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FZBounded Modules

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Abstract

In this article the concept of Fuzzy bounded (for pithiness, FZbounded) module has been introduced, also direct sum of this type of module are discussed. Moreover, the FZpure submodules in the class of FZbounded module are studied.

Keywords: FZbounded modules, FZfully stable module, FZpure module, and FFZregular module.

1. INTRODUCTION

Theroughout this article all rings are commutative with identity and all modules are unitary . Here FZbounded modules are defined in section two of this thesis ,where X an F-module is called FZbounded module if there exists $x_t \subseteq X$ such that F-ann$_R$X =ann$_R x_t$,$\forall t \in [0,1]$. Some results arouned this cocept are presented.

In §$_3$ the direct sum of two FZbounded modules are provided and we have mentioned that the direct summand of FZbounded module doesnot give FZbounded module .

In §$_4$ the behavior of FZpure submodule in FZbounded module is investigated and variance result are proved.

2. F-Bounded Modules

During this section,the concept of FZbounded module has been introduced with sevral results about this concept.

We, first opened our item by the following definition in [8].

Definition(2.1):-A fuzzy set X in a module M over R is known as the name FZmodule if and only if , ∀ x,y ∈M ,r∈R

1-X(x-y) ≥ min{X(x),X(y)}.

2- X(rx) ≥ X(x).

3- X(0)=1 ,where 0 is the zero element of M.

And if A ,X are two FZmodules of M over R ,then A is to be FZsubmodule if and only if A ⊂X .As special case when X(x) = 1 ,∀ x∈M .then A is called FZsubmodule.

The following introduce our main definition.

Definition(2.2): Let X be an FZmodule of M .X is said to be FZbounded module if, ∃ x_t⊆X such that F- ann_RX =$ann_R x_t$ ∀t∈[0,1].

Where F-ann_RX={$r_\ell \subseteq R\ , r_l x_t = 0_1\ \forall x_t \subseteq X$} ,where $0_1(x) = \begin{cases} 1\ if\ x = 1 \\ 0\ othere\ wise \end{cases}$

Remark(2.3): If X is FZmodule of M ,then F-ann_RX=$ann_R X_t$.

Proof:- Let x_t⊆F-ann_RX ∀t∈(0,1] ,then ∃r_ℓ⊆R such that $r_\ell x_t$=0_1 ∀t∈(0,1]

$(rx)_t$=0_t ≤0_1 ∀ℓ ∈(0,1] where t= min{ℓ,t}. [1]

Therefore rx= 0 ,thus x∈$ann_R X_t$=M

(⇐) It is clear

Definition(2.4):- If X is an FZmodule ,then X is named FZcyclic module if , ∃ x_t⊆X such that y_k⊆X written as y_k=$r_\ell x_t$ for some r_ℓ⊆R .

i.e $X = (x_t)$, $\forall x_t \subseteq X$ and $k \in (0,1]$.

Proposition(2.5) : Let X be an FZmodule of M . X is FZbounded module $\Leftrightarrow$. X_t is bounded module.

Proof :- By reamark (2.3),we get the results.

Proposition (2.6):- Let X be an FZmodule of M .If X is FZcyclic, then X is FZbounded module.

The convers of this proposition is not always true as an example

Example(2.7):- Let $X : Q \to [0,1]$ formed as :-

$X(x) = \begin{cases} 1 & if\ x \in Q \\ 0 & otherwise \end{cases}$ By remark (2.3), X_t= Q and Q is bounded but not cyclic

Thus X is FZbounded module not FZcyclic module.

Definition(2.8):- Let X be an FZmodule .X is said to be FZfully stable module if F-ann_R(F-$ann_R(x_t)$) = (x_t), $\forall t \in (0,1]$.

We build the following lemma to prove our next proposition

Lemma(2.9):- Let X be an FZmodule ,then X is FZcyclic if X is FZfully stable module.

Proof:- Since X is FZbounded module ,then F-$ann_R X$ =F-$ann_R(x_t)$.

But F-ann_R(F-$ann_R(x_t)$) = (x_t) =F-ann_R(F-$ann_R X$) = X.

Therefor X = (x_t) ,thus X is FZcyclic module.

Proposition(2,10):- Let X be an FZfully stable module ,then X is FZbounded module $\Leftrightarrow$ X is FZcyclic module.

Definition (2.11):- A ring R is called F-Zintegral domain if ,$\exists$ $0_1 \neq a_s \subseteq R$, $0_1 \neq b_f \subseteq R$ such that $a_s b_f = 0_1, \forall s,f \in (0,1]$.

Definition(2.12):- If R is FZintegral domain and X is an FZmodule ,then for any $x_t \subseteq X$ is called FZtorsion module ,if $F\text{-ann}_R(x_t) \neq 0_1$,And denoted by FT(X).

On other hand if $F\text{-ann}_R(x_t) = 0_1$. Then X is called torsion free FZmodule.

Remark(2.13):- If X is FZmodule , then FT(X) is FZsubmodule of X .

Prposition(2.14):- Let X be an FZtorsion free module . Then X is FZbounded module.

Proof :- $F\text{-ann}_R(x_t) = 0_1$ $\forall 0_1 \neq x_t \subseteq X$ (by hypothesis)

Since $F\text{-ann}_R X = \cap(F\text{-ann}_R(x_t))$ $\forall x_t \subseteq X$. ,thus $F\text{-ann}_R X = 0_1$.

Therefore X is FZbounded module.

The convers of this proposition is not always true. To illustrate this ,note .

Example(2.15):- Let $X: Z_n \to [0,1]$ for n>1

Define by $X(x) = \begin{cases} 1 & if\ x \in Z_n \\ 0 & otherwise \end{cases}$

$X_t = Z_n$ is not torsion free, [1],and it is bounded [2] .

Thus X is FZbounded and not FZtorsion free module.

Definition (2.16)**:-** Let X be an FZmodule . X is called FZmultiplication module if for all non-empty FZsubmodule A of X , ∃ FZideal I of R such that A=IX.

Definition (2.17) :- Let X be an F-module .X is called FZfaithful module if F-ann_RX = 0_1 , where F-ann_RX = $\{r_\ell : r_\ell x_t = 0_1 , \forall x_t \subseteq X , \forall \ell, t \in (0,1]$

Corollary(2.18):- Let R be an FZintegral domain and X is FZfaithful multiplication module ,then X is FZbounded module.

Proof:- Since X is FZfaithful and multiplication module , then X_t is faithful multiplication module [1] .

This implies that X_t is torsion free R-module [2].

Therefore X_t is bounded module [2]

Thus X is FZbounded module (by proposition (2.5)).

Corollary (2.19) :- Let X be an FZbounded fully stable module ,then X is FZZmultiplication module.

Definition (2.20) :- Let X be an FZmodule Then X is called FZdivisible module if and only if r_ℓX =X for each $r_\ell \subseteq$ R , $\ell \in (0,1]$.

Lemma (2.21) :- Let X be FZdivisible module .Then X is FZfaithful module .

Proof :- Since X is FZdivisible module ,then r_ℓX = X ,$\ell \in (0,1]$.

Let $r_\ell \subseteq$ F-ann_RX ,then r_ℓX= 0_1 ,thus X = 0_1.

Corollary (2.22) :- Let R be an FZintegral domain. Let X be an FZdivisible and FZmultiplication module.then X is F-bounded.

3. Direct Sum of FZBounded module

In this iteam ,the direct sum of two FZbounded modules are descriped .The FZbounded module class under the direct sum is closure but every direct summand of FZbounded module not holds ingeneral.

Definition (3.1) :- Let X be an FZmodule of M_1 ,and Y be an FZmodule of M_2 .Then Y⊕Y is an FZmodule of $M_1 \oplus M_2$.Where X⊕Y :- $M_1 \oplus M_2 \rightarrow [0,1]$,define by (X⊕Y)(a,b) = min{A(a),B(b) }, ∀ $(a,b) \in M_1 \oplus M_2$

Proposition (3.2) :- Let X and Y be two FZbounded module ,then Y⊕Y is ZZbounded module .

Proof :- First , since X and Y are FZmodules .Then X⊕Y is FZmodule [3]

Let X and Y are FZbounded modules Then ,∃ $x_t \subseteq X$,and $y_t \subseteq Y$ $\forall t \in (0,1]$.

$F\text{-}ann_R X = F\text{-}ann_R(x_t)$,and $F\text{-}ann_R Y = F\text{-}ann_R(y_t)$.

So, $(x_t,y_t) \subseteq X \oplus Y$ [4]

We claime that $F\text{-}ann_R(X \oplus Y) = F\text{-}ann_R((x_t,y_t))$

Let $r_\ell(x_t,y_t) = (0_1,0_1)$ $\forall$ $\ell \in (0,1]$, so $(r_\ell x_t, r_\ell y_t) = ((rx)_t,(ry)_t) = (0_1,0_1)$,where t =min{t,ℓ}.[5].

Thus $(rx)_t = 0_1$,and $(ry)_t = 0_1$

Therefore r_ℓ $x_t = 0_1$, and $r_\ell y_t = 0_1$, that is$r_\ell \subseteq F\text{-}ann_R(x_t)$, and $r_\ell \subseteq F\text{-}ann_R(y_t)$.

Now, if $(m_s,\overline{m}_s) \subseteq X \oplus Y$,then r_ℓ $(m_s,\overline{m}_s) = ((rm)_s,(r\overline{m})_s) = (0_1,0_1)$, where s = min{ℓ,s}.

This implies that $r_\ell \subseteq F\text{-}ann_R(X \oplus Y)$

Therefore $F\text{-}ann_R(X \oplus Y) = F\text{-}ann_R((x_t,y_t))$.

Note that a direct summand of FZbounded module need not to be FZbounded in general .

Example (3.3) :- Let X: M→[0,1] where M =Z⊕Z_P^∞ define by:

$$X(x) = \begin{cases} 1 & if\ x \in M \\ 0 & otherwise \end{cases}$$

It is clear that X_t = M and M is bounded module [2] ,since ann_Z M = 0 = ann_Z ((1,0)) , but Z_P^∞ is not bounded Z-module

Thus X is not FZbounded module . (by proposition (2.5)).

From Proposition (3.2) and by mathematical induction we have the following :

Corollary (3.4) :- Afinite direct sum of FZbounded module is bounded.

Remark (3.5) :- An infinite direct sum of FZbounded modules need not be FZbounded.

Example (3.6) :- Let M =Z_p where p is prime number and X : M →[0,1] define by

$$X(x) = \begin{cases} 1 & if\ x \in M \\ 0 & otherwise \end{cases}$$

It is clear that M = X_t ,which is bounded as a FZmodule ,then X is FZbounded module .

Now , define Y: ⊕Z_p^∞→ [0,1] as follows:-

$$Y(y) = \begin{cases} 1 & if\ y \in Z_p^\infty \\ 0 & otherwise \end{cases}$$

It is clear that Y_t = ⊕Z_p^∞ , and Z_p^∞ , and Z_p^∞ is not bounded as a FZmodule [2].

Thus Y is not bounded FZmodule.

4. FZPure Submodule of FZBounded modules

Here, the behavior of F-submodule of F-bounded module are discussed whene the FZsubmodule is FZpure.

Now, we interduce the following

Definition (4.1) :- Let A be an FZsubmodule of an FZmodule X ,then X/A is called FZquesiont module and difine by:- X/A ={x_t: x_t +A , $x_t \subseteq$ X ,∀t∈(0,1]}.

Definition (4.2) :- An FZsubmodule A of an FZmodule X is called FZpure module if IX∩A= IA where I is an FZideal [6].

In case a ring R is FZprinciple ideal domain ,then A is FZpure submodule if and only if r_ℓX∩A= r_ℓA ∀ℓ ∈(0,1] and ∀ $r_\ell \subseteq$ R [7].

Proposition(4.3) :- Let A be a FZpure module of X such that X/A is FZbounded module and F-ann$_R$X = (A:X) ,Then X is FZbounded module , where (A:X) = {$r_\ell \subseteq$ R: r_ℓX ⊆ A }.

Proof:- Let X/A be FZbounded module ,then ,∃ x_t A ⊆ X/A ∀x_t ⊆X .

F-ann$_R$X/A =F-ann(x_t +A) , ∀ t∈(0,1] and since F-ann$_R$ X/A = (A:X). but F-ann$_R$X =(A:X)

Thus F-ann$_R$ X= F-ann$_R$ (x_t:+A).

Now, to show that F-ann(x_t+A) = F-ann$_R$(m_n) , $m_n \subseteq$ X , n ∈(0,1] .

Let $r_\ell \subseteq$ F-ann$_R$(x_t+A) ,then $r_\ell(x_t$+A) , implies $r_\ell x_t$ ⊆ A.

S0, $r_\ell x_t$ ⊆ A∩r_ℓX ,we get $r_\ell x_t \subseteq r_\ell$A.

Therefore $r_\ell x_t = r_\ell y_{\mathcal{K}}$, $y_{\mathcal{K}} \subseteq$ A ,$\mathcal{K}$ ∈(0,1] , that is r_ℓ(x_t - $y_{\mathcal{K}}$) = 0_1.

Let m_n = x_t-$y_{\mathcal{K}}$, then $r_\ell m_n$ =0_1 .

Thus $r_\ell \subseteq$ F-ann$_R$ (m_n)

Hence F-ann$_R$(x_t+A) ⊆ F-ann$_R$ (m_n)

Also, $d_i \subseteq$ F-ann$_R$ (m_n) ,then d_i m_n = 0_1 = d_i(x_t - $y_{\mathcal{K}}$) , ∀ i ∈(0,1] .

Which implies $d_i x_t$ - $d_i y_{\mathcal{K}} \subseteq$ A

Hence $d_i x_t$ ⊆ A , therefore $d_i \subseteq F - ann_R(x_t + A)$.

Definition (4.4) :- An FZmodule X is called FFZregular module if for every FZsubmodule of X is FZpure submodule [7].

Corollary (4.5) :- Let X be an FFZregular module and A be a FZsubmodule of X such that X/A is bounded and F-ann$_R$ X = (A:X).Then X is a FZbounded module.

Definition (4.6) :- FZmodule X is said to be FZfinitely generated if there exists $x_{t_1}+x_{t_2}+\ldots\ldots+x_{t_n}$,such that X={$a_1(x_1)_{t_1}+a_2(x_2)_{t_2}+\ldots\ldots\ldots\ldots+a_n(x_n)_{t_n}$ .Where $a_i \subseteq$ R,and $a(x)_t$,∀t∈(0,1],and $(ax)_t(y)=\begin{cases} t \text{ if } y = ax \\ 0 \text{ otherwise} \end{cases}$ [7].

Corollary (4.7) :- Let A be an FZsubmodule of FZmodule X ,if every FZfinitely generated submodule of A is FZpure submodule of FZmodule X such that X/A is FZbounded module and F-ann$_R$X=(A:X) .Then X is FZbounded.

Proof :- Since every FZfinitely generated submodule of A is FZpure in X. [5]

Therefore X is FZbounded module.

Here,we give a definition of FZtorsion module in another case.

Definition (4.8) :- Let R be an F-integral domain .Let X be an FZmodule over R ,then the FZtorsion module is denoted and define by:- FT(X) = $\{x_t \subseteq X, r_\ell x_t = 0_1, r_\ell \subseteq R, \forall\, t, \ell \in (0,1]\}$

If FT(X) = X ,then X is called FZtorsion module . If FT(X) = 0_1 ,then X is called FZtorsion free module.

Definition (4.9) :- An FZideal I of R is called FZprinciple ideal if, $\exists\ r_\ell \subseteq I$ such that I =(r_ℓ) ,$\forall\ m_n \subseteq I$. That is I ={ $m_n \subseteq I : m_n = a_j x_t$, for some $a_j \subseteq R$ }[7].

Lemma (4.10) :- Let R is FZprinciple ideal domain and X is FZmodule .If X/A is FZtorsion free module ,then A is FZpure submodule.

Proof :- Assum that A is not FZpure module ,then there exists $r_\ell \subseteq R$ such that $r_\ell\, X \cap A \neq r_\ell\, A$, that means $r_\ell\, X \cap A \not\subset r_\ell A$, then there exists $x_t = r_\ell\, X \cap A$, and $x_t = r_\ell\, m_n$ and $m_n \nsubseteq A$.

Hence $r_\ell m_n + A = A$,then $r_\ell(m_n + A) = A$,imp;ies that $m_{\ n} + A \subseteq F - T(\frac{X}{A})$ = $\overline{0_1} = A$

Thus $m_n \subseteq A$,which is a contradication.

Threrfore A is FZpure module.

Corollary (4.11) :- Let X be FZfaithful module over FZprinciple ideal ring R and A is FZsubmodule of X such that X/A is FZtorsion free .Then X is FZbounded module.

Proposition (4.12) :- Let R be FZprinciple ideal ring .X is FZmodule ,A is FZdivisible submodule of X such that X/A is a FZbounded module and F-ann$_R$X = (A:X) .then A is F-bounded.

Proof:- Let A be FZdivisible submodule of X ,then $r_\ell A = A$ for some $0_1 \neq r_\ell \subseteq R$.

We have to show that $r_\ell X \cap A = r_\ell A$,since A is FZdivisible submodule ,then $r_\ell X \cap A = r_\ell X \cap r_\ell A = r_\ell A$,so A is FZpure submodule . Therefore X is FZbounded module.

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