Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 7 (2018), pp. 977–987

http://www.ripublication.com/gjpam.htm

Topics on return probability on locally compact groups

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Abstract

In this article we study the classification of locally compact compactly generated groups according to return probability to the origin. We focus on those of return probability decays like $exp(-n^{1/3})$.

AMS subject classification: 11B39, 33C05.
Keywords: Random walk, Geometry on groups, Return probability, Markov Chain.

1. Preliminaries

Let G be a locally compact, separable, compactly generated, unimodular group. Let e be the unit element of G. We denote by λ the Haar measure on G. Let K be a compact generating set of G, symmetric and neighborhood of e,

For all $x \in G - \{e\}$, the length $l_K(x)$ of x associated to K is the minimal number n of elements $s_1, \ldots, s_n \in K$ such that $x = s_1 s_2 \ldots s_n$, and we put by convention $l_K(e) = 0$.

The word distance associated to K is defined by $d_K(x, y) = l_K(x^{-1}y)$.

The volume growth function associated to K is defined by $V_{G,K}(n) = \lambda(B_K(n))$, where $B_K(n) = \{x \in G; l_K(x) \leq n\}$ is the ball centered at e and with radius n with respect to the word distance. For more details about the word distance, one can consult [26].

We define the asymptotic behavior of the volume growth in the following sense;

if f and g are two non-negative functions defined on the positive real axis, we use the notation $f \lesssim g$ if there exist constants $a, b > 0$, such that for x large enough, $f(x) \leq ag(bx)$. If the symmetric relation also holds, we write $f \simeq g$.

When a function is defined only on the integers, we extend it to the positive real axis by linear interpolation. We will use the same name for the original function and its extension. If $f \lesssim g$ holds without $f \simeq g$, we write $f \lnsim g$.

The asymptotic behavior of f is the coset with respect to this relation.

It is well known that, the asymptotic behavior of $V_{G,K}(n)$ is independent on the choice of K, so we can denote it in the sequel by $V_G(n)$.

Definition 1.1. (see [2]) We say that G has (polynomial) growth degree d if there is a constant $C > 0$ such that $\lambda(B_K(n)) \leq Cn^d$ for every $n > 1$ and d is the minimal number with that property.

For a group G, three behaviors may occur:

- G has an exponential volume growth, that is: $V_G(n) \simeq exp(n)$,
- G has a polynomial volume growth, in this case: $V_G(n) \simeq n^d$, for some $d \geq 0$,
- G has an intermediate volume growth; that is:

$$\forall d \in \mathbb{N} \quad n^d \lnsim V_G(n) \lnsim exp(n).$$

For all locally compact compactly generated group G, the lim $\lim_n (V_G(n))^{1/n}$ exist. If this limit is strictly greater than 1, the group has exponential volume growth; if it is at most 1, we will say that G has subexponential volume growth.

The description of the possible behaviors of the volume growth function is due to Guivarc'h [14] and Jenkins [16].

2. Return probability to the origin

The notation μ denotes a probability measure on G associated to a density F with respect to the Haar measure.

We will say that F verify the natural assumptions if it satisfy the following conditions:

1. F is bounded and in $L^1(G, \lambda)$,
2. F is symmetric,
3. there exists a relatively compact symmetric open neighborhood U containing the neutral e and generating G, such that $F_{|U} > C$ for a constant $C > 0$.
4. F have a finite second moment with respect to the word distance associated to the compact set $\bar{U}$, witch is the closure of U. (Remark that this condition doesn't depend on the choice of the compact generating set).

In the sequel, we consider

- $\Omega = G^{\mathbb{N}}$, equipped with the product Borelian structure,
- $P = \delta_e \otimes \nu^{\otimes \mathbb{N}}$ is the product probability on Ω, where δ_e is the Dirac measure at e,

- the canonical projection $X_n : \Omega \to G$, with X_0 is the sure variable equal to e, and for all positive integer n, X_n has the law μ,
- $Z_n(\omega) = \prod_{i=0}^{n} X_i(\omega); \omega \in \Omega$ defines as in [20, 9] the random walk on G associated to μ.

For a Borelian $A \subset G$, the probability that the walk started at the neutral reaches A at time n is $P(Z_n \in A)$. The asymptotic behavior of the random walk is given by $P(Z_{2n} \in U)$ which is also the asymptotic of $F^{*2n}(e)$, when F is a density verifying the natural assumptions (see [9, 15]). We denote by $p_{2n}(x, y) = F^{*2n}(x^{-1}y)$, that is the probability that the random walk starting from x joins y in time $2n$. So with the above notations and assumptions, we have:

Proposition 2.1. (see [25])

$$P(Z_{2n} \in U) \simeq \nu^{*2n}(U) \simeq F^{*2n}(e).$$

It is well known that (see [20, 9]) the asymptotic behavior of $F^{*2n}(e)$ is an invariant of the group, it doesn't depend on the choice of F. In the sequel we denote by $\Phi_G(n)$ the asymptotic behavior of the probability of return on the group G.

2.1. Random walks and the norm of the convolution operator

As we explained, on a locally compact compactly generated group G, to any probability measure μ we can associate a random walk.

In the next we give the most important relationship between random walk and convolution operator;

Proposition 2.2.

$$||\mu||_2 = lim_n P(Z_{2n} \in U)^{1/2n}$$

Proof. The measure $\mu * \mu$ is type positive, so we can apply theorem1 in [3] and we obtain the desired equality. ■

We have also the next relation between the norm of μ and the convolution powers of F.

Proposition 2.3. (see [9])

$$lim_n (F^{*2n}(e))^{1/2n} = ||\mu||_2$$

Also if $m \in \mathbb{N}$

$$lim_n \frac{F^{*2(n+m)}(e)}{F^{*2n}(e)} = ||\mu||_2^{2m}.$$

It is well known that for finitely generated groups that [28], G is non amenable if and only if $\Phi_G(n) \simeq exp(-n)$. In a recent work, Gretete and Barmaki extended this theorem to locally compact groups [10]. Recall that a topological group G is said to be amenable if there exists a continuous linear functional m defined on the space of all Borel measurable bounded functions and such that $m(f) \geq 0$ when $f \geq 0$, $m(1) = 1$ and $m(Lxf) = m(f)$ where L_x is the left translation by x, that is $L_x f(g) = f(xg)$.

3. Relationship between volume growth and return probability

We can resume the known relations between the volume growth and return probability in the following theorem,

Theorem 3.1. see [37, 4, 18] Let G be a finitely generated group. then.

1. If $exp(n^{\alpha}) \lesssim V_G(n)$ then $\Phi_G(n) \lesssim exp(-n^{\alpha/(2+\alpha)})$
2. If $V_G(n) \lesssim exp(n^{\beta})$ then $\Phi_G(n) \gtrsim exp(-n^{\beta/(2-\beta)})$
3. $V_G(n) \simeq n^d$ if and only if $\Phi_G(n) \simeq n^{-d/2}$.

For the first inequality one can see [29] and the second and third inequality involves from [4] Corollary 7.4.

The following proposition is a direct consequence of the second implication in the above theorem;

Proposition 3.2. If G is a finitely generated group of exponential volume growth, then $\Phi_G(n) \lesssim exp(n^{-1/3})$.

4. Groups with polynomial volume growth

Since 1986, Varopoulos in [36] proved that, for a finitely generated group G having polynomial growth of degree d, the return probability decay satisfy $\Phi_G(n) \simeq n^{-d/2}$.

By the celebrated results of Bass and Gromov (see [19, 22]) the groups of polynomial volume growth are exactly those groups containing a nilpotent subgroup of finite index.

The converse of the above implication is true by Gromov's growth theorem [12] and the work of Varopoulos (see Theorem VI.5.1 in [38]).

We resume all these results in the following;

Theorem 4.1. (see [12, 38]) Let G be a finitely generated group, the following properties are equivalent:

1. The group G contains a nilpotent subgroup N of finite index and of growth degree d (G is virtually nilpotent),
2. the group G have a polynomial growth of degree d,
3. the return probability decay $\phi_G(n) \simeq n^{-d/2}$.

It follows from the above theorem that for a nilpotent discrete group G of degree d, $\Phi_G(n) \simeq n^{-d/2}$. An important example in this class of groups is the discrete Heisenberg group $H_{2d+1}(\mathbb{Z})$. For this group we have: $\Phi_G(n) \simeq n^{-d}$.

In the particular case when $d = 3$ and μ is the symmetric measure associated to the canonical basis of $H_3(\mathbb{Z})$, Gretete in [11] gave the more precise result: $\Phi_{H_3(\mathbb{Z})}(n) = \frac{1}{4n^2} + O(\frac{1}{n^3})$. An important question is to generalize this result to any positive integer d.

For a finitely generated subgroup G of a connected Lie group it is well known that G have either polynomial or exponential volume growth. So a such group is amenable of subexponential volume growth if and only if there exist an integer d satisfying $\Phi_G(n) \simeq n^{-d/2}$.

It is well known that, a finitely generated solvable group with subexponential growth (see [12, 30, 39]) is virtually nilpotent, so we can apply the above theorem to a such group.

5. Return probability on wreath product

In this section we present the known results about return probability on certain groups that all obtained by the same algebraic construction known as a wreath product. Examples of such groups were considered by Kaimanovich and Vershik in [27]. They also appear in two papers of Varopoulos [34, 35], in papers of Pittet, Saloff Coste and Coulon [21, 4], also in two papers of Revelle [31, 32] and a paper of Grigorchuk and Zuk [13].

Let M, N be two finitely generated groups, and let G be the wreath product $G = M \wr N = (\sum_{n \in N} M) \rtimes N$. This is the semidirect product of N with of the direct sum of countably many copies of M indexed by N where the action of N is by index translation; see for example [21] for a precise definition.

Let $\mathcal{PE}$ the class of polycyclic groups with exponential volume growth.

We can resume the known results about return probability on wreath product in the tree following theorems;

Theorem 5.1. Let $G = M \wr N$.

1. If N satisfies $V_N(n) \simeq n^d$ for some $d \geq 1$ and M is finite non triviale, then $\Phi_G(n) \simeq exp(- n^{d/d+2})$
2. If N satisfies $V_N(n) \simeq n^d$ for some $d \geq 1$ and M have polynomial volume growth, then $\Phi_G(n) \simeq exp(- n^{d/d+2}(ln(n))^{2/d+2})$
3. If N satisfies $V_N(n) \simeq n^d$ for some $d \geq 1$ and $M \in \mathcal{PE}$, then $\Phi_G(n) \simeq exp(- n^{(d+1)/(d+3)})$
4. If $N \in \mathcal{PE}$ and M is nontrivial, finite or polycyclic. Then we have $\Phi_G(n) \simeq exp(- \frac{n}{log^2(n)})$.

Theorem 5.2. Let $G = M \wr (M \wr (\cdots (M \wr \mathbb{Z}^d) \cdots))$ where k successive wreath products are taken.

1. If M is finite nontrivial, then $\Phi_G(n) \simeq exp(-n(\log_{k-1} n)^{-2/d})$

2. If M satisfies $V_N(n) \simeq n^b$ for some $b \geq 1$, then
$$\Phi_G(n) \simeq exp\left(-n\left(\frac{(\log_{k-1} n)}{(\log_k n)}\right)^{-2/d}\right).$$

Theorem 5.3. If the heat decay of a finitely generated group Γ is equivalent to $exp(-n^{\alpha} ln^{\beta}(n))$ then then the heat decay on $G = \Gamma \wr \mathbb{Z}$ is

$$\Phi_G(n) \simeq exp(-n^{(1+\alpha)/(3-\alpha)} ln^{2\beta/(3-\beta)}(n)).$$

For more details and historical of these theorems one can see [8].

6. Groups with return probability: $\Phi_G(n) = exp(-n^{1/3})$

The first example of a finitely generated group with return probability decays like $exp(-n^{1/3})$ was given by Varopoulos. He showed that the return probability on the wreath product $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ is equivalent to the expectation $E(2^{-R_n})$ where R_n is the random number of visited sites during time n for a random walk on $\mathbb{Z}$. See [34], Appendix II and [35]. The return probability of the wreath product $M \wr N$ of two finitely generated groups M and N with M infinite behaves like $exp(-n^{1/3})$ if and only if M is a finite non-trivial group and M is a finite extension of $\mathbb{Z}$. This follows from technics developed in [21].

Alexopoulos established the lower bound $\Phi_G(n) \gtrsim exp(-n^{1/3})$ for polycyclic groups [1]. In [23], Pittet and Saloff Coste gave a generalization of this lower bound to the class of finitely generated solvable groups of finite Prufer rank. Recall that a finitely generated group has finite Prufer rank if there is a minimal positive integer r, such that any of its finitely generated subgroup admits a generating set of cardinality less or equal to r, they demonstrate the following theorem;

Theorem 6.1. (see [23]) Let G be a finitely generated virtually solvable group of finite Prufer rank. The heat decay of G satisfies $\Phi_G(n) \simeq exp(-n^{1/3})$ if and only if G is not virtually nilpotent.

The group of upper triangular $n \times n$ matrices with coefficients in the ring $\mathbb{Z}[1/d]$ where $d \in \mathbb{N}$ and with units on the diagonal is finitely generated solvable of finite Prufer rank. See also [4, 22] for examples and special cases of the theorem.

Theorem 6.2. (see [4] page 24) Baumslag Solitar $G = \mathbb{Z}[1/q] \rtimes \mathbb{Z}$, we have $\Phi_G(n) \simeq exp(-n^{1/3})$.

In [33], Roman Tessera characterized groups with return probability behaves like $exp(-n^{1/3}$ using the isomperimetric profile. Recal that according to the cited reference, the isoperimetric profile is defined as

$$j_G(n) = sup\{J(A); \mu(A) \leq n\}$$

where

$$J(A) = sup\left\{\frac{\|f\|_2}{sup_{s\in S}\|f - \lambda(s)f\|_2}; f \in L^2(A)\right\}$$

and S is any symmetric finite generating set of G and λ is the regular representation of G. Note that the asymptotic behavior of $j_G(n)$ does not depend on the choice of the generating set S.

Theorem 6.3. (see [33] page 25) Let G be a finitely generated group. Then:

$$j_G(v) \simeq ln(v) \iff \Phi_G(n) \simeq exp(-n^{1/3})$$

Questions: Whether there exists a finitely generated group with a heat decay strictly slower than $exp(-n^{1/3})$, but faster than $t^{-d/2}$, is an open question.

It is not known if there is a finitely generated group with subexponential growth, not virtually nilpotent with: $\Phi_G(n) \simeq exp(-n^{1/3})$.

An important example of solvable locally compact compactly generated group is $sol(K)$ where K is a local field. In [26] M. Sami proves that in the case $K = Q_p$ the field of $p-$adique numbers the return probability satisfy $\Phi_{sol(Q_p)} \simeq exp(-n^{1/3})$, and in [9] the second author generalized this result to the case of local fields.

Recall that the group $sol(K) = K^2 \rtimes K^*$ where K^2 acts on K^* by the automorphisms $a \mapsto \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$.

7. Return probability on groups with intermediate volume growth

The groups with intermediate growth play an important role in the geometry of groups. The existence of such groups was discovered by Grigorchuk in the mid 1980s, he constructed these groups to answer a Neumann conjecture about the existence or not of an amenable group that is not elementary amenable.

The first Grigorchuk group Gr is the 2- rooted group $Gr = \langle a, b, c, d\rangle$ with a is the rooted element, and b, c, d defined by

$$\psi(b) = (a, c), \psi(c) = (a, d), \psi(d) = (1, b)$$

Little is known about return probability on these groups, we can resume known results about these groups in the following theorems due to Anna Erschler.

Theorem 7.1. (see [7]) Let Gr be the first Grigorchuk group then the volume growth of Gr satisfy: There exists two constants $1/2 < \alpha < \beta < 1$ such that

$$exp(n^{\alpha}) \lesssim V_{Gr}(n) \lesssim exp(n^{\beta})$$

This theorem, combined with Theorem 4.1 implies that

$$exp(-n^{\beta/(2-\beta)}) \lesssim \Phi_{Gr}(n) \lesssim exp(-n^{\alpha/(2+\alpha)}).$$

Theorem 7.2. (see [6] page 19) Consider the Grigorchuk group G_ω. we can choose ω and a generating set S such that for any $\epsilon > 0$,

$$\exp\left(\frac{n}{log^{2+\epsilon}(n)}\right) \leq V_{G_\omega}(n) \leq \exp\left(\frac{n}{log^{1-\epsilon}(n)}\right)$$

8. The case of metabelian groups

Definition 8.1. A metabelian group G is an extension of a abelian group by another abelian group.

In this class of groups, the behavior of return probability can vary widely, which provides a good source of examples, allowing to understand the geometry of solvable groups. Among solvable groups the simplest are the metabelian groups, the ones whose commutator group is abelian.

Example. Let G_a be the subgroup of the affine group generated by the transformations:

$$u_1(x) = x + 1, u_2(x) = x - 1, v_a(x) = ax, v_{a^{-1}}(x) = a^{-1}x.$$

These groups are metabelian and have exponential volume growth. They are not discrete in the affine group, and most are not polycyclic. When a is an integer, G_a is finitely presented as $G_a = \langle u, v : uvu^{-1} = v^a\rangle$; these are also known as Baumslag-Solitar groups.

When a is transcendental, G_a is isomorphic to the wreath product $\mathbb{Z} \wr \mathbb{Z}$ and using the second property in Theorem 5.1 we obtain $\Phi_{G_a}(n) \simeq exp(-n^{1/3}log^{2/3}(n))$.

Another important example of metabelian group is the lamplighter group:

$L = \mathbb{Z} \wr \mathbb{Z}^d$, for this group the behavior of return probability is also obtained by Theorem 5.1, that is $\Phi_L(n) \simeq exp(-n^{d/(d+2)})$.

In an unpublished work, Lison Jacoboni compute the return probability on metabelian groups with finite Krull dimension, one can resume these results in the following theorems, for more details, see [17].

Theorem 8.2. Let G be a finitely generated metabelian group of Krull dimension k, then,

$$k \leq 1 \iff \Phi_G(n) \gtrsim exp(-n^{1/3})$$

Theorem 8.3. Let G be a metabelian group of Krull dimension k, $k \geq 1$. Assume that $[G, G]$ is torsion, then, $\Phi_G(n) \gtrsim exp(-n^{\frac{k}{k+2}})$.

Corollary 8.4. Let G be a finitely generated split metabelian group of Krull dimension k. If $[G; G]$ is torsion, then, $\Phi_G(n) \simeq exp(-n^{\frac{k}{k+2}})$.

As a result, a metabelian group with Krull dimension k has return probability decays like $exp(-n^{1/3})$ if and only if $k = 1$.

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