Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 7 (2018), pp. 937–942

http://www.ripublication.com/gjpam.htm

On The Independent Domination Edge Lift Stable Trees

Sharada B.
Department of studies in Computer Science,
University of Mysore, Mysore, INDIA.

Nayaka S.R. and Puttaswamy
Department of Mathematics,
PES College of Engineering,
Mandya-571 401, INDIA.

Abstract

A subset D of vertices in a graph G is an independent dominating set if every vertex in $V\backslash D$ is adjacent to at least one vertex in D and $\langle D\rangle$ is an independent set. A graph is independent domination edge lift stable if the lifting of an edge leaves the independent domination number of the graph unchanged. In this paper, we characterize the trees which are independent domination edge lift stable.

AMS subject classification: Primary 05C50, 05C69.
Keywords: Independent Domination number, Independent Domination edge lift stable, Edge lifting.

1. Introduction

Let $G = (V, E)$ be a graph with n vertices and m edges. The process of edge lifting or sometimes called edge splitting was introduced by *Lováz* [6, 7]. Later *Desormeaux* et al.[9] studied this concept related to the effects on the domination number of a graph. He has shown that edge lifting may increase or decrease domination number by one or it may leave the domination number unchanged. Much work have been done on the graphs where the parameters such as connectedness, chromatic number of a graph increase or decrease or remain same when an edge or vertex is removed or added. *Goddard* et al.

[2] began the study of the effect of edge lifting on the total domination number of a graph. Further, this concept is much studied by *Chen* and *Sohn* [1], *Loizeaux* and *van der Merwe* [5] and others. Sharada et al. [8] considered the effect of edge lifting on the independent domination number of a graph. In this article, we continue the study of the effects of edge lifting on the independent domination numbers of trees.

Let u and v be any two vertices in G at a distance 2 apart and let x be a common neighbor of both u and v. Then, uxv is an induced path in G. An edge lifting defined on uxv is the process of removing the edges ux and vx while adding the edge uv to $E(G)$. We say that the edges ux and vx are lifted off the vertex x. The graph obtained by lifting off the vertex is called the edge lifted graph, denoted by G_x^{uv}. Therefore, $V(G_x^{uv}) = V(G)$ and $E(G_x^{uv}) = (E(G)\backslash\{ux, vx\}) \cup \{uv\}$. In fact the process of edge lifting is a combination of the operations edge removal and edge addition. In this process we remove two edges from the graph and add one edge to it. In this paper by a graph, we mean simple, finite and undirected graph. Here, we restrict ourselves to the graphs of order at least 3 and are non-complete.

2. Basic Definitions and Notations

For any definitions and graph theoretic terminologies that are not defined herein, we refer the reader to Haynes et al. [4] and Harary [3]. Let $G = (V, E)$ be any graph having n vertices and m edges. For $v \in V(G)$, the open neighborhood of v is the set $N_G(v)$ consisting of all vertices that are adjacent to v and the closed neighborhood of v is the set $N_G[v] = \{v\} \cup N_G(v)$. For any subset S of V, its open neighborhood is the set $N(S) = \cup_{v\in S} N(v)$ and its closed neighborhood is the set $N[S] = N(S) \cup S$. The degree of a vertex v in G is $d_G(v) = |N_G(v)|$. The minimum and maximum degree among the vertices of G is denoted by $\delta(G)$ and $\Delta(G)$, respectively. A vertex of degree one is called a leaf, its neighbor a support vertex and its incident edge a pendant edge. A strong support vertex is a support vertex with at least two leaf neighbors.

A subset D of vertices in a graph G is a dominating set if every vertex not in D is adjacent to at least one vertex in D. The minimum cardinality of a dominating set is called the domination number, denoted by $\gamma(G)$. A dominating set D whose induced sub-graph is an independent set in G is called an independent dominating set. The cardinality of a minimum independent dominating set is called the independent domination number, denoted by $i(G)$. A graph is independent domination edge lift stable if the lifting of an edge keeps the independent domination number unchanged. In this paper, we characterize the trees which are independent domination edge lift stable.

Definition 2.1. [8] A graph G is said to be independent domination edge lift stable if for any induced path uxv in G, we have $i(G_x^{uv}) = i(G)$.

3. Main results

Proposition 3.1. [8] Let P_n be a path on n vertices. Then P_n is independent domination edge lift stable if and only if $n \neq 3k$, $k \geq 2$ is an integer.

In this section, our aim is to give the constructive characterization of the independent domination edge lift stable trees. Hence, we first define the family $\mathcal{T}$ of trees as follows.

The family $\mathcal{T}$:
Let $\mathcal{T}$ be the family of trees containing a path P_4, which is closed under the following operation $\mathcal{O}$. We first label the vertices of $T \in \mathcal{T}$ as follows. Initially if $T = P_4$, then $sta(v) = A$ if v is a support vertex of T and $sta(v) = B$ if v is a leaf of T. Once the status is assigned for a vertex, it remains unchanged as the tree is constructed.

Operation $\mathcal{O}$: Let $T_1 = T$ be a path P_4 in $\mathcal{T}$. Extend it by attaching a path v, w, x of length 2 and the edge uv where $sta(u) = B$ and $u \in T$. Then $sta(v) = sta(w) = A$ and $sta(x) = B$.

For $T \in \mathcal{T}$, let $A(T)$ and $B(T)$ be the subsets of $V(T)$ containing the vertices of status A and B, respectively. From the construction of the family $\mathcal{T}$, we have the following observations:

Observation 3.2. Let $T \in \mathcal{T}$ and $v \in V(T)$. Then

1. If $sta(v) = A$, then v is adjacent to exactly one vertex of $A(T)$ and one vertex of $B(T)$.
2. If $sta(v) = B$, then $N(v) \subseteq A(T)$, where $N(v)$ is the neighborhood of the vertex v.
3. If v is a support vertex, then $sta(v) = A$.
4. If v is a leaf of T, then $sta(v) = B$.
5. $|B(T)| \leq |A(T)$, which follows from the construction of $\mathcal{T}$.
6. $B(T)$ is an $i(T)-$set of T.
7. Every independent dominating set of T intersects $B(T)$.

Lemma 3.3. If $T \in \mathcal{T}$ then $B(T)$ is an $i(T)-$set. Further, If T is obtained from $T' \in \mathcal{F}$ using operation $\mathcal{O}_1$ then $i(T) = i(T') + 1$.

Proof. From observation 3.2, $B(T)$ is an independent dominating set of T and hence we have $i(T) \leq |B(T)|$. Conversely, let S be an $i(T)-$set. For if $S = B(T)$, we are done. For the purpose of contradiction, assume $B(T) \neq S$. Then, let D be the set of all vertices common in both $B(T)$ and $V\backslash S$. i.e., $D = B(T) \cap V\backslash S$. Now, we consider a map $f : S \cap A(T) \to D$ that carries each vertex of $S \cap A(T)$ to a neighbor vertex in $B(T)$. From the observation 1, it follows that the map f is injective. Next, let x be any vertex in D. Then $sta(x) = B$ and $x \notin S$. Since S dominates $V(T)$, there is a vertex say, $y \in S$ which is adjacent to x. Again, from observation 2 we must have $y \in A(T)$. Thus, we have $f(y) = x$ and so f is onto and hence bijective. Therefore it follows that $|S| = |S \cap B(T)| + |S \backslash B(T)| \geq |D| + |S \backslash B(T)| = |B(T)|$. i.e., $|B(T)| = |S| = i(T)$. Hence $B(T)$ is an $i(T)-$set of T.

For if, the tree T is obtained from T' using the operation $\mathcal{O}$, then T contains exactly one vertex with status B more than T' has. Hence $i(T) = |B(T)| = |B(T')| + 1 = i(T') + 1$. ■

Lemma 3.4. Let T be any tree in the family $\mathcal{T}$. Then

1. Every vertex of T belongs to some $i(T)-$set.

2. If $v \in B(T)$, then there exists an independent dominating set of $V(T) \backslash \{v\}$ of size $i(T) - 1$ not containing the vertex v.

Proof. Since $T \in \mathcal{T}$, it can be obtained from the sequence $T_1, T_2, \ldots, T_n$ of trees where $T_1 = P_4$, $T = T_n$ and for $1 \leq i \leq n-1$, T_{i+1} can be obtained from T_i by using the operation $\mathcal{O}$. Now, we prove the lemma by using the mathematical induction on n.

Suppose $n = 1$. Then $T = P_4$. Suppose $V(T) = \{v_1, v_2, v_3, v_4\}$ where $A(T) = \{v_2, v_3\}$ and $B(T) = \{v_1, v_4\}$. Clearly $i(T) = 2$ and every vertex of T is in some $i(T)-$set. For if $v \in B(T)$, the set $S = \{v_2\}$ or $S = \{v_3\}$ is an independent dominating set of $V(T) \backslash \{v\}$ of size 1 not containing the vertex v. This proves the base case.

Now, suppose $n \geq 2$. Assume that the result holds for all trees in $\mathcal{T}$ that can be constructed from a sequence of at most $n-1$ trees. Let T be any tree obtained by a sequence of n trees. It follows from our induction hypothesis that every vertex of T_{n-1} belongs to some $i(T_{n-1})-$set. Also from Lemma 2.1, we have $i(T) = i(T_{n-1}) + 1$, since $T = T_n$ is constructed from $T_{n-1}$by using the operation $\mathcal{O}$, that is, by adding a path $P_3 : \{v, w, x\}$ to the vertex $u \in V(T_{n-1})$. Then $sta(v) = sta(w) = A$ and $sta(u) = sta(x) = B$. Let S be $i(T_{n-1})-$set, then $S \cup \{w\}$ is an $i(T)-$set. Thus, every vertex of $V(T_{n-1}) \cup \{w\}$ belongs to some $i(T)-$set. Now, from induction hypothesis, there is an independent dominating set S' of $V(T_{n-1}) \backslash \{u\}$ of size $i(T_{n-1}) - 1$ not containing the vertex u. Then $S' \cup \{v, x\}$ is an $i(T)-$set. Hence every vertex of T belongs to some $i(T)-$set.

Further, $S' \cup \{v\}$ and $S' \cup \{w\}$ are independent dominating sets of size $i(T) - 1$ not containing x and u, respectively. For any vertex $y \in B(T_{n-1})$, let D be an independent dominating set of $V(T_{n-1}) \backslash \{y\}$ of size $i(T_{n-1}) - 1$ not containing y. Then, the set $D \cup \{v\}$ or $D \cup \{w\}$ is an independent dominating set $V(T) \backslash \{y\}$ with cardinality $i(T) - 1$. ■

Lemma 3.5. If $T \in \mathcal{T}$ and T_x^{uv} is the tree obtained by the independent domination edge lifting of uv off x then $i(T_x^{uv}) = i(T)$. That is, T is independent domination edge lift stable.

Proof. We prove by mathematical induction on the length n of the sequence of trees needed to construct T. Suppose $n = 1$. Then $T = P_4$ and $T_x^{uv} = P_3 \cup \{u\}$, where $u \in V(P_4)$. Hence $i(T_x^{uv}) = i(T) = 2$. Also, there exists an $i(T_x^{uv})$-set containing any vertex of $B(T)$. This proves the base case.

Suppose $n \geq 2$ and assume that the result is true for all trees that are constructed from a sequence of k trees with $k \leq n-1$. Let T be a tree obtained from a sequence of n trees. Let us call T_{n-1} by T^*. Now, we have T is obtained by T^* by adding a path $P_3 : \{v, w, x\}$ to the vertex $u \in V(T^*)$. Then $sta(v) = sta(w) = A$ and $sta(u) = sta(x) = B$. Further, if T^{**} is obtained from T^* by edge lifting of uv off x then from the induction hypothesis it follows that, $i(T^{**}) = i(T^*)$ and there is an $i(T^{**})-$set containing any vertex of $B(T^*)$. Let T' be obtained from the tree T^{**} by adding a path P_3 to a vertex u of T^{**} with $sta(u) = B$. Clearly, any independent dominating set of T^{**} can be extended to that of T' by adding w or x. Hence $i(T') \leq i(T^{**}) + 1$. But, from the induction hypothesis, we have $i(T^{**}) = i(T^*)$ and by Lemma 2.1, $i(T) = i(T^*) + 1$. i.e., $i(T^*) = i(T) - 1$. Therefore $i(T') - 1 \leq i(T^{**}) = i(T^*) = i(T) - 1 \leq i(T') - 1$. Thus, we must have equality throughout this chain of inequalities. ■

Theorem 3.6. Let T be any tree of order n. Then $T \in \mathcal{T}$ if and only if T is independent domination edge lift stable.

Proof. The proof of the necessary part of the above theorem follows from Lemma 3.5. To prove this theorem it is enough if we show that the condition is sufficient. Let T be a tree of order n. We have to show that if T is independent domination edge lift stable then $T \in \mathcal{T}$. We prove this by mathematical induction on n. First, suppose $n = 4$. Then T is either path P_4 or a star $K_{1,3}$. By proposition 3.1 P_4 is independent domination edge lift stable nut not $K_{1,3}$. Also by construction $P_4 \in \mathcal{T}$ and so the result is true for $n = 4$.

Assume that all the trees T' which are independent domination edge lift stable with $|T'| < n$ belongs to $\mathcal{T}$. Let T be an independent domination edge lift stable tree of order n. Choose a longest path in T say $P_r : \{v_1, v_2, v_3, \ldots, v_r\}$. Then $deg(v_1) = deg(v_r) = 1$. Since we must have $r \geq 5$, it follows that $deg(v_2) = deg(v_3) \geq 2$. Define $T' = T \backslash \{v_1, v_2, v_3\}$. Now, we first prove that T' is independent domination edge lift stable. Clearly $i(T') = i(T) - 1$ and $i(T_x^{uv}) = i(T)$ as the tree T is independent domination edge lift stable. Consider an arbitrary induced path uxv in T'. Then uxv will be an induced path in T and hence lifting off the vertex x by uv keeps the independent domination number unchanged and so $i(T_x^{'uv}) = i(T_x^{uv}) - 1$. This establishes that $i(T_x^{'uv}) = i(T) - 1 = i(T')$. Therefore, T' is independent domination edge lift stable and $|V(T')| < n$. Thus, $T' \in \mathcal{T}$.

Finally, it remains to show that T is obtained from T' using the operation $\mathcal{O}$. Let v_4 be the vertex in T adjacent to v_3, we have to prove that $sta(v_4) = B$. On contrary, assume that status of v_4 is not B. Then, v_4 is not a leaf. Hence $|N(v_4)| \geq 2$ and hence we can choose an arbitrary vertex $u \in N(v_4)$. Choose an induced path P_3 of which one vertex is u. Then T_x^{uv} will be a tree having two components of which one is path P_4 with u as one of the end vertex. Further, $i(T_x^{uv}) = i(T) + 1$, which leads to a contradiction.

This contradiction shows that v_4 must be a leaf in T and hence $sta(v_4) = B$ and T is obtained from T' by using the operation $\mathcal{O}$. ■

References

[1] X. Chen, M.Y. Sohn *A note on the total domination vertex critical graphs*. Ars Comb 88 (2008), 289–294.

[2] W. Goddard, T.W. Haynes, M. A. Henning, L.C. Van derMerwe *The diameter of total domination vertex critical graphs*. Discrete Math 286(2004), 255–261.

[3] F. Harary, *Graph Theory*, Addison-Wesley, Reading MA, (1969).

[4] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, (1998).

[5] M. Loizeaux, L.C. Van derMerwe, *A total domination vertex critical graph of diameter two*. Bull Inst. Comb Appl 48(2006), 63–65.

[6] L. Lovász Combinatorial problems and exercises, 2nd edn. North-Holland, New York (1993).

[7] L. Lovász In: Lecture, conference of graph theory, Prague (1974).

[8] Sharada B. Nayaka S.R. and Puttaswamy *Edge Lifting and Independent domination in Graphs*, In Communication.

[9] J. Wyatt, Desormeaux, T.W. Haynes, M. A. Henning, *Edge lifting and total domination in graphs*, J Comb Optim 25 (2013), 47–59.