Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 14, Number 5 (2018), pp. 733-741

http://www.ripublication.com

On a New Subclass of Multivalant Functions Defined by Al-Oboudi Differential Operator

[1]Dr.M.Thirucheran, [2]T.Stalin

P.G. and Research Department of Mathematics, L.N. Govt. College, Ponneri-601204-Madras University, Chennai, India.

[1,2]E-mail: drmrhirucheran@gmail.com; goldstaleen@gmail.com,

Abstract

The main object of this paper is investigating a new subclass of normalized multivalent analytic function in the open unit disk U which is defined by Al-Oboudi Differential Operator. We obtain the co-efficient inequality and extreme points and integral means of inequalities for this class are given.

Keywords: Multivalent analytic functions, differential operators, generalized Al-Oboudi differential operator.

1. INTRODUCTION AND DEFINITIONS

The study of Geometric function theory, the multivalent function is a focal area as we can see in recent years; many new articles are written by eminent authors in this area. Now operators of normalized analytic functions become very popular, namely for Differential and Integral. Many articles discuss on operators and new generalization of various authors. Perhaps Ruscheweyh [1] was leading the way in the differential operator who introduced on 1975. It followed by Salagean [2] in 1983 giving another version of differential and integral operator. Many properties have been discussed and studied many researchers for this two operators. In 2004, Al-Oboudi [3] generalized Salagean operator followed by S.Sumer Eker and S.Owa and S.Sumer and Bilal Sekar ([4], [5]). In this study we use these operators to find another type of Differential Operators and obtain co-efficient inequalities and extreme points, and integral means of inequalities.

Definition 1.1

Let A denote the class of functions f normalized by

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \tag{1.1}$$

Which are analytic in the open unit disc $\mathrm{U} = \{\mathrm{z} \in C : |z| < 1\}$.

For $f \in A$, Al-Oboudi [3] introduces the following operator.

$$D^0 f(z) = f(z), \tag{1.2}$$

$$D' f(z) = (1-\delta) f(z) + \delta z f'(z) = D_\delta f(z), \quad \delta \geq 0 \tag{1.3}$$

$$D^n f(z) = D_\delta (D^{n-1} f(z)), \qquad n \in N = 1,2,3,.... \tag{1.4}$$

$$\therefore D^n f(z) = z + \sum_{j=2}^{\infty} [1 + (j-1)\delta]^n a_j z^j \ , \quad n \in N_0 = N \cup \{0\}. \tag{1.5}$$

If $\delta = 1$, then we get Salagean [2] differential operator.

Definition 1.2

Let A_p denote the class of function of $f(z)$ of p - valant analytic function

$$f(z) = z^p + \sum_{j=p+1}^{\infty} a_j z^j \tag{1.6}$$

Which are analytic in the open unit disc $\mathrm{U} = \{\mathrm{z} \in C : |z| < 1\}$.

For $f \in A$, Al-Oboudi [3] introduces the following operator.

$$D^0 f(z) = f(z), \tag{1.7}$$

$$D' f(z) = D\, f(z) = z^p + \sum_{j=p+1}^{\infty} \left(\frac{1 + (j-1)\delta}{p} \right) a_j z^j = D_\delta f(z) \tag{1.8}$$

$$D^n f(z) = D_\delta (D^{n-1} f(z)), \qquad n \in N = 1,2,3,.... \tag{1.9}$$

$$\therefore D^n f(z) = z^p + \sum_{j=p+1}^{\infty} \left(\frac{1 + (j-1)\delta}{p} \right)^n a_j z^j, \quad n \in N_0 = N \cup \{0\}. \tag{1.10}$$

If $\delta = 1$, then we get Salagean [2] differential operator.

Definition: 1.3

Let $N_p(m,n,\alpha,\delta,b)$ denoted the sub class of A_p consisting of functions f which satisfy the inequality

$$\text{Re}\left(1+\frac{1}{b}\left(\frac{D^m f(z)}{D^n f(z)}-1\right)\right)>\alpha \qquad (1.11)$$

For some $0\le\alpha<1, b\in C-\{0\}$, m$\in$N, $n\in N\cup\{0\}$,and all z $\in$U.

2. COEFFICIENT INEQUALITIES

Theorem 2.1

Let $f(z)\in A_p$ satisfies $\sum_{j=p+1}^{\infty}\psi_p(m,n,\alpha,\delta,j,b)|a_j|\le 2(1-\alpha)b$ (2.1)

For some $0\le\alpha<1, b\in C-\{0\}$, m$\in$N, $n\in N\cup\{0\}, \delta\ (\delta\ge 0)$.

Then $f(z)\in N_p(m,n,\alpha,\delta,b)$.

Where

$$\psi_p(m,n,\alpha,\delta,j,b)=\left|\left(\frac{1+(j-1)\delta}{p}\right)^m-(1+\alpha b)\left(\frac{1+(j-1)\delta}{p}\right)^n\right|$$
$$+\left(\left(\frac{1+(j-1)\delta}{p}\right)^m+((2-\alpha)b-1)\left(\frac{1+(j-1)\delta}{p}\right)^n\right) \qquad (2.2)$$

Proof

Suppose that $\psi_p(m,n,\alpha,\delta,j,b)\le 2(1-\alpha)b$ is true for some α $(0\le\alpha<1)$, m $\in$N, $n\in N\cup\{0\}, b\in C-\{0\}, \delta\ (\delta\ge 0)$, then It is suffices to prove that $\left|\frac{F(z)-1}{F(z)+1}\right|<1$ .For $f\in A$, then define the function F(z) by

Let

$$F(z)=1+\frac{1}{b}\left(\frac{D^m f(z)}{D^n f(z)}-1\right)-\alpha \qquad (2.3)$$

$$\therefore\ F(z)-1=1+\frac{1}{b}\left(\frac{D^m f(z)}{D^n f(z)}-1\right)-\alpha-1$$

$$=\left(\frac{D^m f(z)-(1+\alpha b)D^n f(z)}{bD^n f(z)}\right) \tag{2.4}$$

$$\mathrm{F(z)}+1=1+\frac{1}{b}\left(\frac{D^m f(z)}{D^n f(z)}-1\right)-\alpha+1$$

$$=\frac{D^m f(z)-[1-(2-\alpha)b]D^n f(z)}{bD^n f(z)} \tag{2.5}$$

$$\left|\frac{F(z)-1}{F(z)+1}\right|=\left|\frac{D^m f(z)-(1+\alpha b)D^n f(z)}{D^m f(z)-[1-(2-\alpha)b]D^n f(z)}\right|$$

$$=\left|\frac{z^p+\sum_{j=p+1}^{\infty}\left(\frac{1+(j-1)\delta}{p}\right)^m a_j z^j-(1+\alpha b)\left(z^p+\sum_{j=p+1}^{\infty}\left(\frac{1+(j-1)\delta}{p}\right)^n a_j z^j\right)}{z^p+\sum_{j=p+1}^{\infty}\left(\frac{1+(j-1)\delta}{p}\right)^m a_j z^j-(1-(2-\alpha)b)\left(z^p+\sum_{j=p+1}^{\infty}\left(\frac{1+(j-1)\delta}{p}\right)^n a_j z^j\right)}\right|$$

$$=\left|\frac{-\alpha b z^p+\sum_{j=p+1}^{\infty}\left(\left(\frac{1+(j-1)\delta}{p}\right)^m-(1+\alpha b)\left(\frac{1+(j-1)\delta}{p}\right)^n\right)a_j z^j}{(2-\alpha)bz^p+\sum_{j=p+1}^{\infty}\left(\left(\frac{1+(j-1)\delta}{p}\right)^m+((2-\alpha)b-1)\left(\frac{1+(j-1)\delta}{p}\right)^n\right)a_j z^j}\right|$$

$$\therefore\quad \left|\frac{F(z)-1}{F(z)+1}\right|<1$$

$$\left| \frac{-\alpha b z^p + \sum_{j=p+1}^{\infty} \left(\left(\frac{1+(j-1)\delta}{p} \right)^m - (1+\alpha b) \left(\frac{1+(j-1)\delta}{p} \right)^n \right) a_j z^j}{(2-\alpha) b z^p + \sum_{j=p+1}^{\infty} \left(\left(\frac{1+(j-1)\delta}{p} \right)^m + ((2-\alpha)b-1) \left(\frac{1+(j-1)\delta}{p} \right)^n \right) a_j z^j} \right| < 1$$

$$\frac{\left| \sum_{j=p+1}^{\infty} \left(\left(\frac{1+(j-1)\delta}{p} \right)^m - (1+\alpha b) \left(\frac{1+(j-1)\delta}{p} \right)^n \right) a_j z^j \right| + \alpha b}{\left| \sum_{j=p+1}^{\infty} \left(\left(\frac{1+(j-1)\delta}{p} \right)^m + ((2-\alpha)b-1) \left(\frac{1+(j-1)\delta}{p} \right)^n \right) a_j z^j \right| + (2-\alpha) b} < 1$$

$$\left| \sum_{j=p+1}^{\infty} \left(\left(\frac{1+(j-1)\delta}{p} \right)^m - (1+\alpha b) \left(\frac{1+(j-1)\delta}{p} \right)^n \right) a_j z^j \right| + \alpha b$$
$$< \left| \sum_{j=p+1}^{\infty} \left(\left(\frac{1+(j-1)\delta}{p} \right)^m + ((2-\alpha)b-1) \left(\frac{1+(j-1)\delta}{p} \right)^n \right) a_j z^j \right| + (2-\alpha) b$$

$$\sum_{j=p+1}^{\infty} \left(\begin{array}{l} \left| \left(\frac{1+(j-1)\delta}{p} \right)^m - (1+\alpha b) \left(\frac{1+(j-1)\delta}{p} \right)^n \right| \\ \quad + \left(\left(\frac{1+(j-1)\delta}{p} \right)^m + ((2-\alpha)b-1) \left(\frac{1+(j-1)\delta}{p} \right)^n \right) \end{array} \right) |a_j| \le 2(1-\alpha) b$$

Hence $\sum_{j=p+1}^{\infty} \psi_p(m,n,\alpha,\delta,j,b) |a_j| \le 2(1-\alpha) b$

This completes the theorem.

3. EXTREME POINTS

If we define the new subclass $\tilde{N}_p(m,n,\alpha,\delta,b) \subset N_p(m,n,\alpha,\delta,b)$, which consists of the function $f(z) = z^p + \sum_{j=p+1}^{\infty} a_j z^j, (a_j \ge 0)$ (3.1)

Whose Taylor-Maclaurin coefficient satisfy the inequality (2.1)

Now we determine the extreme points of the subclass $\tilde{N}_p(m,n,\alpha,\delta,b)$.

Theorem 3.1

Let $f_p(z) = z^p$ and

$$f_j(z) = z^p + \frac{2(1-\alpha)b\,\varepsilon_j}{\psi_p(m,n,\alpha,\delta,j,b)} z^j, (j = p+1, p+2, \ldots\ldots\ldots, |\varepsilon_j| = 1, \tag{3.2}$$

Then $f \in \tilde{N}_p(m,n,\alpha,\delta,b)$ if and only if it can be expressed in the form

$$f(z) = \lambda_p f_p(z) + \sum_{j=p+1}^{\infty} \lambda_j f_j(z) \tag{3.3}$$

Where $\lambda_j > 0$ and $\lambda_p = 1 - \sum_{j=p+1}^{\infty} \lambda_j$.

Proof

Suppose that

$$f(z) = \lambda_p f_p(z) + \sum_{j=p+1}^{\infty} \lambda_j f_j(z)$$

$$= z^p + \sum_{j=p+1}^{\infty} \lambda_j \frac{2(1-\alpha)b\,\varepsilon_j}{\psi_p(m,n,\alpha,\delta,j,b)} z^j$$

Then

from (2.1)

$$\sum_{j=p+1}^{\infty} \psi_p(m,n,\alpha,\delta,j,b)|a_j| = \sum_{j=p+1}^{\infty} \psi_p(m,n,\alpha,\delta,j,b)\left|\lambda_j \frac{2(1-\alpha)b\,\varepsilon_j}{\psi_p(m,n,\alpha,\delta,j,b)}\right|$$

$$= 2(1-\alpha)b \sum_{j=p+1}^{\infty} |\lambda_j|$$

$$\leq 2(1-\alpha)b \qquad = 2(1-\alpha)b(1-\lambda_p)$$

Which is shows that f satisfies the condition (2.1)

$$\therefore f \in \tilde{N}_p(m,n,\alpha,\delta,b).$$

Conversely ,

Suppose that $f \in \tilde{N}_p(m,n,\alpha,\delta,b)$.

Since ,

$$a_j \leq \frac{2(1-\alpha)b}{\psi(m,n,\alpha,\delta,b)}, (j = p+1, p+2, \ldots) \tag{3.4}$$

Let

$$\lambda_j \le \frac{\psi(m,n,\alpha,\delta,b)}{2(1-\alpha)b\varepsilon_j} a_j \quad \text{and} \quad \lambda_p = 1 - \sum_{j=p+1}^{\infty} \lambda_j$$

Then we obtain

$$f(z) = \lambda_p f_p(z) + \sum_{j=p+1}^{\infty} \lambda_j \; f_j(z)$$

This is completes the proof of theorem 3.1

Corollary 3.2

The extreme points of $\tilde{N}_p(m,n,\alpha,\delta,b)$ are functions $f_p(z) = z^p$ and

$$f_j(z) = z^p + \frac{2(1-\alpha)b\,\varepsilon_j}{\psi_p(m,n,\alpha,\delta,j,b)} z^j, (j = p+1, p+2, \ldots\ldots..) \tag{3.5}$$

4. INTEGRAL MEANS OF INEQUALITIES

Definition 4.1

Let two functions f and g are analytic in U, *we* say that $f(z)$ is subordinate to $g(z)$ if there exist a function $w(z)$ analytic in U satisfying $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$

It is denoted by $f(z) \prec g(z)$.

Lemma 4.2

Let $g(z)$ is univalent in U. Then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.[7]

Theorem 4.3

If f and g are analytic in U with $f(z) \prec g(z)$, then $\int_0^{2\pi} |f(z)|^{\mu} d\theta \le \int_0^{2\pi} |g(z)|^{\mu} d\theta$ for $\mu > 0$ and $z = re^{i\theta}, 0 < r < 1$.(Littlewood [6]

Theorem 4.4

Let $f \in \tilde{N}_p(m,n,\alpha,\delta,b)$ and suppose that $f_j(z)$ is defined by

$f_j(z) = z^p + \frac{2(1-\alpha)b\varepsilon_j}{\psi_p(m,n,\alpha,\delta,j,b)} z^j$,$(j = p+1, p+2,........,|\varepsilon_j| = 1$. If there exist an analytic function $w(z)$ given by $\{w(z)\}^{j-p} = \frac{\psi_p(m,n,\alpha,\delta,b)}{2(1-\alpha)b\varepsilon_j} \sum_{j=p+1}^{\infty} a_j z^{j-p}$, then for $z = re^{i\theta}, 0 < r < 1$, $\int_0^{2\pi} |f(re^{i\theta})|^\mu d\theta \le \int_0^{2\pi} |g(re^{i\theta})|^\mu d\theta$, $\mu > 0$.

Proof

We must show that

$$\int_0^{2\pi} \left| 1 + \sum_{j=p+1}^{\infty} a_j z^{j-p} \right|^\mu d\theta \le \int_0^{2\pi} \left| 1 + \frac{2(1-\alpha)b\varepsilon_j}{\psi_p(m,n,\alpha,\delta,j,b)} z^{j-p} \right|^\mu d\theta$$

By the help of Littlewood subordination theorem, its suffices to shoe that

$$1 + \sum_{j=p+1}^{\infty} a_j z^{j-p} \prec 1 + \frac{2(1-\alpha)b\varepsilon_j}{\psi_p(m,n,\alpha,\delta,j,b)} z^{j-p}$$

Let

$$1 + \sum_{j=p+1}^{\infty} a_j z^{j-p} = 1 + \frac{2(1-\alpha)b\varepsilon_j}{\psi_p(m,n,\alpha,\delta,j,b)} (w(z))^{j-p}$$

Therefore

$$(w(z))^{j-p} = \frac{\psi(m,n,\alpha,\delta,b)}{2(1-\alpha)b\varepsilon_j} \sum_{j=p+1}^{\infty} a_j z^{j-p}$$

Hence $w(0) = 0$.

Furthermore, using (2.1)

$$|(w(z))|^{j-p} = \left| \frac{\psi(m,n,\alpha,\delta,b)}{2(1-\alpha)b\varepsilon_j} \sum_{j=p+1}^{\infty} a_j z \right|^{j-p}$$

$$\le \frac{\psi(m,n,\alpha,\delta,b)}{2(1-\alpha)b} \sum_{j=p+1}^{\infty} |a_j| \, |z|^{j-p}$$

$$\le |z| < 1.$$

Hence theorem completed.

REFERENCES

[1] St.Ruscheweyh(1975), “ *New criteria for Univalent functions”*, Proc.Amer.Math.Soc.49,109-115.

[2] G.Salagean (1983), "*Subclasses Of Univalent Functions*" , Lecture Notes In Maths ,1013,Springer-Verlag,Berlin,362-372.

[3] F.M.Al-Oboudi , "*On univalent functions defined by a generalized Salagean operator"*, International Journal Of Mathematics And Mathematical Sciences, vol.2004,No.27, pp. 1429-1436,2004.

[4] S.Sumer Eker and S.Owa, "*New applications of classes of analytic functions involving the Salagean operator*", in proceedings of the international symposium on complex function theory and applications,pp.21-34, Transilvania University of Printing House, Brasov, Romania, September 2006

[5] S.Sumer Eker and Bilal Sekar, "*On a class of Multivalant Functions defined by Salagean Opreator"*, General Mathematics,Vol.15,Nr.2-3(2007),154-163

[6] J.E. Littlewood, "On inequalities in the theory of functions", Proceedings of society, vol.23, No.1, pp.481-519, 1925.

[7] P.L.Duren, "*Univalent functions"*, Springer New York, NY, USA 1983.