii) hold. If $x \to y \in F$ and $x \in F$, then $(1 \to x) \to y \in F$, which implies $y = 1 \to y \in F$ and $x, y \in F$ then $x \odot y \in F$. Thus, F is an implicative filter.

Conversely, if F is an implicative filter, then by the definition 3.1, we have (i) and (ii) and if $(z \to y) \to x \in F, y \in F$ then by $x \leq z \to x$ and , $y \leq z \to y$. It follows that, $(z \to y) \to x \leq (z \to y) \to (z \to x) \leq y \to (z \to x)$ and hence, we have $y \to (z \to x) \in F$, which implies $z \to x \in F$. □

Proposition 3.9 Let F be an implicative filter of residuated Wajsberg algebra A . Then the following are equivalent for all $x, y, z \in A$,

(i) $F \in A$

(ii) $x \to (x \to y) \in F$ implies $(x \to y) \in F$

(iii) $x \to (y \to z) \in F$ implies $(x \to y) \to (x \to z) \in F$.

Proof. **(i) ⇒ (ii):** Let $F \in A$ and let $x \to (x \to y) \in F$. Since $x \to x = 1 \in F$, it follows from (iii) of definition that $x \to y \in F$, which proves (ii).

(ii)⇒ (iii): Suppose (ii) holds and let $x \to (y \to z) \in F$. Using (vii) from proposition 2.2, (i) from proposition 2.3 and (ii) from definition 2.1, we have $x \to (y \to z) \leq$

$x \to ((x \to y) \to (x \to z))$. Thus, by proposition 3.4 and (vii) from proposition 2.2, we get $x \to (x \to ((x \to y) \to z)) = x \to ((x \to y) \to (x \to z)) \in F$.It follows from (ii) and (vii) from proposition 2.2, we have $x \to ((x \to y) \to z)) = (x \to y) \to (x \to z) \in F$.

(iii)⇒ (i): Assume that (iii) holds and let $x \to (y \to z) \in F$ and $x \to y \in F$. From (iii), we have $(x \to y) \to (x \to z) \in Fandx \to y \in F$. It follows from (ii) that $x \to z \in F$. Thus, $F \in A$. □

Proposition 3.10 Let A be aresiduated lattice Wajsberg algebra, F be a subset of$A$. F is a filter of A if and only if

(i) $1 \in F$

(ii) If $x \in F$ and $x \leq y$ then $y \in F$

(iii) If $x, y \in F$ then $x \odot y \in F$.

Proof. If F is a filter of A, then (i) and (ii) hold. For $x, y \in F$ by $x \to (y \to (x \odot y)) = (x \odot y) \to (x \odot y) = 1 \in F$.It follows that $x \odot y \in F$. Hence (iii) holds. Conversely, suppose that (i), (ii) and (iii) holds. If $x \in F$ and $x \to y \in F$, then $x \odot (x \to y) \in F$. Hence, $y \in F$ for $x \odot (x \to y) = x \wedge \mathrm{y} \leq y$. □

Proposition 3.11 Let A be a residuated lattice Wajsberg algebra and let $F$and G be a filters of A , $F \subseteq G$. If F is an implicative filter then G is also an implicative filter.

Proof. Let $x \to (x \to y) \in G$, it is enough to prove $x \to y \in G$.

By $x \to (x \to (x \to (x \to y)) \to y)) = (x \to (x \to y)) \to (x \to (x \to y)) = 1 \in F$

It follows that $x \to ((x \to (x \to y)) \to y) \in F \subseteq G$.

That is, $(x \to (x \to y)) \to (x \to y) \in G$ and hence $x \to y \in G$. □

Remark 3.12 Let F be an implicative filter of residuated lattice Wajsberg algebra of A satisfying the condition $x \odot (y \odot z) = y \odot (x \odot z)$.

Proposition 3.13 Let F be an implicative filter of residuated lattice Wajsberg algebra of A satisfying the following condition for all$x, y \in A$, $(x \to 1) \to (y \to 1) = (x \to y) \to 1$.

Proof. From (i) and (vii) from proposition 2.2, we have

$$(x \to 1) \to (y \to 1) = (x \to 1) \to (y \to ((x \to y) \to (xy))$$
$$= (x \to 1) \to ((x \to y) \to (y \to (x \to y))$$

$$= (x \to 1) \to ((x \to y) \to (x \to (y \to y))$$
$$= (x \to 1) \to ((x \to y) \to (x \to 1))$$
$$= (x \to y) \to ((x \to 1) \to (x \to 1))$$
$$= (x \to y) \to 1. \ \square$$

Proposition3.14 Let F be a implicative filter of residuated lattice Wajsberg algebra of A, then the following axioms are equivalent for all $x, y, z \in A$,

(i) F is an implicative filter

(ii) F is a filter, $y \to (y \to x) \in F$ implies $(y \to x) \in F$

(iii) F is a filter, $z \to (y \to x) \in F$ implies $(z \to y) \to (z \to x) \in F$

(iv) $1 \in F$ and $z \to (y \to (y \to x)) \in F$ and $z \in F$ $imply$ $(y \to x) \in F$.

Proof. (**i**) $\Rightarrow$ (**ii**)**:** Let F be an implicative filter by proposition 3.5 F is a filter. If $y \to (y \to x) \in F$, since $y \to y = 1 \in F$ by hypothesis, we get $y \to x \in F$

(**ii**) $\Rightarrow$ (**iii**)**:** Let, $z \to (y \to x) \in F$, by proposition 2.2, we have

$$z \to \big(z \to ((z \to y) \to x)\big) = z \to ((z \to y) \to (z \to x))$$
$$\geq z \to (y \to x)$$

Since F is a filter and $z \to (y \to x) \in F$, $z \to \big(z \to ((z \to y) \to x)\big) \in F$. By hypothesis, we conclude that $z \to ((z \to y) \to x) \in F$ and also by proposition 2.2 $(z \to y) \to (z \to x) \in F$.

(**iii**) $\Rightarrow$ (**iv**)**:** Since F is a filter, $1 \in F$, let $z, z \to (y \to (y \to x))) \in F$, since F is a filter then $y \to (y \to x) \in F$. On the other hand, $y \to x = 1 \to (y \to x) = (y \to y) \to (y \to x)$. The hypothesis implies that $(y \to y) \to (y \to x) \in F$.

(**iv**) $\Rightarrow$ (**i**)**:** let $z \to (y \to x) \in F$ and $z \to y \in F$. We must show that $z \to x \in F$ by proposition2.2 $z \to (y \to x) = y \to (z \to x) \leq (z \to y) \to \big(z \to (z \to x)\big)$. Since F is a filter and z$\to (y \to x) \in F$, we get $(z \to y) \to (z \to (z \to x)) \in F$. $z \to y \in F$ and (iv) imply that $z \to x \in F$. $\square$

Definition 3.15 Let F be a non-empty subset of L and let $a \in A$. Define $F_a = \{x \in A / a \to x \in F\}$

Proposition 3.16 Let F be an implicative filter in residuated lattice Wajsberg algebra if and only if F_a is also a implicative filter in residuated lattice Wajsberg algebra, for all $a \in F$

Proof. Let $x, x \to \in F_a$ for all $a \in A$. Then $a \to (x \to y) \in F$ and $a \to x \in F$, it follows that $a \to y \in F$, we have $y \in F_a$. Let $x,\ x \odot y \in F_a$ for all $a \in A$,then $(a \odot (x \odot y) \in F$ and $a \odot x \in F$, it follows that $a \odot y \in F$, we have$y \in F_a$. Conversely, suppose F_a is an implicative filter in residuated lattice Wajsberg algebra for all $a \in A$. Let $x \to (y \to z) \in F$and $x \to y \in F$, then $y \to z \in F_a$ and$y \in F_a$. Which imply that $z \in F_a$.That is, $x \to z \in F$, and also, $x \odot (y \odot z) \in F$ and $x \odot y \in F$ then $y \odot z \in F_a$ and $y \in F_a$ which imply that $z \in F_a$. That is, $x \odot z \in F$. Hence, F is an implicative filter in residuated lattice Wajsberg algebra. □

CONCLUSION

In this paper**,** we have proved the notion of implicative filter of residuated lattice Wajsberg algebra and discussed some of their properties with illustrations. Also, we have proved that every implicative filter is a filter in residuated lattice Wajsberg algebra and subset of implicative filter is also an implicative filter. Further, we extend this idea as in fuzzy concept.

REFERENCES

[1] R. Balbes and Ph. Dwinger, *Distributive Lattices*, University of Missouri Press, 1974.

[2] M. Basheer Ahamed and A. Ibrahim, *Fuzzy implicative filters of lattice Wajsberg algebras*, Advances in Fuzzy Mathematica 6 (2) (2011), 235-243.

[3] M. Basheer Ahamed and A. Ibrahim, *Anti fuzzy implicative filters in lattice W-algebras*, International Journal of Computational Science and Mathematics, 4 (1) (2012), 49-56.

[4] R. P. Dilworth, *Non-commutative residuated lattices*, Transactions of the American Mathematical Society 46 (1939), 426-444.

[5] J. M. Font, A. J. Rodriguez and A. Torrens, *Wajsberg algebras*, Stochastica, Vol. VIII, 1(1984), 5-31.

[6] P. M. Idziak, *Lattice operations in BCK-algebras*, Mathematica Japonica, 29(1984), 839-846

[7] M. Wajsberg, *Beitragezum Metaaussagenkalkul I*, Monashefte fur Mathematikun Physik 42 (1935), 221-242.

[8] M. Word, R. P. Dilworth, *Residuated lattices*, Transactions of the American Mathematical Society 45 (1939), 335-354.

[9] Young Bae Jun ,*Implicative filters of lattice implication algebras*, Bull. Korean Math.Soc.34 (1997) , No .2,pp. 193-198.

[10] Young Bae Jun,*Fantastic filters of lattice implication algebras*, Internat. J.Math. and Math Sci, Vol.24, No.4 (2000) 277-281.

[11] Young Bae Jun, Yang Xu and Keyun Qin,*Positive implicative and Associative filters of lattice implication algebras.* Bull. Korean. Soc.35 (1998), No .2, 53-61.

[12] L. A. Zadeh, *Fuzzy sets*, Inform. Control 8 (1965), 338-353.