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Dissipative Numerical Method for an Euler-Bernoulli Beam with Rigid Body

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Abstract

This article concerns the numerical analysis of an Euler-Bernoulli beam problem with a rigid body. We use the finite elements method based on the cubic Hermite polynomials for the approximation of (1)-(4) in space and Crank-Nicholson scheme for its approximation in time. We also verify that the property of dissipativity is preserved in the numerical scheme, analogous to the continuous case. Finally we derive error bounds for both discretizations semi and fully.

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Keywords: dissipative numerical method, Euler-Bernoulli beam, dissipative finite elements method, a-priori estimates, Crank-Nicholson scheme.

1. Introduction

This paper deals with the numerical analysis of an Euler-Bernoulli beam problem with a rigid body. The model consists of a flexible homogeneous beam clamped at left end $x = 0$ and a rigid body (antenna) (see [5, 6]) at the free end. The motions of the system $\{$beam + rigid body$\}$ are described by the following equations:

$$y_{tt} + y_{xxxx} = 0, \qquad 0 \le x \le 1, \qquad t > 0, \tag{1}$$

$$y(t,0) = y_x(t,0) = 0, \qquad t > 0, \tag{2}$$

$$\mu_1 y_{tt}(t,1) - y_{xxx}(t,1) = -\alpha y_t(t,1) + \beta y_{xt}(t,1), \qquad t > 0, \tag{3}$$

$$\mu_2 y_{xtt}(t,1) + y_{xx}(t,1) = \gamma y_t(t,1) - \nu y_{xt}(t,1), \qquad t > 0, \tag{4}$$

where t, x are respectively the time and the position, and $y(t, x)$, the transversal deviation of the beam at time t. Next, let $\mu_1, \mu_2 > 0$ be feedback constants and $\alpha,\ \nu > 0$ positive constants such that

$$4\alpha\nu > (\beta + \gamma)^2. \tag{5}$$

Without loss of generality, the length of the beam is taken to be unity.

In the literature, many works have dealt with the numerical analysis of Euler-Bernoulli beams and the more elaborate approaches are based on finite elements methods.

In [3], the authors use the spectral element methods in time and in space for solving an Euler-Bernoulli beam problem subject to forced lateral vibrations but with no mass attached. In [1], the authors present a semi-discrete scheme by using cubic splines and fully discrete Galerkin scheme based on the Crank-Nicholson method for the strongly damped, extensible beam equation with both ends hinged. All these strategies used in the cited models are for models without boundary control. However, in [2, 4, 7], the authors developed a numerical method based on the finite elements method applied to PDE with boundary control, which preserves the dissipativity of the system, implying immediately unconditional stability of numerical systems obtained.

In this paper, we study the well-posedness of the weak form of (1)-(4). Next we use the finite elements method based on the cubic Hermite polynomials for the approximation of (1)-(4) in space and Crank-Nicholson scheme for its approximation in time. Moreover, we verify that the dissipativity is preserved in the numerical scheme. And we establish error bounds for the both, semi-discrete and fully discrete schemes.

The paper consists of three additional sections. In the section 2, we recall that the system (1)-(4) is dissipative under the condition (5) and is well-posed in the sense of C_0-semigroup of contractions . Next, the section 3 concerns the numerical resolution of (1)-(4). In this part, we use the dissipative finite elements method (see [2, 4, 7]). And, the last section is devoted to a-priori errors estimates for the semi and fully discrete schemes.

2. Well-posedness of the system

Consider the Hilbert space

$$\mathcal{H} = H_E^2[0,1] \times L^2[0,1] \times \mathbb{C} \times \mathbb{C}, \tag{6}$$

where $H_E^2[0,1] := \{f \in H^2[0,1] : f(0) = 0,\ f'(0) = 0\}$ and the hermitian inner product defined by:

$$\langle Y_1, Y_2 \rangle_{\mathcal{H}} := \int_0^1 \Big(f_1''(x)\overline{f_2''}(x) + g_1(x)\overline{g_2}(x) \Big) dx + \mu_1 \xi_1 \bar{\xi}_2 + \mu_2 \eta_1 \bar{\eta}_2, \tag{7}$$

where $Y_1 = (f_1, g_1, \xi_1, \eta_1)^\top$, $Y_2 = (f_2, g_2, \xi_2, \eta_2)^\top$ and $\|.\|_{\mathcal{H}}$, the associated norm.

The system (1)–(4) becomes on $\mathcal{H}$:

$$\begin{aligned} &\frac{d}{dt}Y(t) = \mathcal{A}Y(t),\\ &Y(0) = Y_0 \in \mathcal{H}, \end{aligned} \tag{8}$$

where $Y(t) := (y(t,.), y_t(t,.), y_t(t,1), y_{xt}(t,1))^\top$ and the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ defined as:

$$\mathcal{A}\begin{bmatrix} f \\ g \\ \xi \\ \eta \end{bmatrix} = \begin{bmatrix} g \\ -f^{(4)} \\ \frac{1}{\mu_1}(f^{(3)}(1) - \alpha g(1) + \beta g'(1)) \\ -\frac{1}{\mu_2}(f''(1) - \gamma g(1) + \nu g'(1)) \end{bmatrix}, \quad \forall (f, g, \xi, \eta)^\top \in D(\mathcal{A}), \tag{9}$$

with

$$D(\mathcal{A}) = \{(f, g, \xi, \eta)^\top \in (H^4 \cap H_E^2[0,1]) \times H_E^2[0,1] \times \mathbb{C} \times \mathbb{C} \,|\, \xi = g(1),\ \eta = g'(1)\}. \tag{10}$$

Theorem 2.1. [6] Operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ defined by (9)-(10) is a closed, densely defined dissipative operator with compact resolvents. $\mathcal{A}$ is invertible with $\mathcal{A}^{-1}$ being compact, and $\mathcal{A}$ generates a C_0-semigroup of contractions $e^{\mathcal{A}t}$ on $\mathcal{H}$.

Proof. See [6]. ■

3. Numerical Resolution

3.1. Weak formulation

In order to derive the weak formulation of (1)–(4), we suppose:

$$y(0) = y_0 \in H_E^2[0,1], \tag{11}$$

$$y_t(0) = y_1 \in L^2[0,1]. \tag{12}$$

Let $t \in [0, +\infty[$. Let $w \in H_E^2[0,1]$ and $x \in [0,1]$.

$$(1) \implies \int_0^1 (y_{tt}(t,x)w(x) + y_{xxxx}(t,x)w(x))dx = 0;$$

By integrating twice by parts and using the relations (2)–(4), we obtain:

$$\begin{aligned} (1) \implies &\int_0^1 \Big(y_{tt}(t,x)w(x) + y_{xx}(t,x)w_{xx}(x)\Big)dx + \mu_1 y_{tt}(t,1)w(1) + \mu_2 y_{xtt}(t,1)w_x(1)\\ &+ \alpha y_t(t,1)w(1) - \gamma y_t(t,1)w_x(1) - \beta y_{xt}(t,1)w(1) + \nu y_{xt}(t,1)w_x(1) = 0. \end{aligned} \tag{13}$$

Let H be a Hilbert space with its inner product defined by:

$$\begin{aligned} H &:= \mathbb{R} \times \mathbb{R} \times L^2[0,1], \\ \langle \hat{\omega}, \hat{\phi} \rangle_H &:= \int_0^1 \omega_3 \phi_3 dx + \mu_1 \omega_2 \phi_2 + \mu_2 \omega_1 \phi_1, \end{aligned} \tag{14}$$

for all $\hat{\omega} = (\omega_1, \omega_2, \omega_3)$, $\hat{\phi} \in H$. Next, consider an other Hilbert space V and its inner product defined as:

$$\begin{aligned} V &:= \left\{ \hat{\omega} = (\omega_x(1), \omega(1), \omega) : \ \omega \in H_E^2[0,1] \right\}, \\ \langle \hat{\omega}, \hat{\phi} \rangle_V &:= \langle (\omega_3)_{xx}, (\phi_3)_{xx} \rangle_{L^2[0,1]}. \end{aligned} \tag{15}$$

V is embedded and densely defined in H. Setting H as a pivot space, we have a Gelfand triple $V \subset H \subset V'$. Let the following bilinear forms:

$$\begin{aligned} &a : V \times V \to \mathbb{R} && b : H \times H \to \mathbb{R} \\ &(\hat{\omega}, \hat{\phi}) \mapsto \langle \hat{\omega}, \hat{\phi} \rangle_V && (\hat{\omega}, \hat{\phi}) \mapsto \alpha \omega_2 \phi_2 + \nu \omega_1 \phi_1 - (\gamma \omega_2 \phi_1 + \beta \omega_1 \phi_2). \end{aligned}$$

We have the following definition:

Definition 3.1. Let $T > 0$. The function $\hat{y} = (y_x(1), y(1), y)$ is said to be weak solution of (1)–(4) on $[0, T]$ if

$$\hat{y} \in L^2(0, T; V) \cap H^1(0, T; H) \cap H^2(0, T; V')$$

and satisfies:

$$_{V'}\langle \hat{y}_{tt}, \hat{w} \rangle_V + a(\hat{y}, \hat{w}) + b(\hat{y}_t, \hat{w}) = 0, \ \hat{w} \in V, \tag{16}$$

for almost every $t \in (0, T)$, with the initial conditions:

$$\hat{y}(0) = \hat{y}_0 = ((y_0)_x(1), y_0(1), y_0) \in V, \tag{17}$$

$$\hat{y}_t(0) = \hat{v}_0 = ((y_1)_x(1), y_1(1), y_1) \in H. \tag{18}$$

3.1.1 Existence and Uniqueness results

In this part, we use intermediate spaces $[X, Y]_\theta$ (see section 2.1 of [11]).

Lemma 3.2. Let X and Y be Hilbert spaces, such that X is dense and continuously embedded in Y. Suppose that:

$$\begin{aligned} y &\in L^2(0, T; X), \\ y_t &\in L^2(0, T; Y). \end{aligned}$$

Then, after possibly a modification on a set of measure zero, we have:

$$y \in C([0, T]; [X, Y]_{\frac{1}{2}}).$$

Moreover, we give an other lemma, **Duality theorem** (see [11] chapter 6 page 29).

Lemma 3.3. Let X and Y be Hilbert spaces, such that X is dense and continuously embedded in Y. For all $\theta \in (0,1)$, we have

$$[X,Y]'_\theta = [Y',X']_{1-\theta}.$$

Lemma 3.4. There exists a set of functions $\{w_k\}_{k=1}^{\infty}$ which forms an orthogonal basis of $H_E^2(0,1)$ and an orthonormal basis of $L^2(0,1)$.

Proof. See [2, 7]. ■

Theorem 3.5.

(a) The weak formulation (16)–(18) has an unique solution $\hat{y}$.

(b) The weak solution $\hat{y}$ has the additional regularity:

$$\hat{y} \in L^\infty(0,T;V),\ \hat{y}_t \in L^\infty(0,T;H), \tag{19}$$

$$\hat{y} \in C([0,T];[V,H]_{\frac{1}{2}}), \tag{20}$$

$$\hat{y}_t \in C([0,T];[V,H]'_{\frac{1}{2}}). \tag{21}$$

Proof. We proceed in four steps.
Step 1: Existence of the weak solution.
Let $(\hat{w}_k)_k$ be a sequence of functions which forms an orthonormal basis of H and an orthogonal basis of V. Existence and construction of such basis are given by lemma 3.4. Consider the following finite dimensional spaces:

$$\hat{W}_n := \langle \hat{w}_1, \dots, \hat{w}_n \rangle,\ \forall n \in \mathbb{N}. \tag{22}$$

For a fixed $n \in \mathbb{N}$, we consider the Galerkin approximation $\hat{y}_n(t) \in \hat{W}_n$:

$$\hat{y}_n(t) = ((y_n)_x(1), y_n(1), y_n) = \sum_{k=1}^{n} d_n^k(t)\hat{w}_k,$$

where $d_n^k(t) \in \mathbb{R}$, solution of (13) on $\hat{W}_n$ and $\hat{y}_n(t)$ verifies:

$$\langle (\hat{y}_n)_{tt}, \hat{w} \rangle_H + a(\hat{y}_n, \hat{w}) + b((\hat{y}_n)_t, \hat{w}) = 0, \tag{23}$$

with initial conditions:

$$\hat{y}_n(0) = \hat{y}_{0n} = \sum_{k=1}^{n} \langle \hat{y}_0, \hat{w}_k \rangle_V \hat{w}_k, \quad \hat{y}_{0n} \underset{n\to\infty}{\longrightarrow} \hat{y}_0 \text{ in } V, \tag{24}$$

$$(\hat{y}_n)_t(0) = \hat{v}_{0n} = \sum_{k=1}^{n} \langle \hat{v}_0, \hat{w}_k \rangle_H \hat{w}_k, \quad \hat{v}_{0n} \underset{n\to\infty}{\longrightarrow} \hat{v}_0 \text{ in } H. \tag{25}$$

Thus a linear system of second order differential equations is obtained. By rewriting it as a system of first order differential equations, the Cauchy-Lipschitz theorem permits us to conclude that there exists a unique solution satisfying $\hat{y}_n \in C^2([0,T];V)$.

Next, we define the energy function for the trajectory $\hat{y}$:

$$\hat{E}(t;\hat{y}) := \frac{1}{2}\Big[\, \|\hat{y}\|_V^2 + \|\hat{y}_t\|_H^2 \,\Big]. \tag{26}$$

Taking $\hat{w} = (\hat{y}_n)_t$ in (23), using the smoothness of $\hat{y}_n$ and (5), we have:

$$\frac{d}{dt}\hat{E}(t;\hat{y}_n) = -\alpha(\hat{y}_{n,2})_t^2 - \nu(\hat{y}_{n,1})_t^2 + (\beta+\gamma)(\hat{y}_{n,1})_t(\hat{y}_{n,2})_t \le 0. \tag{27}$$

Hence

$$\hat{E}(t;\hat{y}_n) \le \hat{E}(0;\hat{y}_{0n}), \quad t \ge 0, \tag{28}$$

and due to the convergence of the sequences $(\hat{y}_{0n})_n$ and $(\hat{v}_{0n})_n$, we obtain

$$\hat{y}_n,\ (\hat{y}_n)_t \text{ are bounded in } C([0,T];V) \text{ and } C([0,T];H) \text{ respectively.} \tag{29}$$

By using the boundedness results, it holds for $\hat{w}$ in V:

$$|a(\hat{y}_n(t),\hat{w}) + b((\hat{y}_n)_t(t),\hat{w})| \le D_1 \|\hat{w}\|_V, \tag{30}$$

almost everywhere on $(0,T)$, with a constant $D_1 > 0$ independent of n.

Let $n \in \mathbb{N}$. Let $\hat{w} \in V$ such that $\hat{w} = \hat{\phi}_1 + \hat{\phi}_2$, where $\hat{\phi}_1 \in \hat{W}_n$ and $\hat{\phi}_2 \in \hat{W}_n^\top$, the orthogonal of $\hat{W}_n$ in H. Then we have:

$$\langle (\hat{y}_n)_{tt}, \hat{w} \rangle_H \le D_1 \left\|\hat{\phi}_1\right\|_V \le D_1 \|\hat{w}\|_V. \tag{31}$$

This shows that $(\hat{y}_n)_{tt}$ is bounded in $C([0,T];V')$. According to EBERLEIN-SMULJAN theorem, there exists a subsequence $(\hat{y}_{n_l})_l$, and functions $\hat{y} \in L^2(0,T;V)$, $\hat{y}_t \in L^2(0,T;H)$, $\hat{y}_{tt} \in L^2(0,T;V')$ such that:

$$(\hat{y}_{n_l})_l \rightharpoonup \hat{y} \text{ in } L^2(0,T;V),\ ((\hat{y}_{n_l})_t)_l \rightharpoonup \hat{y}_t \text{ in } L^2(0,T;H),\ ((\hat{y}_{n_l})_{tt})_l \rightharpoonup \hat{y}_{tt} \text{ in } L^2(0,T;V'). \tag{32}$$

Furthermore, (32) yields

$$((\hat{y}_{n_l,i})_t)_l \longrightarrow (\hat{y}_{n,i})_t, \tag{33}$$

for $i = 1,2$ and almost every t in $[0,T]$. Let $n_0 \in \mathbb{N}$ and $\hat{\varphi} \in L^2(0,T;\hat{W}_{n_0})$ such that

$$\hat{\varphi}(t,x) = \sum_{j=1}^{n_0} \alpha_j(t) w_j(x), \tag{34}$$

where $\alpha_j \in L^2(0,T;\mathbb{R})$. For all $n_l \geq n_0$, equation (23) yields:

$$\int_0^T \Big(\langle (\hat{y}_{n_l})_{tt}, \hat{\varphi} \rangle_H + a(\hat{y}_{n_l}, \hat{\varphi}) + b((\hat{y}_{n_l})_t, \hat{\varphi})\Big) dt = 0. \tag{35}$$

Passing to the limit in (35), and using (32), one obtains:

$$\int_0^T {}_{V'}\langle \hat{y}_{tt}, \hat{\varphi} \rangle_V + a(\hat{y}, \hat{\varphi}) + b(\hat{y}_t, \hat{\varphi}) dt = 0. \tag{36}$$

However, the set of functions of the form (34) is dense in $L^2(0,T;V)$, then (36) is satisfied for $\hat{\varphi}$ in $L^2(0,T;V)$. Then (16) holds almost everywhere on $[0,T]$. Thus $\hat{y}$ solves the weak formulation.

Step 2: Regularity.

From the construction of the weak solution and (29), $\hat{y}$ verifies (19). Using lemma 3.2, we obtain (20), after, possibly, a modification on a set of measure zero. And regularity (21) follows from lemmas 3.2 and 3.3.

Step 3: Verification of initial conditions.

Integrating (16) by parts (in time), with $\hat{w} \in C^2([0,T];V)$ such that $\hat{w}(T) = 0$ and $\hat{w}_t(T) = 0$, we obtain:

$$\begin{aligned}\int_0^T [\langle \hat{y}, \hat{w}_{tt} \rangle_H + a(\hat{y}, \hat{w}) + b(\hat{y}_t, \hat{w})] d\tau &= \int_0^T [\langle \hat{y}, \hat{w}_{tt} \rangle_H - {}_{V'}\langle \hat{y}_{tt}, \hat{w} \rangle_V] d\tau \\ &= -\langle \hat{y}(0), \hat{w}_t(0) \rangle_H + {}_{V'}\langle \hat{y}_t(0), \hat{w}(0) \rangle_V.\end{aligned} \tag{37}$$

Similarly, for a fixed n and from (23):

$$\int_0^T [\langle \hat{y}_n, \hat{w}_{tt} \rangle_H + a(\hat{y}_n, \hat{w}) + b((\hat{y}_t)_n, \hat{w})] d\tau = -\langle \hat{y}_{0n}, \hat{w}_t(0) \rangle_H + \langle \hat{v}_{0n}, \hat{w}(0) \rangle_H. \tag{38}$$

Due to (24)–(25) and (32), passing to the limit in (38) along the convergent subsequence $(y_{n_l})_l$, one has:

$$\int_0^T [\langle \hat{y}, \hat{w}_{tt} \rangle_H + a(\hat{y}, \hat{w}) + b(\hat{y}_t, \hat{w})] d\tau = -\langle \hat{y}_0, \hat{w}_t(0) \rangle_H + \langle \hat{v}_0, \hat{w}(0) \rangle_H. \tag{39}$$

Comparing (37) and (39), we deduce $\hat{y}(0) = \hat{y}_0$ and $\hat{y}_t(0) = \hat{v}_0$.

Step 4: Uniqueness of the solution.

Consider $\hat{y}$ a solution of (16) with zero initial conditions. Let $s \in (0,T)$, and set

$$\hat{Y}(t) := \begin{cases} -\displaystyle\int_t^s \hat{y}(\tau) d\tau, \ t < s, \\ 0 \text{ else }. \end{cases}$$

We have:

$$\int_0^s \Big({}_{V'}\langle \hat{y}_{tt}(\tau), \hat{Y}(\tau) \rangle_V + a(\hat{y}(\tau), \hat{Y}(\tau)) + b(\hat{y}_t(\tau), \hat{Y}(\tau)) \Big) d\tau = 0. \tag{40}$$

Performing integration by part in time, (40) becomes:

$$-\frac{1}{2}\int_0^s \frac{d}{dt}\,\|y(\tau)\|_H^2 + \frac{1}{2}\int_0^s \frac{d}{dt}a(\hat{Y}(\tau),\hat{Y}(\tau))\,d\tau - \int_0^s b(\hat{y}(\tau),\hat{y}(\tau))\,d\tau = 0. \quad (41)$$

From (41), we obtain:

$$\frac{1}{2}\int_0^s \frac{d}{dt}\,\left\|\hat{y}(\tau)\right\|_H^2 - \frac{1}{2}\int_0^s \frac{d}{dt}a(\hat{Y}(\tau),\hat{Y}(\tau))\,d\tau = -\int_0^s b(\hat{y}(\tau),\hat{y}(\tau))\,d\tau \le 0. \quad (42)$$

Then we have:

$$\left\|\hat{y}(s)\right\|_H^2 + a(\hat{Y}(0),\hat{Y}(0)) \le 0. \quad (43)$$

Hence $\hat{y}(s) = 0$ and $\hat{Y}(0) = 0$ (a is coercive). Since $s \in (0,T)$ is arbitrary, $\hat{y} \equiv 0$. ■

3.1.2 High regularity results

In this subsection, we demonstrate that even stronger continuity holds for the weak solution $\hat{y}$ solving (16)–(18).

Theorem 3.6. After, possibly a modification on a set of measure zero, the weak solution $\hat{y}$ of (16)–(18) satisfies

$$\hat{y} \in C([0,T];V), \quad (44)$$

$$\hat{y}_t \in C([0,T];H). \quad (45)$$

A definition and a lemma are stated before demonstrating this theorem.

Definition 3.7. Let Y be a Banach space. Then

$$C_w([0,T];Y) := \{w \in L^\infty(0,T;Y) : t \mapsto \langle f, w(t)\rangle \text{ is continuous on } [0,T], \forall f \in Y'\}$$

denotes the space of weakly continuous functions with values in Y.

The following lemma was stated and proved in [11].

Lemma 3.8. Let X, Y be Banach spaces, $X \subset Y$ with continuous injection, X reflexive. Then we have

$$L^\infty(0,T;X) \cap C_w(0,T;Y) = C_w(0,T;X).$$

Proof. **Proof of theorem 3.6**
We adapt this proof to standard strategies to the situation at hand (see section 8.4 of [11] and section 2.4 of [12]). Using lemma 3.8 with $X = V$ and $Y = H$, it follows from (19) and (20) that $\hat{y} \in C_w([0,T];V)$. Similarly, (19) and (21) imply $\hat{y}_t \in C_w([0,T];H)$.
Now, consider the scalar cut-off function $\xi \in C^\infty(\mathbb{R})$ such that it equals 1 on some

interval $J \subset\subset [0, T]$, and 0 on $\mathbb{R} \setminus [0, T]$. Then the function $\xi \hat{y} : \mathbb{R} \to V$ is compactly supported. Let $\eta^\varepsilon : \mathbb{R} \to \mathbb{R}$ be a standard mollifier in time. For example

$$\eta^\varepsilon := \frac{1}{\varepsilon}\eta\Big(\frac{t}{\varepsilon}\Big),$$

where

$$\eta(t) := \begin{cases} e^{-\frac{1}{1-|t|^2}}, & |t| < 1, \\ 0, & |t| \geq 1. \end{cases}$$

Let $\hat{y}^\varepsilon := \eta^\varepsilon * \xi \hat{y} \in C_c^\infty(\mathbb{R}, V)$. $\hat{y}^\varepsilon$ converges to $\hat{y}$ in V, and $\hat{y}_t^\varepsilon$ to $\hat{y}_t$ in H almost everywhere on J. Then $\hat{E}(t; \hat{y}^\varepsilon)$ converges to $\hat{E}(t; \hat{y})$ almost everywhere on J. Since $\hat{y}^\varepsilon$ is smooth and using (5), one has $\dfrac{d}{dt}\hat{E}(t; \hat{y}^\varepsilon) = -\alpha(\hat{y}_t^\varepsilon)_2^2 - \nu(\hat{y}_t^\varepsilon)_1^2 + (\beta+\gamma)(\hat{y}_t^\varepsilon)_1(\hat{y}_t^\varepsilon)_2 \leq 0$. When $\varepsilon \to 0$,

$$\frac{d}{dt}\hat{E}(t; \hat{y}) = -\alpha(\hat{y}_t)_2^2 - \nu(\hat{y}_t)_1^2 + (\beta + \gamma)(\hat{y}_t)_1(\hat{y}_t)_2 \leq 0, \tag{46}$$

in the sense of distributions on J. Since J is arbitrary, (46) holds on all compact subintervals of $(0, T)$.
For a fixed t, let $\lim_{n\to\infty} t_n = t$ and the sequence $(\pi_n)_n$ defined by:

$$\pi_n := \frac{1}{2}\Big[\ \big\|\hat{y} - \hat{y}(t_n)\big\|_V^2 + \big\|\hat{y}_t - \hat{y}_t(t_n)\big\|_H^2\ \Big]. \tag{47}$$

Due to the t-continuity of the energy function $\hat{E}$, and using the weak continuity of $\hat{y}$ and $\hat{y}_t$, we obtain:

$$\lim_{n\to\infty} \pi_n = 0.$$

This implies that:

$$\lim_{n\to\infty} \big\|\hat{y}(t) - \hat{y}(t_n)\big\|_V = 0 \text{ and } \lim_{n\to\infty} \big\|\hat{y}_t(t) - \hat{y}_t(t_n)\big\|_H = 0. \tag{48}$$

Finally we have $\hat{y} \in C([0, T]; V)$ and $\hat{y}_t \in C([0, T]; H)$. ■

3.2. Dissipative finite elements method

Recall that the energy function is

$$\hat{E}(t; y) := \frac{1}{2}\Big[\int_0^1 y_t(x,t)^2\,dx + \int_0^1 y_{xx}(x,t)^2\,dx + \mu_1 y_t(t,1)^2 + \mu_2 y_{tx}(t,1)^2\Big], \tag{49}$$

where $y \in C^2([0,\infty); H^2_E[0,1])$. The goal of this section is to develop a stable and convergent numerical method which faithfully describes the behaviour of the system. We know, in fact, that the energy of the system decreases in time:

$$\frac{d}{dt}\hat{E}(t;y) := -\alpha y_t(1)^2 - \nu y_{xt}(1)^2 + (\beta+\gamma) y_{xt}(1) y_t(1) \leq 0.$$

Therefore, it is important that the corresponding numerical method preserves the structural property of dissipativity: for long-time computations, the numerical scheme must be convergent in classical sense, but also yields the correct large time limit. Moreover, the dissipativity of the scheme implies immediately unconditional stability.

3.2.1 Semi discrete scheme

Space discretization Let $V_h \subset H^2_E[0,1]$ be an arbitrary chosen finite dimensional space. We obtain the following approximating problem:

Problem G^h : Find $y_h \in C^2([0,+\infty), V_h)$ i.e.
$\hat{y}_h = ((y_h)_x(1), y_h(1), y_h) \in C^2([0,+\infty), V)$ verifying:

$$\begin{aligned}
&\int_0^1 ((y_h)_{tt}(t)(x)w_h(x) + ((y_h)(t))_{xx}(x)(w_h)_{xx}(x))dx \\
&\quad + \mu_1 (y_h)_{tt}(t)(1)w(1) + \mu_2 (y_h)_{xtt}(t)(1)w_x(1) \\
&\quad + \alpha (y_h)_t(t)(1)w(1) - \gamma (y_h)_t(t)(1)w_x(1) \\
&\quad - \beta (y_h)_{xt}(t)(1)w(1) + \nu (y_h)_{xt}(t)(1)w_x(1) = 0,
\end{aligned} \tag{50}$$

for all $w \in V_h$, with the following initial conditions:

$$\begin{aligned}
y_h(0,.) &= y_h^0 \in V_h, \\
(y_h)_t(0,.) &= y_h^1 \in V_h.
\end{aligned} \tag{51}$$

Discretize $[0,1]$ in p subintervals of same length h. $V_h \subset H^2[0,1]$ then its elements are globally $C^1[0,1]$. Let us consider:

$$V_h := \Big\{\phi \in C^1[0;T] : \phi_{|[x_k;x_{k+1}]} \in \mathbb{P}_3(\mathbb{R}),\ k=0,\dots,p-1\Big\},$$

with $x_k = kh$, $k = 0,1,\dots,p$. Then $\dim V_h = 2p$ and set:

$$V_h = \langle \phi_1, \phi_2, \dots, \phi_{2p} \rangle, \tag{52}$$

where ϕ_i, $i = 1,\dots,2p$, are the associated basis functions at nodes x_j, $j = 1,\dots,p$.

In this basis, $y_h(t,x) = y_h(t)(x) = \sum_{j=1}^{p} \Big(Y_j(t)\phi_{2j-1}(x) + (Y_j(t))_x \phi_{2j}(x)\Big)$. Replacing y_h by its expression in (50), one obtains:

$$\sum_{j=1}^{p}\Big[\Big(\int_0^1 \phi_{2i-1}\phi_{2j-1}dx + \mu_1\phi_{2i-1}(1)\phi_{2j-1}(1)\Big)(Y_j)_{tt}(t) + \int_0^1 \phi_{2i-1}\phi_{2j}dx\,(Y_j)_{xtt}(t)$$
$$+ \int_0^1 (\phi_{2i-1})_{xx}(\phi_{2j-1})_{xx}dx\, Y_j(t) + \int_0^1 (\phi_{2i-1})_{xx}(\phi_{2j})_{xx}dx\,(Y_j)_x(t)$$
$$+ \alpha\phi_{2i-1}(1)\phi_{2j-1}(1)(Y_j)_t$$
$$- \beta\phi_{2i-1}(1)(\phi_{2j})_x(1)(Y_j)_{xt}\Big] = 0,$$
$$\sum_{j=1}^{p}\Big[\int_0^1 \phi_{2i}\phi_{2j-1}dx\,(Y_j)_{tt}(t) + \Big(\int_0^1 \phi_{2i}\phi_{2j}dx + \mu_2(\phi_{2i})_x(1)(\phi_{2j})_x(1)\Big)(Y_j)_{xtt}(t)$$
$$+ \int_0^1 (\phi_{2i})_{xx}(\phi_{2j-1})_{xx}dx\, Y_j(t) + \int_0^1 (\phi_{2i})_{xx}(\phi_{2j})_{xx}dx\,(Y_j)_x(t)$$
$$- \gamma(\phi_{2i})_x(1)\phi_{2j-1}(1)(Y_j)_t$$
$$+ \nu(\phi_{2i})_x(1)(\phi_{2j})_x(1)(Y_j)_{xt}\Big] = 0, \tag{53}$$

for all $i = 1,\dots,2p$. Finally, we obtain an ordinary differential equation:

$$\mathbb{A}Y_{tt}(t) + \mathbb{B}Y_t(t) + \mathbb{K}Y(t) = 0, \tag{54}$$

where the matrices,

$$\mathbb{A} = (a_{ij})_{1\le i,j\le 2p}, \quad \mathbb{B} = (b_{ij})_{1\le i,j\le 2p}, \quad \mathbb{K} = (k_{ij})_{1\le i,j\le 2p}, \text{ and } \quad Y = \begin{pmatrix} Y^i \\ Y^i_x \end{pmatrix}_{1\le i\le p},$$

are defined by:

$$a_{2i-1,2j-1} = \int_0^1 \phi_{2i-1}\phi_{2j-1}dx + \mu_1\phi_{2i-1}(1)\phi_{2j-1}(1),\ a_{2i-1,2j} = \int_0^1 \phi_{2i-1}\phi_{2j}dx,$$
$$a_{2i,2j-1} = \int_0^1 \phi_{2i}\phi_{2j-1}dx,\ a_{2i,2j} = \int_0^1 \phi_{2i}\phi_{2j}dx + \mu_2(\phi_{2i})_x(1)(\phi_{2j})_x(1),$$
$$k_{2i-1,2j-1} = \int_0^1 (\phi_{2i-1})_{xx}(\phi_{2j-1})_{xx}dx,\ k_{2i-1,2j} = \int_0^1 (\phi_{2i-1})_{xx}(\phi_{2j})_{xx}dx,$$
$$k_{2i,2j-1} = \int_0^1 (\phi_{2i})_{xx}(\phi_{2j-1})_{xx}dx,\ k_{2i,2j} = \int_0^1 (\phi_{2i})_{xx}(\phi_{2j})_{xx}dx,$$
$$b_{2i-1,2j-1} = \alpha\phi_{2i-1}(1)\phi_{2j-1}(1),\ b_{2i-1,2j} = -\beta\phi_{2i-1}(1)(\phi_{2j})_x(1),$$
$$b_{2i,2j-1} = -\gamma(\phi_{2i})_x(1)\phi_{2j-1}(1),\ b_{2i,2j} = \nu(\phi_{2i})_x(1)(\phi_{2j})_x(1), \tag{55}$$

for all $i, j = 1, 2, \dots, p$.

Derivation of element matrices The element matrices are:

$$\mathbb{A}_e = h\begin{pmatrix} \frac{13}{35} & \frac{11}{210}h & \frac{9}{70} & -\frac{13}{420}h \\ \frac{11}{210}h & \frac{1}{105}h^2 & \frac{13}{420}h & -\frac{1}{140}h^2 \\ \frac{9}{70} & \frac{13}{420}h & \frac{13}{35} & -\frac{11}{210}h \\ -\frac{13}{420}h & -\frac{1}{140}h^2 & -\frac{11}{210}h & \frac{1}{105}h^2 \end{pmatrix},$$

$$\mathbb{K}_e = \frac{1}{h^3}\begin{pmatrix} 12 & 6h & -12 & 6h \\ 6h & 4h^2 & -6h & 2h^2 \\ -12 & -6h & 12 & -6h \\ 6h & 2h^2 & -6h & 4h^2 \end{pmatrix}, \quad \mathbb{B}_e = \mathbb{O}_4.$$

Remark 3.9. In the definitive expressions of matrices, one shall take into account the following parameters:

$$\begin{aligned} a_{2p-1,2p-1} = \frac{13}{35}h + \mu_1, \quad a_{2p,2p} = \frac{1}{105}h^2 + \mu_2, \\ b_{2p-1,2p-1} = \alpha,\ b_{2p-1,2p} = -\beta,\ b_{2p,2p-1} = -\gamma,\ b_{2p,2p} = \nu. \end{aligned} \tag{56}$$

Dissipativity of the method In this paragraph, we demonstrate the dissipativity of the semi-discrete scheme.

Theorem 3.10. The solution $y_h \in C^2([0,\infty); H_E^2[0,1])$ of the problem G^hsatisfies:

$$\forall t > 0,\ \frac{d}{dt}\hat{E}(t; y_h) = -\alpha(y_h)_t(1)^2 - \nu(y_h)_{xt}(1)^2 + (\beta+\gamma)(y_h)_{xt}(1)(y_h)_t(1) \leq 0. \tag{57}$$

Proof. For all $y_h \in C^2([0,\infty); H_E^2[0,1])$ solution of problem G^h, we have:

$$\hat{E}(t; y_h) = \frac{1}{2}\Big[\int_0^1 ((y_h)_t(t,x)^2 + (y_h)_{xx}(t,x)^2)\,dx + \mu_1(y_h)_t(t,1)^2 + \mu_2(y_h)_{tx}(t,1)^2\Big]. \tag{58}$$

Replacing w_h by $(y_h)_t$ in (50) and using (5), one obtains:

$$\frac{d}{dt}\hat{E}(t; y_h) = -\alpha\Big((y_h)_t(1) - \frac{\beta+\gamma}{2\alpha}(y_h)_{xt}(1)\Big)^2 - \frac{4\alpha\nu - (\beta+\gamma)^2}{4\alpha}(y_h)_{xt}(1)^2 \leq 0.$$

■

3.2.2 Fully discrete scheme

Let $v_h := (y_h)_t$ and set $\mathbb{V} := \mathbb{Y}_t = [V_1\ V_2\ \dots\ V_{2p}]^\top$ its representation in the basis $\{\phi_i\}_i$. The relation (54) becomes:

$$\begin{aligned} \mathcal{N}\mathbb{Z}_t &= \mathcal{M}\mathbb{Z}, \\ \mathbb{Z}_0 &= [\mathbb{Y}_0\ \mathbb{V}_0]^\top, \end{aligned} \tag{59}$$

where $\mathbb{Z} = [\mathbb{Y}\ \mathbb{V}]^\top$, $\mathcal{N} = \begin{bmatrix} \mathbb{I}_{2p} & \mathbb{O}_{2p} \\ \mathbb{O}_{2p} & \mathbb{A} \end{bmatrix}$ and $\mathcal{M} = \begin{bmatrix} \mathbb{O}_{2p} & \mathbb{I}_{2p} \\ -\mathbb{K} & -\mathbb{B} \end{bmatrix}$.

The time interval is discretized into s equidistant subintervals of same length, $s \in \mathbb{N}^*$. Let $\Delta t := T/s$ be the time step and $t_k = k\Delta t$, for all $k \in \{0, 1, \dots, s\}$, the nodes of the discretization.
Let us note $Z_h^k = [y_h^k\ v_h^k]^\top$, the approximation of the solution Z_h at time t_k. Let $\mathbb{Y}_k$ and $\mathbb{V}_k$ be the vector representations of y_h^k and v_h^k in the considered basis in V_h.
Applying Crank-Nicholson scheme to the system (59), we obtain for all $k = 0, 1, \dots, s-1$:

$$\begin{aligned} \mathbb{M}\ \mathbb{Z}_{k+1} &= \mathbb{S}\ \mathbb{Z}_k, \\ \mathbb{Z}_0 &= [\mathbb{Y}_0\ \mathbb{V}_0]^\top, \end{aligned} \tag{60}$$

with $\mathbb{M} = \dfrac{\mathcal{N}}{\Delta t} - \dfrac{\mathcal{M}}{2} = \begin{bmatrix} \dfrac{\mathbb{I}_{2p}}{\Delta t} & -\dfrac{\mathbb{I}_{2p}}{2} \\ \dfrac{\mathbb{K}}{2} & \dfrac{\mathbb{A}}{\Delta t} + \dfrac{\mathbb{B}}{2} \end{bmatrix}$ and $\mathbb{S} = \dfrac{\mathcal{N}}{\Delta t} + \dfrac{\mathcal{M}}{2} = \begin{bmatrix} \dfrac{\mathbb{I}_{2p}}{\Delta t} & \dfrac{\mathbb{I}_{2p}}{2} \\ -\dfrac{\mathbb{K}}{2} & \dfrac{\mathbb{A}}{\Delta t} - \dfrac{\mathbb{B}}{2} \end{bmatrix}$.

Thus, for all $k = 0, 1, \dots, s-1$,

$$\frac{\mathbb{Y}_{k+1} - \mathbb{Y}_k}{\Delta t} = \frac{\mathbb{V}_{k+1} + \mathbb{V}_k}{2}, \tag{61}$$

$$\mathbb{A}\frac{\mathbb{V}_{k+1} - \mathbb{V}_k}{\Delta t} = -\mathbb{K}\frac{\mathbb{Y}_{k+1} + \mathbb{Y}_k}{2} - \mathbb{B}\frac{\mathbb{V}_{k+1} + \mathbb{V}_k}{2}. \tag{62}$$

Dissipativity of the method We show in this paragraph that (61)-(62) dissipates the norm. Let us recall that the natural norm of $Z_h = Z_h(t) = [y_h\ v_h]^\top$ is defined as follows:

$$\|Z_h\|^2 = \frac{1}{2}\Big[\int_0^1 v_h^2\,dx + \int_0^1 (y_h)_{xx}^2\,dx + \mu_1 v_h(1)^2 + \mu_2 (v_h)_x(1)^2\Big]. \tag{63}$$

We have the following theorem:

Theorem 3.11. For $k \in \mathbb{N}$,

$$\left\|Z_h^{k+1}\right\|^2 - \|Z_h^k\|^2 = -\frac{\Delta t}{4}\Big[\alpha\Big(v_h^{k+1}(1) + v_h^k(1)\Big)^2 + \nu\Big((v_h^{k+1})_x(1) + (v_h^k)_x(1)\Big)^2 - (\beta+\gamma)\Big(v_h^{k+1}(1) + v_h^k(1)\Big)\Big((v_h^{k+1})_x(1) + (v_h^k)_x(1)\Big)\Big] \le 0. \tag{64}$$

Proof. We have:

$$\left\|Z_h^{k+1}\right\|^2 - \|Z_h^k\|^2 = \frac{1}{2}\Big[\int_0^1 \Big((y_h^{k+1})_{xx}^2 - (y_h^k)_{xx}^2\Big)dx + \int_0^1 \Big((v_h^{k+1})^2 - (v_h^k)^2\Big)dx + \mu_1\Big(v_h^{k+1}(1)^2 - v_h^k(1)^2\Big) + \mu_2\Big((v_h^{k+1})_x(1)^2 - (v_h^k)_x(1)^2\Big)\Big].$$

Using Crank-Nicholson scheme, one obtains $\dfrac{y_h^{k+1} - y_h^k}{\Delta t} = \dfrac{v_h^{k+1} + v_h^k}{2}$. Multiplying the latter by $v_h^{k+1} - v_h^k$ and integrate it over [0; 1], we have:

$$\frac{1}{2}\int_0^1 \Big((v_h^{k+1})^2 - (v_h^k)^2\Big)dx = \int_0^1 \frac{y_h^{k+1} - y_h^k}{\Delta t}(v_h^{k+1} - v_h^k)dx. \tag{65}$$

Additionally, We apply Crank-Nicholson scheme to (50) and replace w_h by y_h^{k+1} and next by y_h^k in (50). Then we obtain two expressions and we subtract one from another. We get:

$$\begin{aligned}\frac{1}{2}\int_0^1 \Big(((y_h^{k+1})_{xx})^2 - ((y_h^k)_{xx})^2\Big)dx = &-\frac{1}{2}\Big[\int_0^1 \Big((v_h^{k+1})^2 - (v_h^k)^2\Big)dx \\ &+ \mu_1\Big(v_h^{k+1}(1)^2 - v_h^k(1)^2\Big) + \mu_2\Big((v_h^{k+1})_x(1)^2 - (v_h^k)_x(1)^2\Big) \\ &- \frac{\alpha\Delta t}{2}\Big(v_h^{k+1}(1) + v_h^k(1)\Big)^2 - \frac{\nu\Delta t}{2}\Big((v_h^{k+1})_x(1) + (v_h^k)_x(1)\Big)^2 \\ &+ \frac{(\beta+\gamma)\Delta t}{2}\Big(v_h^{k+1}(1) + v_h^k(1)\Big)\Big((v_h^{k+1})_x(1) + (v_h^k)_x(1)\Big)\Big].\end{aligned} \tag{66}$$

Hence, using (5), we get:

$$\left\|Z_h^{k+1}\right\|^2 - \|Z_h^k\|^2 = -\frac{\Delta t}{4}\Big[\alpha\Big(v_h^{k+1}(1) + v_h^k(1) - \frac{\beta+\gamma}{2\alpha}\Big((v_h^{k+1})_x(1) + (v_h^k)_x(1)\Big)\Big)^2 + \frac{4\alpha\nu - (\beta+\gamma)^2}{4\alpha}\Big((v_h^{k+1})_x(1) + (v_h^k)_x(1)\Big)^2\Big] \le 0.$$

■

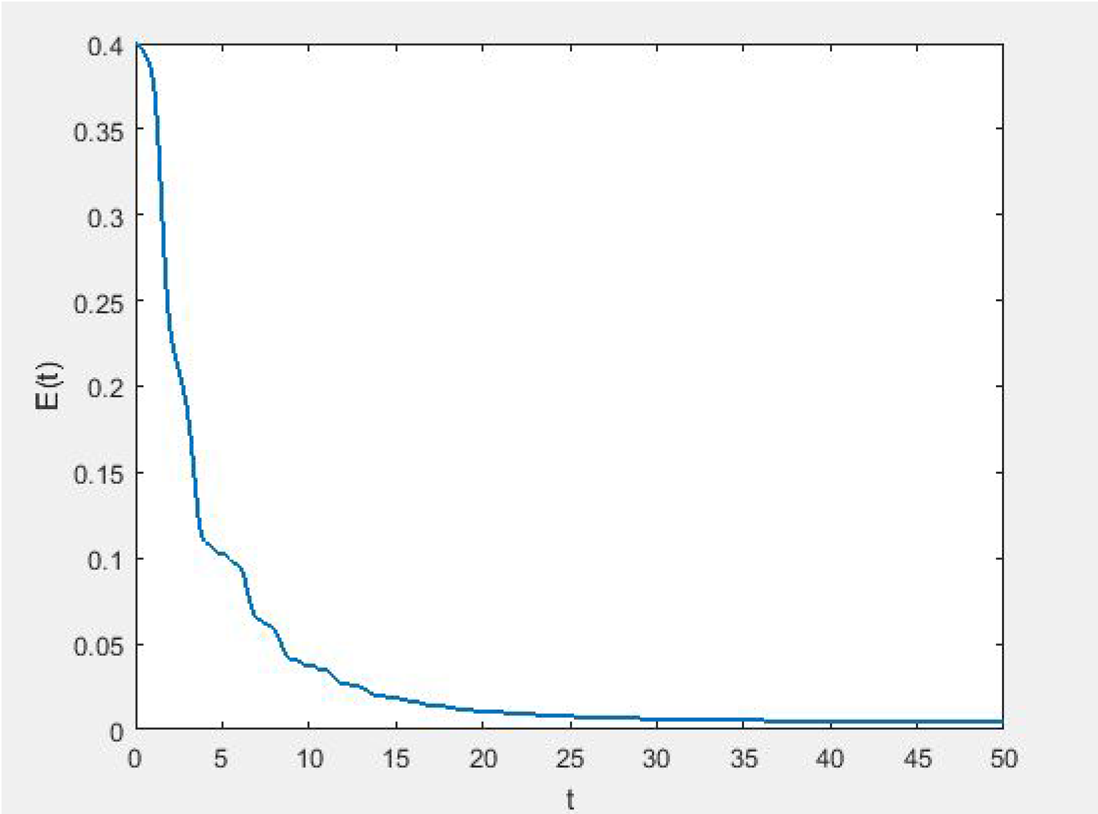

Figure 1: Energy dissipation $E(t)$.

4. A-priori errors estimates

4.1. A-priori error estimates of semi-discrete scheme

This section concerns the derivation of a-priori error estimates of the Galerkin solution of (50)-(51), where the space discretization V_h is the space of Hermite polynomials $\mathbb{P}_3$. These estimates are based on the method used in [1]. The Hermite interpolation of the weak solution $y \in V_h$ is denoted by $\tilde{y}$ such that $\tilde{y}(t,x) = \sum_{j=1}^{p} \Big(y(t,x_j)\phi_{2j-1}(x) + y_x(t,x_j)\phi_{2j}(x) \Big)$. Suppose that

$$y \in C([0,T]; H_E^4(0,1)), \quad y_t \in L^2(0,T; H_E^4(0,1)), \quad y_{tt} \in L^2(0,T; H_E^2(0,1)). \tag{67}$$

For almost every t, we have (cf. [9], [1]):

$$\begin{aligned} \|y - \tilde{y}\|_{H^2(0,1)} \le Ch^2 \|y\|_{H^4(0,1)}, \ \|y_t - \tilde{y}_t\|_{H^2(0,1)} \le Ch^2 \|y_t\|_{H^4(0,1)}, \\ \|y_{tt} - \tilde{y}_{tt}\|_{L^2(0,1)} \le Ch^2 \|y_{tt}\|_{H^2(0,1)}. \end{aligned} \tag{68}$$

The error of the semi-discrete solution y_h is defined as $\epsilon^h := y_h - \tilde{y} \in V_h$. Using (50)-(51), it follows for all $w \in V_h,\ \ t > 0$:

$$\int_0^1 \epsilon_{tt}^h w dx + \int_0^1 \epsilon_{xx}^h w_{xx} dx + \mu_1 \epsilon_{tt}^h(1)w(1) + \mu_2 \epsilon_{xtt}^h(1)w_x(1) + \alpha\epsilon_t^h(1)w(1) - \gamma\epsilon_t^h(1)w_x(1) - \beta\epsilon_{xt}^h(1)w(1) + \nu\epsilon_{xt}^h(1)w_x(1) = \int_0^1 (y_{tt} - \tilde{y}_{tt})w dx + \int_0^1 (y_{xx} - \tilde{y}_{xx})w_{xx} dx. \tag{69}$$

Taking $w = \epsilon_t^h$ and using (5), we obtain:

$$\frac{d}{dt}\hat{E}(t;\epsilon^h) \le \int_0^1 (y_{tt} - \tilde{y}_{tt})\epsilon_t^h dx + \int_0^1 (y_{xx} - \tilde{y}_{xx})\epsilon_{txx}^h dx, \tag{70}$$

for almost every $t \in [0, T]$. Integrating the previous expression over $[0, t]$, it follows:

$$\hat{E}(t;\epsilon^h) \le \hat{E}(0;\epsilon^h(0)) + \int_0^t \Big(\int_0^1 (y_{tt} - \tilde{y}_{tt})\epsilon_t^h dx + \int_0^1 (y_{xx} - \tilde{y}_{xx})\epsilon_{txx}^h dx\Big)ds. \tag{71}$$

Performing integration by part, the relation (71) becomes:

$$\hat{E}(t;\epsilon^h) \le \hat{E}(0;\epsilon^h(0)) + \int_0^t \Big(\int_0^1 (y_{tt} - \tilde{y}_{tt})\epsilon_t^h dx - \int_0^1 (y_{txx} - \tilde{y}_{txx})\epsilon_{xx}^h dx\Big)dt + \int_0^1 (y_{xx}(t,x) - \tilde{y}_{xx}(t,x))\epsilon_{xx}^h(t,x)dx - \int_0^1 (y_{xx}(0,x) - \tilde{y}_{xx}(0,x))\epsilon_{xx}^h(0,x)dx. \tag{72}$$

Applying the Cauchy-Schwarz and Young inequalities to the second member of (72), we get:

$$\Big(1 - \frac{1}{4\eta}\Big)\hat{E}(t;\epsilon^h(t)) \le \Big(1 + \frac{1}{4\eta}\Big)\hat{E}(0;\epsilon^h(0)) + \frac{1}{4\eta}\int_0^t \hat{E}(s;\epsilon^h(s))ds$$

$$+ \eta\Big(\|y_{tt} - \tilde{y}_{tt}\|^2_{L^2(0,T;L^2(0,1))} + \|y_t - \tilde{y}_t\|^2_{L^2(0,T;H^2(0,1))} + 2\,\|y - \tilde{y}\|^2_{C([0,T];H^2(0,1))}\Big).$$

for all $\eta > 0$. Taking $\eta = 1$, and using (68), one obtains:

$$\hat{E}(t;\epsilon^h(t)) \le \frac{5}{3}\hat{E}(0;\epsilon^h(0)) + \frac{1}{3}\int_0^t \hat{E}(s;\epsilon^h(s))ds + Ch^4\Big(\|y_{tt}\|^2_{L^2(0,T;H^2(0,1))} + \|y_t\|^2_{L^2(0,T;H^4(0,1))} + \|y\|^2_{C([0,T];H^4(0,1))}\Big). \tag{73}$$

Gronwall's inequality applied to (73) gives:

$$\hat{E}(t;\epsilon^h(t)) \leq \; C\Big[\hat{E}(0;\epsilon^h(0)) + h^4\Big(\|y_{tt}\|^2_{L^2(0,T;H^2(0,1))} + \|y_t\|^2_{L^2(0,T;H^4(0,1))} + \|y\|^2_{C([0,T];H^4(0,1))}\Big)\Big].$$

Using the previous expression and the triangle inequality, we obtain the following result:

Theorem 4.1. Suppose (67), and take V_h the space of the piecewise cubic Hermite polynomials. For $y_h \in C^2([0,T];V_h)$ solving (50)-(51), we have for all t in $[0,T]$:

$$\hat{E}(t;y_h - y)^{\frac{1}{2}} \leq \; C\Big[\hat{E}(0;\epsilon^h(0))^{\frac{1}{2}} + h^2\Big(\|y_{tt}\|_{L^2(0,T;H^2(0,1))} + \|y_t\|_{L^2(0,T;H^4(0,1))} + \|y\|_{C([0,T];H^4(0,1))}\Big)\Big]. \quad (74)$$

Furthermore, if y_h^0 and y_h^1 are Hermite interpolations of y_0 and y_1, then we have:

$$\hat{E}(t;y_h - y)^{\frac{1}{2}} \leq \; Ch^2\Big(\|y_{tt}\|_{L^2(0,T;H^2(0,1))} + \|y_t\|_{L^2(0,T;H^4(0,1))} + \|y\|_{C([0,T];H^4(0,1))}\Big). \quad (75)$$

4.2. A-priori error estimates of Fully scheme

In this paragraph, a-priori error estimates are given for the scheme (61)-(62). Assume that $y \in H^4(0,T;H_E^2(0,1))$. Let $\breve{y} \in V_h$ be the projection of the weak solution, such that for all $t \in [0;T]$, $a(y(t) - \breve{y}(t), w_h) = 0$, for all $w_h \in V_h$,. One verifies that $\breve{y} \in H^4(0,T;H_E^2(0,1))$ because the projection $y \mapsto \breve{y}$ is bounded in $H_E^2(0,1)$. Moreover, let $y^e := y - \breve{y}$ be the projection error. Suppose that $y \in H^2(0,T;H_E^4(0,1))$. We have (cf. [8]):

$$\begin{gathered}\left\|y^e\right\|_{H^2(0,1)} \leq Ch^2\left\|y\right\|_{H^4(0,1)}, \quad \left\|y_t^e\right\|_{H^2(0,1)} \leq Ch^2\left\|y_t\right\|_{H^4(0,1)}, \\ \left\|y_{tt}^e\right\|_{H^2(0,1)} \leq Ch^2\left\|y_{tt}\right\|_{H^4(0,1)}.\end{gathered} \quad (76)$$

Let $U(t_k) = (y(t_k);y_t(t_k))^\top$ denote the weak solution of (16) at time t_k and $U^k = (y^k;z^k)$ the $k-$th iteration of the fully discrete scheme approximating $U(t_k)$. Then we define the error by:

$$\Psi^k := y^k - \breve{y}(t_k),\; \Phi^k := z^k - \breve{y}_t(t_k),\text{ and } U_e^k := (\Psi^k;\Phi^k), \forall k \in \{0,1,\ldots,s\}.$$

Theorem 4.2. Assume $y \in H^4(0,T;H_E^2(0,1)) \cap H^2(0,T;H_E^4(0,1))$. Let $k \in \{1,\ldots,s\}$. Then we have:

$$\left\|U^k - U(t_k)\right\| \leq C\Big[\left\|U_e^0\right\| + h^2\left\|y\right\|_{H^2(0,T;H^4(0,1))} + (\Delta t)^{\frac{3}{2}}\Big(\|y_{tt}\|_{H^2(0,T;H^2(0,1))} + \|y_{tt}\|_{L^2(0,T;H^4(0,1))}\Big)\Big]. \quad (77)$$

Proof. Let $k \in \{0, 1, \dots, s\}$. Taylor's theorem yields:

$$\forall x \in [0,1],\ \frac{\breve{y}(t_{k+1},x) - \breve{y}(t_k,x)}{\Delta t} = \frac{\breve{y}_t(t_{k+1},x) + \breve{y}_t(t_k,x)}{2} + \Delta t R_1^k(x), \tag{78}$$

$$\text{where } R_1^k(x) = \int_{t_{k+\frac{1}{2}}}^{t_{k+1}} \frac{\breve{y}_{ttt}(t,x)}{2(\Delta t)^2}(t_{k+1}-t)^2 dt + \int_{t_k}^{t_{k+\frac{1}{2}}} \frac{\breve{y}_{ttt}(t,x)}{2(\Delta t)^2}(t_k - t)^2 dt$$
$$- \int_{t_{k+\frac{1}{2}}}^{t_{k+1}} \frac{\breve{y}_{ttt}(t,x)}{2\Delta t}(t_{k+1}-t) dt + \int_{t_k}^{t_{k+\frac{1}{2}}} \frac{\breve{y}_{ttt}(t,x)}{2\Delta t}(t_k - t) dt.$$

From (78), we obtain:

$$\frac{\Psi^{k+1} - \Psi^k}{\Delta t} + \Delta t R_1^k = \frac{\Phi^{k+1} + \Phi^k}{2}. \tag{79}$$

Multiplying the previous expression by $(\Phi^{k+1} - \Phi^k)$ and integrating it over $[0,1]$, it follows:

$$\int_0^1 \frac{\Psi^{k+1} - \Psi^k}{\Delta t}(\Phi^{k+1} - \Phi^k) dx = \frac{1}{2}\int_0^1 \Big((\Phi^{k+1})^2 - (\Phi^k)^2\Big) dx$$
$$- \Delta t \int_0^1 R_1^k(x)(\Phi^{k+1} - \Phi^k) dx.$$

Moreover, replacing t by $t_{k+\frac{1}{2}}$ in (13) and using the Taylor expansion, we obtain:

$$\begin{aligned}
&\int_0^1 \frac{y_t(t_{k+1},x) - y_t(t_k,x)}{\Delta t} w(x) dx + \int_0^1 \frac{y_{xx}(t_{k+1},x) + y_{xx}(t_k,x)}{2} w_{xx}(x) dx \\
&+ \mu_1 \frac{y_t(t_{k+1},1) - y_t(t_k,1)}{\Delta t} w(1) + \mu_2 \frac{y_{tx}(t_{k+1},1) - y_{tx}(t_k,1)}{\Delta t} w_x(1) \\
&+ \alpha \frac{y_t(t_{k+1},1) + y_t(t_k,1)}{2} w(1) + \nu \frac{y_{tx}(t_{k+1},1) + y_{tx}(t_k,1)}{2} w_x(1) \\
&- \beta \frac{y_{tx}(t_{k+1},1) + y_{tx}(t_k,1)}{2} w(1) - \gamma \frac{y_t(t_{k+1},1) + y_t(t_k,1)}{2} w_x(1) \\
&= \Delta t R_2^k(w),
\end{aligned} \tag{80}$$

where the functional $R_2^k : H_E^2[0,1] \to \mathbb{R}$ is given by:

$$\begin{aligned}
R_2^k(w) = &\int_0^1 \Big(\int_{t_{k+\frac{1}{2}}}^{t_{k+1}} \frac{y_{tttt}(t,x)}{2(\Delta t)^2}(t_{k+1}-t)^2 dt + \int_{t_k}^{t_{k+\frac{1}{2}}} \frac{y_{tttt}(t,x)}{2(\Delta t)^2}(t_k - t)^2 dt \Big) w(x) dx \\
&+ \int_0^1 \Big(\int_{t_{k+\frac{1}{2}}}^{t_{k+1}} \frac{y_{ttxx}(t,x)}{2\Delta t}(t_{k+1}-t) dt - \int_{t_k}^{t_{k+\frac{1}{2}}} \frac{y_{ttxx}(t,x)}{2\Delta t}(t_k - t) dt \Big) w_x(x) dx \\
&+ \mu_1 \Big(\int_{t_{k+\frac{1}{2}}}^{t_{k+1}} \frac{y_{tttt}(t,1)}{2(\Delta t)^2}(t_{k+1}-t)^2 dt + \int_{t_k}^{t_{k+\frac{1}{2}}} \frac{y_{tttt}(t,1)}{2(\Delta t)^2}(t_k - t)^2 dt \Big) w(1) \\
&+ \mu_2 \Big(\int_{t_{k+\frac{1}{2}}}^{t_{k+1}} \frac{y_{ttttx}(t,1)}{2(\Delta t)^2}(t_{k+1}-t)^2 dt + \int_{t_k}^{t_{k+\frac{1}{2}}} \frac{y_{ttttx}(t,x)}{2\Delta t}(t_k - t)^2 dt \Big) w_x(1) \\
&+ \alpha \Big(\int_{t_{k+\frac{1}{2}}}^{t_{k+1}} \frac{y_{ttt}(t,1)}{2\Delta t}(t_{k+1}-t) dt - \int_{t_k}^{t_{k+\frac{1}{2}}} \frac{y_{ttt}(t,1)}{2\Delta t}(t_k - t) dt \Big) w(1) \\
&+ \nu \Big(\int_{t_{k+\frac{1}{2}}}^{t_{k+1}} \frac{y_{tttx}(t,1)}{2\Delta t}(t_{k+1}-t) dt - \int_{t_k}^{t_{k+\frac{1}{2}}} \frac{y_{tttx}(t,1)}{2\Delta t}(t_k - t) dt \Big) w_x(1) \\
&- \beta \Big(\int_{t_{k+\frac{1}{2}}}^{t_{k+1}} \frac{y_{tttx}(t,1)}{2\Delta t}(t_{k+1}-t) dt - \int_{t_k}^{t_{k+\frac{1}{2}}} \frac{y_{tttx}(t,1)}{2\Delta t}(t_k - t) dt \Big) w(1) \\
&- \gamma \Big(\int_{t_{k+\frac{1}{2}}}^{t_{k+1}} \frac{y_{ttt}(t,1)}{2\Delta t}(t_{k+1}-t) dt - \int_{t_k}^{t_{k+\frac{1}{2}}} \frac{y_{ttt}(t,1)}{2\Delta t}(t_k - t) dt \Big) w_x(1). \quad (81)
\end{aligned}$$

Now, from (62) and (80), we have for all $w_h \in V_h$:

$$\begin{aligned}
&\int_0^1 \frac{\Phi^{k+1} - \Phi^k}{\Delta t} w_h dx + \int_0^1 \frac{\Psi_{xx}^{k+1} + \Psi_{xx}^k}{2} (w_h)_{xx} dx + \mu_1 \frac{\Phi^{k+1}(1) - \Phi^k(1)}{\Delta t} w_h(1) \\
&+ \mu_2 \frac{\Phi_x^{k+1}(1) - \Phi_x^k(1)}{\Delta t} (w_h)_x(1) + \alpha \frac{\Phi^{k+1}(1) + \Phi^k(1)}{2} w_h(1) \\
&+ \nu \frac{\Phi_x^{k+1}(1) + \Phi_x^k(1)}{2} (w_h)_x(1) \\
&- \beta \frac{\Phi_x^{k+1}(1) + \Phi_x^k(1)}{2} w_h(1) - \gamma \frac{\Phi^{k+1}(1) + \Phi^k(1)}{2} (w_h)_x(1) = G^k(w_h) - \Delta t \, R_2^k(w_h),
\end{aligned} \quad (82)$$

where the functional $G^k(w_h)$ is defined as:

$$\begin{aligned} G^k(w_h) :=& \int_0^1 \frac{y_t^e(t_{k+1},x)-y_t^e(t_k,x)}{\Delta t} w_h dx + \mu_1 \frac{y_t^e(t_{k+1},1)-y_t^e(t_k,1)}{\Delta t} w_h(1) \\ &+ \mu_2 \frac{y_{tx}^e(t_{k+1},1)-y_{tx}^e(t_k,1)}{\Delta t}(w_h)_x(1) + \alpha \frac{y_t^e(t_{k+1},1)+y_t^e(t_k,1)}{2} w_h(1) \\ &+ \nu \frac{y_{tx}^e(t_{k+1},1)+y_{tx}^e(t_k,1)}{2}(w_h)_x(1) - \beta \frac{y_{tx}^e(t_{k+1},1)+y_{tx}^e(t_k,1)}{2} w_h(1) \\ &- \gamma \frac{y_t^e(t_{k+1},1)+y_t^e(t_k,1)}{2}(w_h)_x(1). \end{aligned} \tag{83}$$

Taking $w_h = \frac{\Delta t}{2}(\Phi^{k+1}+\Phi^k) \in V_h$ in (82) and using (5), and next, by differentiating (79) twice and integrating it over $[0,1]$, we obtain:

$$\begin{aligned} \left\|U_e^{k+1}\right\|^2 - \left\|U_e^k\right\|^2 \leq& -\frac{(\Delta t)^2}{2}\int_0^1 (R_1^k)_{xx}\left(\Psi_{xx}^{k+1}+\Psi_{xx}^k\right)dx \\ &+ \frac{\Delta t}{2} G^k(\Phi^{k+1}+\Phi^k) - \frac{(\Delta t)^2}{2} R_2^k(\Phi^{k+1}+\Phi^k). \end{aligned} \tag{84}$$

We show that:

$$\left\|R_1^k\right\|^2_{H^2(0,1)} \leq \Delta t \int_{t_k}^{t_{k+1}} \|\breve{y}_{ttt}(t)\|^2_{H^2(0,1)} dt \leq C\Delta t \int_{t_k}^{t_{k+1}} \|y_{ttt}(t)\|^2_{H^2(0,1)} dt. \tag{85}$$

Rewriting the second term of $R_2^k(w)$ (using the fact that $w(0) = w_x(0) = 0$), we get:

$$\begin{aligned} &\int_0^1 \left(\int_{t_{k+\frac12}}^{t_{k+1}} \frac{y_{ttxx}(t,x)}{2\Delta t}(t_{k+1}-t)dt - \int_{t_k}^{t_{k+\frac12}} \frac{y_{ttxx}(t,x)}{2\Delta t}(t_k-t)dt \right) w_{xx}(x)dx \\ &= \int_{t_{k+\frac12}}^{t_{k+1}} \frac{t_{k+1}-t}{2\Delta t}\left(y_{ttxx}(t,1)w_x(1) - y_{ttxxx}(t,1)w(1) + \int_0^1 y_{ttxxxx}(t,x)w(x)dx \right) dt \\ &\quad - \int_{t_k}^{t_{k+\frac12}} \frac{t_k-t}{2\Delta t}\left(y_{ttxx}(t,1)w_x(1) - y_{ttxxx}(t,1)w(1) + \int_0^1 y_{ttxxxx}(t,x)w(x)dx \right) dt. \end{aligned}$$

Then we have:

$$\begin{aligned} R_2^k(\Phi^k) \leq C\Big[&\left\|\Phi^k\right\|^2_{L^2(0,1)} + |\Phi^k(1)|^2 + |\Phi_x^k(1)|^2 + \Delta t \int_{t_k}^{t_{k+1}} \Big(\|y_{tttt}(t)\|^2_{H^2(0,1)} \\ &+ \|y_{ttt}(t)\|^2_{H^2(0,1)} + \|y_{tt}(t)\|^2_{H^4(0,1)} \Big) dt \Big]. \end{aligned} \tag{86}$$

Next, from (83) follows:

$$|G^k(\Phi^{k+1}+\Phi^k)| \le C\Big(\|\Phi^{k+1}+\Phi^k\|^2_{L^2(0,1)} + |\Phi^{k+1}(1)+\Phi^k(1)|^2 + |\Phi_x^{k+1}(1)+\Phi_x^k(1)|^2 + \frac{1}{\Delta t}\int_{t_k}^{t_{k+1}} (\|y_{tt}^e(t)\|^2_{L^2(0,1)} + |y_{tt}^e(t,1)|^2 + |y_{ttx}^e(t,1)|^2)dt + \|y_t^e\|^2_{C([t_k;t_{k+1}];H^2(0,1))}\Big). \tag{87}$$

Using the relations (85)-(87), (84) becomes:

$$\|U_e^{k+1}\|^2 - \|U_e^k\|^2 \le C\Big[\Delta t(\|U_e^{k+1}\|^2 + \|U_e^k\|^2) + \Delta t\, \|y_t^e\|^2_{C([t_k;t_{k+1}];H^2(0,1))} + \int_{t_k}^{t_{k+1}} \Big(\|y_{tt}^e(t)\|^2_{L^2(0,1)} + |y_{tt}^e(t,1)|^2 + |y_{ttx}^e(t,1)|^2\Big)dt + (\Delta t)^3 \int_{t_k}^{t_{k+1}} \Big(\|y_{tttt}(t)\|^2_{H^2(0,1)} + \|y_{ttt}(t)\|^2_{H^2(0,1)} + \|y_{tt}(t)\|^2_{H^4(0,1)}\Big)dt\Big]. \tag{88}$$

Let $m \in \{0,1,\dots,s\}$. Suppose that $\Delta t \le \frac{1}{2C}$ (with C from (88)). Summing (88) over $k \in \{0,1,\dots,m\}$, we get:

$$\frac{1}{2}\|U_e^{m+1}\|^2 \le \frac{3}{2}\|U_e^0\|^2 + C\Big(\Delta t\sum_{k=1}^{m}\|U_e^k\|^2 + \|y_t^e\|^2_{C([0,T],H^2(0,1))} + \|y_{tt}^e\|^2_{L^2(0,T;H^2(0,1))} + (\Delta t)^3(\|y_{tttt}\|^2_{L^2(0,T;H^2(0,1))} + \|y_{ttt}\|^2_{L^2(0,T;H^2(0,1))} + \|y_{tt}\|^2_{L^2(0,T;H^4(0,1))})\Big). \tag{89}$$

Finally, using the discrete-in-time Gronwall inequality and (89), we obtain:

$$\|U_e^{m+1}\|^2 \le C\Big(\|U_e^0\|^2 + h^4(\|y_t\|^2_{C([0,T],H^4(0,1))} + \|y_{tt}\|^2_{L^2(0,T;H^4(0,1))}) + (\Delta t)^3(\|y_{tttt}\|^2_{L^2(0,T;H^2(0,1))} + \|y_{ttt}\|^2_{L^2(0,T;H^2(0,1))} + \|y_{tt}\|^2_{L^2(0,T;H^4(0,1))})\Big). \tag{90}$$

The result now follows from the previous expression, (76) and the triangle inequality. ■

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