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Upper Pendant Domination in Graphs

Nayaka S. R. and Puttaswamy
Department of Mathematics,
P.E.S. College of Engineering, Mandya.

Purushothama S.
Department of Mathematics,
Maharaja Institute of Technology, Mysore.

Abstract

Let G be any graph. A dominating set S in G is called a pendant dominating set if $\langle S \rangle$ contains at least one pendant vertex. The minimal pendant dominating set with maximum cardinality is called the upper pendant dominating set. The cardinality of an upper pendant dominating set is called upper pendant domination number, denoted by $\Gamma_{pe}(G)$. In this paper, we initiate the study of this parameter. Exact value of $\Gamma_{pe}(G)$ is determined for some families of standard graphs and we estimate some upper and lower bounds. Further, we study the inter-relation with other parameters and the properties of this parameter.

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1. Introduction and Definitions

Let $G = (V, E)$ be any graph of order n and size m. The concept of paired domination is an interesting concept introduced by Teresa W. Haynes with the following application in mind. if we think of each vertex v as the possible location for a guard capable of protecting each vertex in its closed neighborhood, then $domination$ requires every vertex to be protected. For total domination, each guard must, in turn, be protected by other guard. But for paired-domination, each guard is assigned another adjacent one, and they are designated as backups for each other. We introduce pendant domination for which at least one guard is assigned a backup. In this paper by graph, we mean a simple, finite and undirected graph without isolated vertices.

For each vertex $v \in V$, the open neighborhood of v is the set $N(v)$ containing all the vertices u adjacent to v and the closed neighborhood of v is the set $N[v]$ containing v and all the vertices u adjacent to v. Let S be any subset of V, then the open neighborhood of S is $N(S) = \cup_{v \in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$.

A set of vertices is called a dominating set if any vertex not in S is adjacent to at least one vertex in S. The minimum cardinality of a dominating set is called the domination number, denoted by $\gamma(G)$. The minimal dominating set of maximum order is called upper dominating set, denoted by $\Gamma(G)$. The dominating set of cardinality $\Gamma(G)$ is called the $\Gamma(G)-$set. The minimum and maximum of the degree among the vertices of G is denoted by $\delta(G)$ and $\Delta(G)$ respectively. A graph G is said to be regular if $\delta(G) = \Delta(G)$. A vertex v of a graph G is called a *cut vertex* if its removal increases the number of components. A *bridge* or *cut edge* of a graph is an edge whose removal increases the number of components. A vertex of degree zero is called an isolated vertex and a vertex of a degree one is called a pendant vertex. An edge incident to a pendant vertex is called a pendant edge. A complete bipartite graph $K_{1,3}$ is a tree called as *claw*. Any graph containing no subgraph isomorphic to $K_{1,3}$ is called a claw-free graph.

The corona of two disjoint graphs G_1 and G_2 is defined to be the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2. If G and H are disjoint graphs, then the join of G and H denoted by $G \vee H$ is the graph such that $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The line graph $L(G)$ of a graph G is the graph whose vertex set corresponds to the edges of G such that two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent.

Any graph G with at least one bridge is called a bridged graph. The n-Barbell graph is the simple graph obtained by connecting two copies of a complete graph K_n by a bridge.The n Pan graph is the graph obtained by joining a cycle graph C_n to a singleton graph K_1 with a bridge.The ladder graph is a Cartesian product of P_2 and P_n where P_n is a path graph.

Definition 1.1. A dominating set S in G is called a pendant dominating set if $\langle S \rangle$ contains at least one pendant vertex. The minimum cardinality of a pendant dominating set is called the pendent domination number denoted by $\gamma_{pe}(G)$.

2. The Upper Pendant Domination Number of a Graph

Definition 2.1. The minimal pendant dominating set with maximum cardinality is called the upper pendant dominating set. The cardinality of an upper pendant dominating set is called an upper pendant domination number, denoted by $\Gamma_{pe}(G)$. Any upper pendant dominating set of cardinality $\Gamma_{pe}(G)$ is called the $\Gamma_{pe}-$set.

Example 2.2.

1. Let G be a complete graph. Then $\Gamma_{pe}(G) = 2$.

2. Let $G \cong K_{m_1,m_2,\ldots m_k}$ be a complete multipartite graph. Then $\Gamma_{pe}(G) = 2$.

3. Let G be a Barbell graph. Then $\Gamma_{pe}(G) = 3$.

Observation 2.3. Let G be totally disconnected. Then upper pendant domination is not defined for G.

From the definition of the pendant domination, it is clear that the parameter $\Gamma_{pe}(G)$ is defined if G has at least one edge. Thus, hereafter by a graph, we mean a graph having at least one edge.

Observation 2.4.

- $\gamma_{pe}(K_n) = \Gamma_{pe}(K_n)$ for all n.

- $\gamma_{pe}(P_n) = \Gamma_{pe}(P_n)$ if and only if $n = 2$ or 3 and $\gamma_{pe}(C_n) = \Gamma_{pe}(C_n)$ if and only if $n \leq 6$.

- $\gamma(K_{m,n}) = \gamma_{pe}(K_{m,n}) = \Gamma(K_{m,n}) = \Gamma_{pe}(K_{m,n})$ if and only if $m, n = 2$ and $\gamma_{pe}(K_{m,n}) = \Gamma_{pe}(K_{m,n})$ for all $m, n \geq 1$.

Theorem 2.5. Let $G \cong K_m(a_1, a_2, \ldots, a_m)$ be a multi star graph. Then

$$\Gamma_{pe}(G) = 2 + max_{1 \leq i \leq m}\, \Sigma_{j=1, j \neq i}^{m} a_j.$$

Proof. Let $G \cong K_m(a_1, a_2, \ldots, a_m)$ be a multi star of order $a_1 + a_2 + \cdots + a_m + m$. Assume that $a_1 \leq a_2 \leq \cdots \leq a_m$. Then, the collection S of all leaves will be an upper dominating set in G and hence $\Gamma(G) = a_1 + a_2 + \cdots + a_m$. Picking an edge uv from the star K_{a_1} and taking the leaves of G not in K_{a_1}, the set $S' = (S - V(K_{a_1})) \cup \{u, v\}$ will be a pendant dominating set in G. As the vertices in S' are leaves and contains exactly one edge, S' will be a minimal pendant dominating set of maximum cardinality. Therefore, $\Gamma_{pe}(G) = |S'| = 2 + \sum_{i=2}^{m} a_i$. In general, by the maximality, we have $\Gamma_{pe}(G) = 2 + max_{1 \leq i \leq m}\, \Sigma_{j=1, j \neq i}^{m} a_j$. ■

Corollary 2.6. For any integer $k \geq 3$, there exists a graph G such that $\Gamma_{pe}(G) = k$.

Theorem 2.7. Let $G \cong P_n$ be a path of order $n \geq 2$. Then

$$\Gamma_{pe}(G) = \begin{cases} 2, & \text{if n=2;} \\ 2 + \left\lceil \dfrac{n-3}{2} \right\rceil, & \text{otherwise.} \end{cases}$$

Proof. Let $G \cong P_n$ be a path and let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Clearly, $\Gamma_{pe}(P_2) = 2$. Suppose $n \geq 3$. Since any upper pendant dominating set should contain a pendant vertex, we may fix an edge $\{v_1, v_2\}$ in G and let $H = V(G) - N[v_1, v_2]$. Then $\Gamma_{pe}(G) = 2 + \Gamma(H)$ where $H \cong P_{n-3}$. Therefore, $\Gamma_{pe}(G) = 2 + \left\lceil \frac{n-3}{2} \right\rceil$. ■

Theorem 2.8. Let C_n be a cycle of order $n \geq 3$. Then $\Gamma_{pe}(C_n) = \left\lceil \frac{n}{2} \right\rceil$.

Proof. Let C_n be a cycle and let $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of C_n. Fix an arbitrary edge, say uv in C_n and let $H = V(G) - N[u, v]$. Then $\Gamma_{pe}(G) = 2 + \Gamma(H)$ where $H \cong P_{n-4}$. Therefore, $\Gamma_{pe}(G) = 2 + \left\lceil \frac{n-4}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil$. ■

Theorem 2.9. Let G be a Pan graph with $n \geq 5$ vertices. Then $\Gamma_{pe}(G) = 2 + \left\lceil \frac{n-3}{2} \right\rceil$.

Proof. Let G be a pan graph and $V(G) = \{v_1, v_2, \ldots, v_n\}$ where v_n is the vertex attached to the vertex v_1 of C_{n-1}. Fix the edge $\{v_1, v_n\}$, then for Γ-set of H, where H is the graph obtained by removing $\{v_1, v_n\}$ and its neighbours from G. Clearly, $H \cong P_{n-3}$ and so $\Gamma_{pe}(G) = 2 + \left\lceil \frac{n-3}{2} \right\rceil$. ■

Theorem 2.10. Let G_1 and G_2 and be any two graphs. Then

$$\Gamma_{pe}(G_1 \vee G_2) = max\{\Gamma_{pe}(G_1), \Gamma_{pe}(G_2)\}.$$

Proof. Let G_1 and G_2 be any two graphs and let S_1, S_2 be the $\Gamma_{pe}-$sets of G_1 and G_2 respectively. By the definition of join of graphs, S_1 and S_2 are minimal pendant dominating sets of $G_1 \vee G_2$ and so $\Gamma_{pe}(G_1 \vee G_2) \geq max\{\Gamma_{pe}(G_1), \Gamma_{pe}(G_2)\}$. Let v be any vertex of $G_1 \vee G_2$. Assume $v \in V(G_1)$. Then $S_1 \cup \{v\}$ fails to be a minimal pendant dominating set. On the other hand, $S_2 \cup \{v\}$ fails to be dominating set in $G_1 \vee G_2$. Thus, $\Gamma_{pe}(G_1 \vee G_2) \leq max\{\Gamma_{pe}(G_1), \Gamma_{pe}(G_2)\}$. Therefore, we have $\Gamma_{pe}(G_1 \vee G_2) = max\{\Gamma_{pe}(G_1), \Gamma_{pe}(G_2)\}$. ■

Proposition 2.11. Let G be any graph of size at least one. Then $\Gamma_{pe}(G \vee \overline{K_n}) = \Gamma_{pe}(G)$.

Proof. Let G be any graph of size at least one and let S be the $\Gamma_{pe}-$set of G. Then S is also a minimal dominating set of $G \vee \overline{K_n}$ and so $\Gamma(G \vee \overline{K_n}) \geq |S|$. On the other hand, for any vertex v of $G \vee \overline{K_n}$, the set $S \cup \{v\}$ will not be minimal. This proves that, $\Gamma(G \vee \overline{K_n}) \leq |S|$ and hence $\Gamma_{pe}(G \vee \overline{K_n}) = |S| = \Gamma_{pe}(G)$. ■

Corollary 2.12. Let G be an $m-$gonal $n-$cone graph. Then $\Gamma_{pe}(G) = \left\lceil \frac{m}{2} \right\rceil$.

Proof. Let G be an $m-$gonal $n-$cone graph. Then G is the graph join of the cycle graph C_m with $\overline{K_n}$. Taking G to be the cycle graph on m vertices in the above theorem, we get

$\Gamma_{pe}(G) = \left\lceil \frac{m}{2} \right\rceil$. ■

Corollary 2.13. For a wheel W_n of order $n \geq 4$, $\Gamma_{pe}(W_n) = \left\lceil \frac{n-1}{2} \right\rceil$.

Proof. Let G be a wheel graph of order $n \geq 4$. Then $G \cong C_{n-1} + \overline{K_n}$. Therefore, ny taking G to be the cycle on $n-1$ vertices in proposition 2.11, we obtain that $\Gamma_{pe}(W_n) = \left\lceil \frac{n-1}{2} \right\rceil$. ■

For a wheel W_n of order n, the line graph $G \cong L(W_n)$ is a bi-regular graph on $2(n-1)$ vertices such that degree of any vertex in G belongs to the set $\{n-1, n\}$.

Theorem 2.14. For a wheel W_{n+1} of order $n \geq 4$, $\Gamma_{pe}(L(W_{n+1})) = \left\lceil \frac{n+1}{2} \right\rceil$.

Theorem 2.15. Let G be a disconnected graph with components $G_1, G_2, \ldots, G_m$. Then $\Gamma_{pe}(G) = min_{1\leq i\leq m}\{\Gamma_{pe}(G_i) + \sum_{j=1, j\neq i}^{m} \Gamma(G_j)\}$.

Proof. We prove this result by using mathematical induction. The result is trivially true for $m = 1$. Suppose $m = 2$. Then $G = G_1 \cup G_2$. Let S_1, S_2 be the $\Gamma_{pe}-$sets of G_1 and G_2 respectively. Then, $S_1 \cup S_2'$ and $S_2 \cup S_1'$ are pendant dominating sets in G, where S_i' denotes the $\Gamma-$set of G_i, $i = 1, 2$. Therefore $\Gamma_{pe}(G) \leq min\{\Gamma_{pe}(G_1) + \Gamma(G_2), \Gamma_{pe}(G_2) + \Gamma(G_1)\}$.

On the other hand, let S be any pendant dominating set in G. Then S has to dominate both $V(G_1)$ and $V(G_2)$ and $\langle S\rangle$ should contain at least one pendant vertex. Moreover, the set S should contain the pendant dominating set of G_1 or G_2. Otherwise $\langle S\rangle$ contains no pendant vertex which is a contradiction. This contradiction shows that $|S| \geq min\{\Gamma_{pe}(G_1) + \Gamma(G_2), \Gamma_{pe}(G_2) + \Gamma(G_1)\}$. Hence, $|S| = min\{\Gamma_{pe}(G_1) + \Gamma(G_2), \Gamma_{pe}(G_2) + \Gamma(G_1)\}$, proving the result for $m = 2$.

Next, suppose $m \geq 3$ and assume that the result is true for $m = k-1$. Let G be any graph with the components $G_1, G_2, \ldots, G_{k-1}, G_k$. Let G' be a graph with $k-1$ components, say $G_1, G_2, \ldots, G_{k-1}$. Then from the induction hypothesis we have $\Gamma_{pe}(G') = min_{1\leq i\leq k-1}\{\Gamma_{pe}(G_i) + \Sigma_{j=1, j\neq i}^{k-1}\Gamma(G_j)\}$. Now, we have $G = G' \cup G_m$. That is, G is the graph having only two components namely G' and G_m. Hence from the case $m = 2$, we obtain that $\Gamma_{pe}(G) = min_{1\leq i\leq k}\{\Gamma_{pe}(G_i) + \Sigma_{j=1, j\neq i}^{m}\Gamma(G_j)\}$. Therefore the result is true for $m = k$ and hence true for any positive integer m. Thus we have

$$\Gamma_{pe}(G) = min_{1\leq i\leq r}\{\Gamma_{pe}(G_i) + \sum_{j=1, j\neq i}^{m} \Gamma(G_j)\}.$$ ■

Let G_1 and G_2 be any two graphs. Then the cartesian product of G_1 and G_2 is denoted by $G_1 \square G_2$ and defined to be the graph G where the vertices $u = (u_1, u_2)$ and

(v_1, v_2) are adjacent if $u_1 = v_1$ and u_2 adjacent to v_2 or $u_2 = v_2$ and u_1 adjacent to v_1. The graph $P_m \Box P_n$ is called a grid graph and $C_n \Box P_2$ is called a prism graph.

Theorem 2.16. Let $G \cong P_m \Box P_n$ be a grid graph. Then $\Gamma_{pe}(G) = n\lfloor \frac{m}{2} \rfloor$.

Proof. Let $G \cong P_m \Box P_n$ be a grid graph and let $V(G) = \{u_{ij} | 1 \leq i \leq m, 1 \leq j \leq n\}$. Choose the minimum dominating set S' in one copy of P_m and let S be the set of all vertices in the row to which the vertex of S' belongs to. Then, S is a minimal pendant dominating set in G and further, for no vertex in $V - S$, the set $S \cup \{v\}$ will be a minimal pendant dominating set. Therefore, $\Gamma_{pe}(G) = |S| = n\lfloor \frac{m}{2} \rfloor$. ∎

Corollary 2.17. Let G be Ladder Graph. Then $\Gamma_{pe}(G) = n$.

Proof. Let $G \cong P_2 \Box P_n$ be a ladder graph and let $V(G) = \{(u_i, v_i) | 1 \leq i \leq n\}$. Fix an edge $e = u_1v_1$ of G and let H be the graph obtained by removing the vertices u_1, v_1 and its neighbors from G. Then, $H \cong P_2 \Box P_{n-2}$ and so $\Gamma_{pe}(P_2 \Box P_n) = 2 + \Gamma(P_2 \Box P_{n-2}) = (n-2) + 2 = n$. ∎

Proposition 2.18. Let $G \cong P_n \Box K_m$. Then $\Gamma_{pe}(G) = n + 1$.

Proof. Let $G \cong P_n \Box K_m$ be a graph of order $2n$ where K_n be a complete graph of order n and $\gamma(K_n) = 1$. Choosing two vertices from one copy of K_n and exactly one vertex from other copies of K_n, we obtain the minimal pendant dominating set of G. In fact, this set would be a minimal pendant dominating set of maximal cardinality. Therefore, $\Gamma_{pe}(G) = n + 1$. ∎

Theorem 2.19. Let G be a stacked book graph. Then

$$\Gamma_{pe}(G) = \begin{cases} 2, & \text{if n = 1;} \\ \dfrac{m(n+1)}{2}, & \text{if } n \geq 3 \text{ and odd;} \\ \dfrac{mn}{2}, & \text{if n is even.} \end{cases}$$

Proof. Let G be a stacked book graph. Then G is the product graph of $K_{1,m}$ with P_n, and hence G contains m copies of the path P_n attached to one copy of P_n obtained by joining the centers of the star $K_{1,m}$ and call it as the graph H. Suppose $n = 1$, then G is a star and so $\Gamma_{pe}(G) = 2$. Assume $n \geq 2$. Let G' be the graph obtained by deleting vertices of H from G. Then G' is the union of m copies of the path P_n. Moreover, $\Gamma_{pe}(G) = \Gamma_{pe}(G')$. From Theorem 2.8, it is clear that the upper pendant dominating set is obtained by choosing upper pendant dominating set in one copy of P_n and upper dominating set from other copies of path P_n. Therefore, $\Gamma_{pe}(G) = (m-1)\left\lceil \frac{n}{2} \right\rceil + 2 + \left\lceil \frac{n-3}{2} \right\rceil$. Suppose n even, then $n = 2k$, for some integer k. Substituting for n, we get $\Gamma_{pe}(G) = \frac{mn}{2}$. Similarly, whenever n odd, we obtain that $\frac{m(n+1)}{2}$. ∎

Corollary 2.20. Let G be a book graph of order $2m$. Then $\Gamma_{pe}(G) = m$.

Proof. Let G be a book graph. Then G is the graph Cartesian product of the star $K_{1,m}$ with P_2. Hence, taking $n = 2$ in the above theorem, we obtain that $\Gamma_{pe}(G) = m$. ■

Theorem 2.21. Let G be a prism graph of order $2n$. Then

$$\Gamma_{pe}(G) = \begin{cases} n-1, & \text{if n is even;} \\ n, & \text{if n is odd.} \end{cases}$$

Proof. Let G be a prism graph of order $2n$, then $G \cong C_n \Box P_2$. Let (u, v) be any pair of adjacent vertices in G. Assume that u and v are from outside cycle. Suppose n odd, select $\dfrac{n-3}{2}$ non-adjacent vertices from outside cycle not in the neighborhood of u and v. Then, exactly $\dfrac{n-1}{2}$ in the cycle inside not dominated by any of the vertices in G. Therefore, the collection of these vertices will be a minimal dominating set in G. Since, we are selecting alternative vertices, the collection will be a minimal dominating set of maximum cardinality. Hence, $\Gamma_{pe}(G) = n$. Next, suppose n even. As in the above case, Let S and S' be the upper dominating sets of the inner and outer cycles on removing the vertices u, v and its neighbors from G. Then, $\Gamma_{pe}(G) = 2 + |S| + |S'|$. That is, $\Gamma_{pe}(G) = 2 + \dfrac{n-4}{2} + \dfrac{n-2}{2} = n - 1$. ■

Since the prism graph $G \cong C_n \Box P_2$ consists of two cycles, upper pendant dominating set may be chose by taking upper pendant dominating set in one copy of C_n and upper dominating set from another copy. Therefore, $\Gamma_{pe}(G) = 2\Gamma_{pe}(C_n) - 1$. Generally, the graph $C_n \Box P_m$ consists of $m-$cycles each of order n.

3. Bounds for $\Gamma_{pe}(G)$

Theorem 3.1. Let G be any connected graph of order n. Then $2 \leq \Gamma_{pe}(G) \leq n$. Further, $\Gamma_{pe}(G) = n$ if and only if $G \cong K_2$ and $\Gamma_{pe}(G) = 2$ if and only if G contains an edge of degree at least $n - 2$.

Proof. Let G be any connected graph of order n. The inequalities are trivial. Suppose $\Gamma_{pe}(G) = n$. Suppose $n > 2$. Then, there exists a vertex $v \in V$ such that $V - \{v\}$ will be an upper pendant dominating set in G. Therefore $n = 2$ and as the graph G is connected, we must have $G \cong K_2$. Next, suppose $\Gamma_{pe}(G) = 2$ and let $\{u, v\}$ be a $\Gamma_{pe}-$set in G. Further, as the set contains a pendant vertex, u and v must be adjacent and the uv must have degree at least $n - 2$. ■

Theorem 3.2. If G is a connected graph on n vertices. Then $\Gamma_{pe}(G) \leq \left\lceil \dfrac{n}{2} \right\rceil + 1$. Equality holds if $G \cong P_{2k}$.

Theorem 3.3. Let G be a connected graph of order n.Then $\Gamma_{pe}(G) = n - 1$ if and only if G is one of the graphs P_3, K_3, P_4.

Proof. Suppose $\Gamma_{pe}(G) = n-1$. Suppose there exists two adjacent vertices u and v in G of degree at most two. Then the set $S = V - \{u, v\}$ is a dominating set in G. Suppose S contains no edge, then S should have exactly one vertex. For otherwise, $\Gamma_{pe}(G) \leq n-2$. If $S = \{w\}$, then $G \cong K_3$ and P_3. Suppose S contains an edge, then S will be a upper pendant dominating set in G. Therefore, $\Gamma_{pe}(G) \leq n-2$, a contradiction. Hence, either u or v must be a pendant vertex in G and so $G \cong K_{1,n-1}$. But we have $\Gamma_{pe}(K_{1,n-1}) = 2$, from which it follows that $n = 3$ showing that $G \cong P_3$. If S contains an edge $e = v_1v_2$ then, e will be adjacent to u and v. Thus, $\Gamma_{pe}(G) \leq n - 2$, a contradiction. Therefore, removing any one edge, we obtain $G \cong P_4$. ■

For any graph G, it is always true that $\gamma(G) \leq \gamma_{pe}(G)$. But the same is not true for upper pendant domination number. For instance, if G contains a vertex of degree $n - 1$, then $\Gamma_{pe}(G) \geq \Gamma_{pe}(G)$. Further, if G is a bi-star, then we have $\Gamma_{pe}(G) \leq \Gamma(G)$. In fact, the difference $\Gamma_{pe}(G) - \Gamma(G)$ can be made arbitrarily large. In the following Theorem, we give the necessary condition for $\Gamma_{pe}(G) \leq \Gamma(G)$.

Theorem 3.4. If G contains a strong support vertex then, $\Gamma_{pe}(G) \leq \Gamma(G)$. Equality holds if $\Gamma-$set induces a cycle in G such that $pn(v, S) \neq \phi$, for each $v \in V(G)$.

Proof. Let G be graph and v be a strong support vertex of G. Then v adjacent to at least 2 leaves. Let S be an upper dominating set of G and let $\mid S \mid= k$. Then clearly $S \cup N(v) \geq 2$. Put $H = G - N[v]$, then $\Gamma(H) \leq \Gamma(G)-2$. Let S' be an upper dominating set of H. Then $S' \cup \{u, v\}$, where $u \in N(v)$, will be an upper pendant dominating set in G. Therefore, $\Gamma_{pe}(G) =\mid S' \mid +2 \leq \Gamma(G) - 2 + 2 = \Gamma(G)$. Hence, $\Gamma_{pe}(G) \leq \Gamma(G)$. Next, let S be an upper dominating set in G such that $\langle S \rangle$ is a cycle in G. ■

Proposition 3.5. Let r, s be any two positive integers. Then there exists a graphs G and H such that $\Gamma_{pe}(G) - \Gamma(G) \geq r$ and $\Gamma(H) - \Gamma_{pe}(H) = s$.

Proof. Let r, s be any two positive integers. From Corollary 2.1, given any positive integer k, there exists a graph of order n such that $\Gamma_{pe}(G) = k$. Put $m = k + r$ and let $\overline{K}_m$ be the totally disconnected graph of order m. Now, consider the graph $G' \cong G \vee \overline{K}_m$. Clearly $\Gamma(G \vee \overline{K}_m) \geq m$ and $\Gamma_{pe}(G \vee \overline{K}_m) = \Gamma_{pe}(G) = k$. Therefore, $\Gamma(G') - \Gamma_{pe}(G') \geq r$. Next, suppose H is a star $K_{1,s+2}$ of order $s+3$. Since $\Gamma_{pe}(H) = 2$ but $\Gamma(H) = s + 1$. Hence, $\Gamma(H) - \Gamma_{pe}(H) = s$. ■

Theorem 3.6. For any connected graph G, we have $\left\lceil \dfrac{n}{1 + \Delta(G)} \right\rceil \leq \Gamma_{pe}(G) \leq n - \Delta(G)$.

Proof. Let S be a Γ_{pe} set of G. First we consider the Lower bound each vertex dominates at-most itself and $\Delta(G)$ of other vertices. Hence $\Gamma_{pe}(G) \geq \left\lceil \dfrac{n}{1 + \Delta(G)} \right\rceil$. For the upper bound, for any graph the pendant domination number is less than or equal to number of vertices. Therefore $\Gamma_{pe}(G) \leq n$. ■

Theorem 3.7. If G is an r-regular graph with $r \geq 2$, then $\Gamma_{pe}(G) \leq \Gamma(G) + r - 2$.

Theorem 3.8. If a graph G has $diam(G) = 2$. Then $\Gamma_{pe}(G) \leq \Delta(G)$. Equality holds if G is a cycle or path.

Proof. Let G be any graph and w be a vertex of G of degree $\Delta(G)$. As $diam(G) = 2$, any vertex in G will be at a distance at most 2 from w. Thus $N[w]$ will be a pendant dominating set in G. Hence $\Gamma_{pe}(G) \leq \Delta(G)$. ■

Theorem 3.9. For any connected graph G. Then $\left\lceil \frac{diam(G)+1}{3} \right\rceil \leq \Gamma_{pe}(G)$.

Proof. Let S be a $\Gamma_{pe}(G)$ set of a connected graph G. Consider an arbitrary path of length $diam(G)$. The diameter path includes the pendant vertex. Furthermore S is $\Gamma_{pe}(G)$ set, joining the neighborhoods of the vertices of S. Hence $diam(G) \leq 2\Gamma_{pe}(G) + \Gamma_{pe}(G) - 1 = 3\Gamma_{pe}(G) - 1$. ■

Theorem 3.10. For any two graphs G and H of order n and m respectively, we have $\Gamma_{pe}(G \circ H) = (n-1)\Gamma(H) + \Gamma_{pe}(H)$.

Proof. Let S be an upper pendant dominating set of a graph H_i, for at least one i, and S' be an upper dominating set of remaining copies of H. Then $S \cup S'$ will be an upper pendant dominating set of $G \circ H$ and hence, $\mid S \cup S' \mid = (n-1)\Gamma(H) + \Gamma_{pe}(H) \leq \Gamma_{pe}(G \circ H)$. Conversely, assume S is an upper pendant dominating set of $G \circ H$. Then S will be an upper dominating set of each copy of H, and in fact S contains an edge from at least one copy of H. Thus, S is just the union of upper dominating sets of copies of H such that at least one of the copy of upper pendant dominating set contains an edge. Thus, $|S| \leq (n-1)\Gamma(H) + \Gamma_{pe}(H)$, and therefore, we must have $\Gamma_{pe}(G \circ H) = (n-1)\Gamma(H) + \Gamma_{pe}(H)$. ■

4. Complementary Graphs

Theorem 4.1. Let G be any graph. If $diam(G) \geq 2$ then $\Gamma_{pe}(\overline{G}) = 3$.

Proof. Let G be a connected graph of diameter at least 2. If $u, v \in V(G)$ with $diam(G) \geq 2$, and w be a any vertex in $V(G)$ then, the set $\{u, v, w\}$ is a pendant dominating set of $\overline{G}$ with maximum cardinality. Therefore, $\Gamma_{pe}(\overline{G}) = 3$. ■

Theorem 4.2. Let G be any graph. If $diam(G) \geq 3$ then $\Gamma_{pe}(\overline{G}) = 2$ or 3.

Proof. If G contains an isolated vertex, then it is trivially true that $\Gamma_{pe}(\overline{G}) = 2$. Let u and v be any two vertices of G such that $d(u, v) = diam(G) \geq 3$. Obviously, u and v

dominates $\overline{G}$, and if u and v are adjacent then we have $\Gamma_{pe}(\overline{G}) = 2$. Suppose u and v are not adjacent, then choose any one vertex $w \in G$ is adjacent to either u or v in $\overline{G}$. In that case, we have $\Gamma_{pe}(\overline{G}) = 3$. ■

Theorem 4.3. Let G be a triangle free graph without isolated vertices. Then $\Gamma_{pe}(\overline{G}) =$ 2 or 3.

Proof. Let G be a triangle free graph. If G contains an isolated vertex then clearly $\Gamma_{pe}(\overline{G}) = 2$ or 3. Suppose G has no isolated vertex, then G contains at least one edge say $e = uv$. As G is triangle free no vertex in G is adjacent to both u and v. Thus $S = \{u, v\}$ will be Γ- set in $\overline{G}$. Now, for any vertex $w \in G$, the set $S \cup \{w\}$ will be a Γ_{pe}-set in $\overline{G}$. Hence $\Gamma_{pe}(\overline{G}) = 3$. ■

Proposition 4.4. Let G be any connected graph having no vertex of degree $n - 1$. Then $4 \leq \Gamma_{pe}(G) + \Gamma_{pe}(\overline{G}) \leq 2n - 1$.

Proof. Let G be a connected graph of maximum degree $n - 2$. Then, neither G nor $\overline{G}$ can be a path of order 2 and so $\Gamma_{pe}(G)$, $\Gamma_{pe}(\overline{G}) \leq n - 2$. On the other hand, since $\Gamma_{pe}(G) \geq$ for any graph G, it follows that $4 \leq \Gamma_{pe}(G) + \Gamma_{pe}(\overline{G}) \leq 2n - 1$. ■

Proposition 4.5. Let G be any graph of order $n \geq 5$ containing no isolated vertex. Then $5 \leq \Gamma_{pe}(G) + \Gamma_{pe}(\overline{G}) \leq n + 3$.

5. Conclusion

In this paper, we have initiated the study of a graph theoretic parameter called upper pendant domination number. We have calculated the exact value for some standard families of graphs and we have established some bounds for this parameter in terms degree, order etc. Further, we have studied some important properties of this parameter, an attempt has been made to find the relation with other domination invariants and also we have studied some properties of the new parameter in the complement of the graphs.

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