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Second Hankel Determinant for Strongly Bi-Starlike of order α

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Abstract

Let A denote the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in the open unit disc $U = \{z : |z| < 1\}$. Let S denote the class of all functions in A that are univalent in U. A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ denote the class of bi-univalent functions in U. In this paper, we obtained the upper bounds for the second Hankel functional $|a_2a_4 - a_3^2|$ for strongly bi-starlike of order α.

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1. Introduction

Let A denote the family of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and S be the class of functions $f \in A$ that are univalent in U and normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in U). \tag{1.1}$$

The Koebe one-quarter theorem [7] ensures that the image of U under every $f \in S$ contains a disk of radius 1/4. Thus, every $f \in S$ has an inverse function f^{-1} satisfying $f^{-1}(f(z)) = z$ $(z \in U)$ and $f(f^{-1}(w)) = w$ $(|w| < r_0(f);\ r_0(f) \geq 1/4)$, where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \ldots \tag{1.2}$$

A function $f \in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U. Let Σ denote the class of bi-univalent functions in U given by (1.1). The concept of bi-univalent functions was introduced by Lewin [11] in 1967 and he showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [2] improved the result to $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$.

The study of bi-univalent functions was investigated by several authors. They proved the results on coefficient estimates for the initial coefficient$|a_2|$,$|a_3|$and$|a_4|$ ([1], [5], [9], [10], [15]). However, the coefficient estimate for various subclasses of Σ are nonsharp, therefore coefficient estimate problem for each of $|a_n|$ is still an open problem.

In 1976, Noonan and Thomas[13] defined q-th Hankel determinants of f for $q \geq 1$ and $n \geq 1$ which is defined by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \qquad (a_1 = 1)$$

This determinant was discussed by several authors with $q = 2$. For case $q = 2$ and $n = 1$, the functional is well known as Fekete-Szego functional, $H_2(1) = a_3 - a_2^2$ and one usually considers the further generalized functional $a_3 - \mu a_2^2$ where μ is some real number, (see [8]). Estimating for the upper bound of $|a_3 - \mu a_2^2|$ is known as the Fekete-Szego problem. Recently, Orhan *et al.* [14] and Zaprawa ([16],[17]) studied the Fekete-Szego problem for some classes of bi-univalent functions.

The second Hankel determinant $H_2(2)$ is given by $H_2(2) = a_2a_4 - a_3^2$. Recently, the upper bound for the functional $H_2(2) = a_2a_4 - a_3^2$ for the subclasses of Σ are found, see ([4] and [6]). The object of the present paper is to seek the upper bound for the functional $|a_2a_4 - a_3^2|$ for $f \in S^*_\Sigma(\alpha)$ which is defined as follows.

Definition 1.1. **[3]** A function $f(z)$ given by (1.1) belongs to the class $S^*_\Sigma(\alpha)$ with $0 < \alpha \leq 1$, if it satisfies the following conditions:

$$f \in \Sigma \text{ and } \left| arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2}, z \in U$$

and

$$g \in \Sigma \text{ and } \left| arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2}, w \in U$$

where $g(w) = f^{-1}(w)$ as in (1.2).

For special case, when $\alpha = 0$, we have

$$S^*_\Sigma(\alpha) = S^*_\Sigma(0) = S^*_\Sigma$$

which known as bi-starlike functions.

2. Preliminary Result

Let P be the family of all functions p analytic in U for which $Re(p(z)) > 0$ and

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots \tag{2.3}$$

for $z \in U$.

Lemma 2.1. [7] If $p \in P$ then $|c_k| \leq 2$ for each k.

Lemma 2.2. [12] If the function $p \in P$ is given as $p(z) = 1 + \Sigma_{n=1}^{\infty} c_n z^n$, then

$$2c_2 = c_1^2 + x(4 - c_1^2), \tag{2.4}$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c^2)(1 - |x|^2)\zeta \tag{2.5}$$

for some complex value x with $|x| \leq 1$ and some complex value ζ with $|\zeta| \leq 1$.

3. Main Result

The main result for the functions $f \in S^*_\Sigma(\alpha)$ is stated as follows

Theorem 3.1. Let the function $f(z)$ given by (1.1) be in the class $S^*_\Sigma(\alpha)$ with $0 < \alpha \leq 1$. Then

$$|a_2 a_4 - a_3^2| = \begin{cases} \alpha^2, & 0 < \alpha \leq \frac{1}{7} \\ \frac{\alpha^2(7\alpha^2 - 28\alpha)}{56\alpha^2 - 42\alpha + 1}, & \frac{1}{7} \leq \alpha \leq \frac{21+\sqrt{889}}{112} \\ \frac{\alpha^2(56\alpha^2 + 4)}{9}, & \frac{21+\sqrt{889}}{112} \leq \alpha \leq 1. \end{cases} \tag{3.6}$$

Proof. Let $f \in S^*_\Sigma(\alpha)$, $0 < \alpha \leq 1$ and $g = f^{-1}$. Then

$$\frac{zf'(z)}{f(z)} = [p(z)]^\alpha \quad \text{and} \quad \frac{wg'(w)}{g(w)} = [q(w)]^\alpha \tag{3.7}$$

where $p, q \in P$ and defined by $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$ and $q(z) = 1 + q_1 z + q_2 z^2 + q_3 z^3 + \cdots$.

Now, upon equating the coefficients in (3.7), we have

$$a_2 = \alpha p_1 \tag{3.8}$$

$$2a_3 - a_2^2 = \alpha p_2 + \frac{\alpha(\alpha-1)}{2} p_1^2 \tag{3.9}$$

$$3a_4 + a_2^3 - 3a_2a_3 = \alpha p_3 + \alpha(\alpha-1)p_1p_2 + \frac{\alpha(\alpha-1)(\alpha-2)}{6} p_1^3 \tag{3.10}$$

and

$$-a_2 = \alpha q_1 \tag{3.11}$$

$$a_2^2 - 4a_3 = \alpha q_2 + \frac{\alpha(\alpha-1)}{2} q_1^2 \tag{3.12}$$

$$-3a_4 - 10a_2^3 + 12a_2a_3 = \alpha q_3 + \alpha(\alpha-1)q_1q_2 + \frac{\alpha(\alpha-1)(\alpha-2)}{6} q_1^3 \tag{3.13}$$

From (3.3) and (3.6), we find that

$$p_1 = -q_1 \tag{3.14}$$

and

$$a_2 = \alpha p_1 \tag{3.15}$$

Now, from (3.4) and (3.7) and (3.10), we get

$$a_3 = \alpha^2 p_1^2 + \frac{\alpha(p_2 - q_2)}{4} \tag{3.16}$$

Also, form (3.5) and (3.8), we find that

$$a_4 = \frac{2\alpha^3 p_1^3}{3} + \frac{\alpha(\alpha-1)(\alpha-2)p_1^3}{18} + \frac{5\alpha^2 p_1(p_2 - q_2)}{8} + \frac{\alpha(\alpha-1)p_1(p_2+q_2)}{6} + \frac{\alpha(p_3 - q_3)}{6} \tag{3.17}$$

Thus, we can easily establish that

$$|a_2a_4 - a_3^2| = \Big| -\frac{\alpha^4 p_1^4}{3} + \frac{\alpha^2(\alpha-1)(\alpha-2)p_1^4}{18} + \frac{\alpha^3 p_1^2(p_2-q_2)}{8} + \frac{\alpha^2(\alpha-1)p_1^2(p_2+q_2)}{6} + \frac{\alpha^2 p_1(p_3-q_3)}{6} - \frac{\alpha^2(p_2-q_2)^2}{16} \Big| \tag{3.18}$$

According to Lemma 2.2 and (3.9), we have

$$\left.\begin{aligned} 2p_2 &= p_1^2 + x(4-p_1^2) \\ 2q_2 &= q_1^2 + y(4-q_1^2) \end{aligned}\right\} \Rightarrow p_2 - q_2 = \frac{4-p_1^2}{2}(x-y) \tag{3.19}$$

$$\left.\begin{aligned} 2p_2 &= p_1^2 + x(4-p_1^2) \\ 2q_2 &= q_1^2 + y(4-q_1^2) \end{aligned}\right\} \Rightarrow p_2 + q_2 = p_1^2 + \frac{4-p_1^2}{2}(x+y) \tag{3.20}$$

and

$$4p_3 = p_1^3 + 2p_1(4-p_1^2)x - p_1(4-p_1^2)x^2 + 2(4-p_1^2)(1-|x|^2)z$$
$$4q_3 = q_1^3 + 2q_1(4-q_1^2)y - q_1(4-q_1^2)y^2 + 2(4-q_1^2)(1-|y|^2)w$$

therefore, we have

$$\begin{aligned} p_3 - q_3 = \frac{p_1^3}{2} + \frac{p_1(4-p_1^2)}{2}(x+y) - \frac{p_1(4-p_1^2)}{4}(x^2+y^2) \\ + \frac{(4-p_1^2)[(1-|x|^2)z - (1-|y|^2)w]}{2} \end{aligned} \tag{3.21}$$

for some x, y, z and w with $|x| \le 1$,$|y| \le 1$,$|z| \le 1$ and $|w| \le 1$. By using (3.14), (3.15) and (3.16) in (3.13) and applying triangle inequality, we have

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| = \Big| &-\frac{\alpha^4 p_1^4}{3} + \frac{\alpha^2(\alpha-1)(\alpha-2)p_1^4}{18} + \frac{\alpha^3 p_1^2}{8}\left[\frac{4-p_1^2}{2}(x-y)\right] \\ &+ \frac{\alpha^2(\alpha-1)p_1^2}{6}\left[p_1^2 + \frac{4-p_1^2}{2}(x+y)\right] \\ &+ \frac{\alpha^2 p_1}{6}\Big[\frac{p_1^3}{2} + \frac{p_1(4-p_1^2)}{2}(x+y) - \frac{p_1(4-p_1^2)}{4}(x^2+y^2) \\ &+ \frac{(4-p_1^2)[(1-|x|^2)z - (1-|y|^2)w]}{2}\Big] - \frac{\alpha^2}{16}\left[\frac{4-p_1^2}{2}(x-y)\right]^2\Big| \\ \le &\frac{\alpha^4 p_1^4}{3} + \frac{\alpha^2 p_1^4}{12} + \frac{\alpha^2(\alpha-1)(\alpha-2)p_1^4}{18} + \frac{\alpha^2(\alpha-1)p_1^4}{6} + \frac{\alpha^2(4p_1 - p_1^3)}{6} \\ &+ \frac{(7\alpha)\alpha^2 p_1^2(4-p_1^2)}{48}(|x|+|y|) + \frac{\alpha^2 p_1(4-p_1^2)(p_1-2)}{24}(|x|^2+|y|^2) \\ &+ \frac{\alpha^2(4-p_1^2)^2}{64}(|x|+|y|)^2 \end{aligned} \tag{3.22}$$

From Lemma 2.1, since $p \in P$, we obtain $|p_1| \le 2$. Letting $p_1 = p$, we may assume without restriction that $p \in [0, 2]$. Thus, for $\gamma_1 = |x| \le 1$ and $\gamma_2 = |y| \le 1$, we obtain

$$|a_2a_4 - a_3^2| \le T_1 + T_2(\gamma_1+\gamma_2) + T_3(\gamma_1^2+\gamma_2^2) + T_4(\gamma_1+\gamma_2)^2 = F(\gamma_1, \gamma_2)$$

where

$$T_1 = T_1(p) = \frac{\alpha^4 p^4}{3} + \frac{\alpha^2 p^4}{12} + \frac{\alpha^2(\alpha-1)(\alpha-2)p^4}{18} + \frac{\alpha^2(\alpha-1)p^4}{6} + \frac{\alpha^2(4p-p^3)}{6} \geq 0$$

$$T_2 = T_2(p) = \frac{(7\alpha)\alpha^2 p^2(4-p^2)}{48} \geq 0$$

$$T_3 = T_3(p) = \frac{\alpha^2 p(4-p^2)}{24}(p-2) \leq 0$$

$$T_4 = T_4(p) = \frac{\alpha^2(4-p^2)^2}{64} \geq 0$$

We now need to maximize $F(\gamma_1, \gamma_2)$ in the closed square $\mathbb{S} = \{(\gamma_1, \gamma_2) : 0 \leq \gamma_1 \leq 1, 0 \leq \gamma_2 \leq 1\}$ for $p \in [0,2]$. Thus, we must investigate the maximum of $F(\gamma_1, \gamma_2)$ according to $p \in (0,2)$,$p = 0$ and $p = 2$ taking into account the sign of $F_{\gamma_1\gamma_1}F_{\gamma_2\gamma_2} - (F_{\gamma_1\gamma_2})^2$.

Firstly, we let $p \in (0,2)$. Since $T_3 \leq 0$ and $T_3 + 2T_4 > 0$ for $p \in (0,2)$, we can conclude that $F_{\gamma_1\gamma_1}F_{\gamma_2\gamma_2} - (F_{\gamma_1\gamma_2})^2 < 0$. Thus, the function F cannot have a local maximum in the interior of the square $\mathbb{S}$. Now, we investigate the maximum of F on the boundary of the square $\mathbb{S}$.

For $\gamma_1 = 0$ and $0 \leq \gamma_2 \leq 1$ (similarly to $\gamma_2 = 0$ and $0 \leq \gamma_1 \leq 1$) we obtain

$$F(0, \gamma_2) = G(\gamma_2) = T_1 + T_2\gamma_2 + (T_3 + T_4)\gamma_2^2$$

Case 1. Let $T_3 + T_4 \leq 0$. In this case for $0 < \gamma_2 < 1$ and any fixed p with $0 < p < 2$, it is clear that $G'(\gamma_2) > 0$ and therefore we can say that $G(\gamma_2)$ is an increasing function. Hence, for fixed $p \in (0,2)$, the maximum of $G(\gamma_2)$ occurs at $\gamma_2 = 1$ and

$$\max\{G(\gamma_2)\} = G(1) = T_1 + T_2 + T_3 + T_4\ .$$

Case 2. Let $T_3 + T_4 < 0$. Since $T_2 + 2(T_3 + T_4) \geq 0$ for $0 < \gamma_2 < 1$ and any fixed p with $0 < p < 2$, it is clear that $T_2 + 2(T_3 + T_4) < T_2 + 2(T_3 + T_4)\gamma_2 < T_2$ and so $G'(\gamma_2) > 0$. Hence, for fixed $p \in (0,2)$, the maximum of $G(\gamma_2)$ occurs at $\gamma_2 = 1$ and

$$F(\gamma_1, \gamma_2) = F(0,1) = G(1) = T_1 + T_2 + T_3 + T_4\ .$$

For $\gamma_1 = 1$ and $0 \leq \gamma_2 \leq 1$ (similarly to $\gamma_2 = 1$ and $0 \leq \gamma_1 \leq 1$) we obtain

$$F(1, \gamma_2) = H(\gamma_2) = T_1 + T_2 + T_3 + T_4 + (T_2 + 2T_4)\gamma_2 + (T_3 + T_4)\gamma_2^2$$

According to the above cases of $T_3 + T_4$, we get

$$\max\{H(\gamma_2)\} = H(1) = T_1 + 2T_2 + 2T_3 + 4T_4\ .$$

Notice that, $G(1) \leq H(1)$ for $p \in (0,2)$, therefore maximum $F(\gamma_1, \gamma_2) = F(1,1)$ on the boundary of the square $\mathbb{S}$. Thus, the maximum of $F(\gamma_1, \gamma_2)$ occurs at $\gamma_1 = 1$ and $\gamma_2 = 1$ in the closed square $\mathbb{S}$.

Let $\Phi : (0,2) \to \mathbb{R}$

$$\Phi(p) = max\ \{F(\gamma_1, \gamma_2)\} = F(1,1) = T_1 + 2T_2 + 2T_3 + 4T_4 \tag{3.23}$$

Substituting the values of T_1, T_2, T_3 and T_4 into the function $\Phi(p)$ given by (3.18), we get

$$\Phi(p) = \frac{\alpha^2}{144}\left[(56\alpha^2 - 42\alpha + 1)p^4 + (168\alpha - 24)p^2 + 144\right] .$$

Let

$$P = 56\alpha^2 - 42\alpha + 1, \quad Q = 168\alpha - 24, \quad \text{and} \quad R = 144 .$$

Then, since

$$\max_{0\le t\le 4} (Pt^2 + Qt + R) = \begin{cases} R & \left(Q \le 0; P \le -\dfrac{Q}{4}\right), \\ \dfrac{4PR - Q^2}{4P} & \left(Q \ge 0; P \le -\dfrac{Q}{8} \text{ or } Q \le 0; P \ge -\dfrac{Q}{4}\right) \\ 16P + 4Q + R & \left(Q > 0; P \ge -\dfrac{Q}{8}\right). \end{cases} \tag{3.24}$$

Thus, we have

$$|a_2a_4 - a_3^2| \le \frac{\alpha^2}{144}\begin{cases} 144 & \left(0 < \alpha \le \dfrac{1}{7}\right), \\ \dfrac{1008\alpha^2 - 4032\alpha}{56\alpha^2 - 42\alpha + 1} & \left(\dfrac{1}{7} \le \alpha \le \dfrac{21+\sqrt{889}}{112}\right), \\ 896\alpha^2 + 64 & \left(\dfrac{21+\sqrt{889}}{112} \le \alpha \le 1\right). \end{cases}$$

This completes the proof of Theorem 3.1. ■

For $\alpha = 1$ in Theorem 3.1, we obtain the coefficient estimate given by the following corollary.

Corollary 3.2. [6] Let $f(z)$ given by (1.1) be in the class S^*_Σ. Then

$$|a_2a_4 - a_3^2| \le \frac{20}{3}$$

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