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Analytic and Entire Duals of a Class of Sequence of Fuzzy Numbers[1]

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Abstract

In this paper we define the analytical and entire duals of some known sequence spaces. Also we define perfect space and we try to obtain some inclusion relations.

Key words and Phrases: Fuzzy number, entire sequence space, Analytical dual, entire dual.

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1. INTRODUCTION

In recent years there has been an increasing interest in mathematical aspects of operations defined on fuzzy sets. The concept of fuzzy sets and fuzzy set operations was first introduced by Zadeh[1] and subsequently several authors have discussed various aspects of theory and applications of fuzzy sets. Bounded and convergent sequences of fuzzy numbers were introduced by Matloka [2] where it was shown that every convergent sequence of fuzzy numbers is bounded. In [3] Nandha has studied the space of all absolutely p-summable convergent sequences of fuzzy numbers and shown that they are all complete metric spaces.Later on sequences of fuzzy numbers have been discussed by many authors [4, 5, 6, 7]. Ozer Talo and Feyzi Basar [8] determined the duals of classical sets of sequences of fuzzy numbers and related matrix transformations.

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The space of all entire sequences was introduced and studied by Ganapathy Iyer [9]. Recently the fuzzy entire sequence space was intoduced by Kavikumar, Asme Bin Khamis and Kandasamy[10]. Also Orlicz space of entire sequences of fuzzy numbers was introduced by Subramanian and Metin Basarir [11]. The main purpose of this article is to introduce Analytical dual and entire dual of sequence space of fuzzy numbers and we try to obtain some inclusion relations.

The rest of the paper is organized as follows:

In section 2, some required definitions and consequences related with the fuzzy numbers are given. In section 3, we find the Analytical and Entire duals of some known sequence space of fuzzy numbers and some inclusion relations are also obtained.

2 DEFINITIONS AND PRELIMINARIES

We begin with giving some required definitions and statements of theorems, propositions and lemmas. A fuzzy number is a fuzzy set on the real axis.i.e. a mapping $u : R \to [0, 1]$ which satisfies the following four conditions.

i. u is normal i.e. there exists an $x_o \in R$ such that $u(x_o) = 1$.

ii. u is fuzzy convex i.e. $u[\lambda x + (1 - \lambda y] \geq \min\{u(x), u(y)\}$ for all $x, y \in R$ and for all $\lambda \in [0, 1]$.

iii. u is upper semi continuous.

iv. The set $[u]_0 = \overline{\{x \in R : u(x) > 0\}}$ is compact (Zadeh [1]) where $\overline{\{x \in R : u(x) > 0\}}$ denote the closure of the set $\{x \in R : u(x) > 0\}$ in the usual topology of R

We denote the set of all fuzzy numbers on R by E' and called it as the space of fuzzy numbers. The λ -level set $[u]_\lambda$ of $u \in E'$ is defined by

$$[u]_\lambda = \begin{cases} \{t \in R : u(t) \geq \lambda\}, & (0 < \lambda \leq 1) \\ \{t \in R : u(t) > \lambda, & \lambda = 0 \end{cases}$$

The set $[u]_\lambda$ is a closed bounded and non-empty interval for each $\lambda \in [0, 1]$ which is defined by $[u]_\lambda = [u^-(\lambda), u^+(\lambda)]$ can be embedded in E'. Since each $r \in R$ can be regarded as a fuzzy number defined by r(x) = 1, (x = r)

0 , (x ≠ r)

Let u, v, w ∈ E' and k ∈ R. The operations addition, scalar multiplication, product and division defined on E' by

$u + v = w \Rightarrow [w]_\lambda = [u]_\lambda + [v]_\lambda$ f or all $\lambda \in [0, 1] \Leftrightarrow$
$w^-(\lambda) == [u^-(\lambda), v^-(\lambda)]$ and

$w^+(\lambda) = [u^+(\lambda), v^+(\lambda)]$ and f or all $\lambda \in [0, 1]$

$[ku]_\lambda = k[u]_\lambda$ f or all $\lambda \in [0, 1]$

and $uv = w \Leftrightarrow [w]_\lambda = [u]_\lambda[v]_\lambda$ f or all $\lambda \in [0, 1]$

where it is immediate that

$$w^-(\lambda) = \min\{u^-(\lambda)v^-(\lambda), u^-(\lambda)v^+(\lambda), u^+(\lambda)v^-(\lambda), u^+(\lambda)v^+(\lambda)\}$$

$$\text{and } w^+(\lambda) = \max\{u^-(\lambda)v^-(\lambda), u^-(\lambda)v^+(\lambda), u^+(\lambda)v^-(\lambda), u^+(\lambda)v^+(\lambda)\}$$

for all $\lambda \in [0,1]$

$u/v = w \Leftrightarrow [w]_\alpha = [u]_\alpha/[v]_\alpha$ f or all $\alpha \in [0, 1]$

Let W be the set of all closed and bounded intervals A of real numbers with endpoints $\underline{A}$ and $\overline{A}$ i.e. $A = [\underline{A}, \overline{A}]$.

Define the relation d on W by

$$d(A, B) = \max\{|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|\}$$

Then it can be observed that d is a metric on W

(W, d) is a complete metric space(Nanda[3]).

Now we can define the metric D on E' by

means of a Hausdroff metric d as

$$D(u, v) = \sup_{\lambda \in [0,1]} d([u]_\lambda, [v]_\lambda) = \sup \max\{|u^-(\lambda) - v^-(\lambda)|, |u^+(\lambda) - v^+(\lambda)|\}$$

(E', D) is a complete metric space. One can extend the natural order relation on the real line to intervals as follows.

$A \leq B$ if and only if $\underline{A} \leq \underline{B}$ and $A \leq B$

The partial order relation on E'is defined as follows.

$u \leq v \Leftrightarrow [u]_\lambda \leq [v]_\lambda \Leftrightarrow u^-(\lambda) \leq v^-(\lambda)$ and $u^+(\lambda) \leq v^+(\lambda)$ for all $\lambda \in [0, 1]$

$u \in E'$ is a non-negative fuzzy number if and only if $u(x) = 0$ for all $x < 0$

It is immediate that $u \geq 0$ if u is a non negative fuzzy number.

One can see that $\bar{0}$

$$D(u, \bar{0}) = \sup_{\lambda\in[0,1]} \max \{|u^-(\lambda), u^+(\lambda)|\}$$

Proposition 2.1 Let $u, v, w \in E'$ and $k \in R$. Then

i. (E', D) is a complete metric space.
ii. $D(ku, kv) = |k|D(u, v)$.
iii. $(u + v, w + v) = D(u, w)$.
iv. $D(u + v, w + z) \leq D(u, w) + D(v, z)$.
v. $|D(u, \bar{0})-D(v, \bar{0})| \leq D(u, v) \leq D(u, \bar{0}) + D(v, \bar{0})$.

Lemma 2.2 The following statements hold (Talo [8]).

i) $D(uv, \bar{0}) \leq D(u, \bar{0})D(v, \bar{0})$

ii. If $u_k \to u$ as $k \to \infty$ then $D(u_k, \bar{0}) \to D(u, \bar{0})$ as $k \to \infty$ where $(u_k) \in w(F)$

In the sequel, we require the following Definitions and lemmas

Definition 2.3 A sequence $u = (u_k)$ of fuzzy numbers is a function u from the set N into the set E'. The fuzzy number u_k denotes the value of the function at $k \in N$ and is called the k^{th} term of the sequence. Let w(F) denote the set of all sequences of fuzzy numbers.

Definition 2.4 A sequence $(u_k) \in w(F)$ is called convergent with limit $u \in E'$ if and only if for every > 0 there exists an $n_0 = n_0(\varepsilon) \in N$ such that

$$D(u_k, u) < \varepsilon \text{ for all } k \geq n_0.$$

Theorem 2.5 ([2]) Let $(u_k), (v_k) \in w(F)$ with $u_k \to a$, $v_k \to b$ as $k \to \infty$. Then,

i. $u_k + v_k \to a + b$ as $k \to \infty$.

ii. $u_k - v_k \to a - b$ as $k \to \infty$.

iii. $u_k v_k \to ab$ as $k \to \infty$.

iv. $u_k/v_k \to a/b$ as $k \to \infty$ where $0 \notin [v_k]_0$ for all $k \in N$ and $\bar{0} \notin [b]_0$.

Definition 2.6 A sequence $(u_k) \in w(F)$ is called bounded if and only if the set of all fuzzy numbers consisting of the terms of the sequence (u_k) is a bounded set.

That is to say that a sequence $(u_k) \in w(F)$ is said to be bounded if and only if there exist two fuzzy numbers m and M such that $m \le u_k \le M$ for all $k \in N$.

Definition 2.7 A sequence $(u_k) \in w(F)$ is called entire sequence if and only if

$$D(u_k, \bar{0})^{1/k} \to 0 \text{ as } k \to \infty$$

Definition 2.8 A sequence $(u_k) \in w(F)$ is called analytic sequence if and only if

$$\sup D(u_k, \bar{0})^{1/k} < \infty$$

3. MAIN RESULTS

In this section we define the analytical and entire duals of the sequence space of fuzzy numbers and we find the duals for some known sequence spaces.

Definition 3.1 Let X(F) be a sequence space of fuzzy numbers. The analytical dual or Λ -dual of X(F) is defined as

$$X^{\Lambda}(F) = \{u = (u_k) : (x_k u_k) \in \Lambda(F) \text{ for every } u = u_k \in w(F)\}$$

The entire dual or Γ -dual of X(F) is defined as

$$X^{\Gamma}(F) = \{u = (u_k) : (x_k u_k) \in \Gamma(F) \text{ for every } u = u_k \in w(F)\}.$$

Proposition 3.2 The analytic and entire duals of sequence of fuzzy numbers has the following properties

i. $X^{\varsigma}(F)(\varsigma = \Gamma \text{ or } \Lambda)$ is a linear subspace of w(F) for every subset of w(F)

ii. $X(F) \subset w(F)$ implies $Y^{\varsigma}(F) \subset X^{\varsigma}(F)(\varsigma = \Gamma \text{ or } \Lambda)$

iii. $X^{\Gamma}(F) \subseteq X^{\Lambda}(F)$ for every $X(F) \subset w(F)$

Definition 3.3 The sequence space X(F) of fuzzy numbers is called a perfect space if $X^{\Lambda\Lambda}(F) = X(F)$ or $X^{\Gamma\Gamma}(F) = X(F)$.

Theorem 3.4 The Λ -dual of $c_0(F)$, $c(F)$, $l_\infty(F)$ and $\Gamma(F)$ are $\Lambda(F)$.

Proof. We give the proof for $c_0(F)$ and $\Gamma(F)$.

First let us prove that $c_0^\Lambda(F) = \Lambda(F)$. Suppose $(x_k) \in w(F)$.

Then there exist M > 0 such that sup < M for all k.

Let $(u_k) \in c_0(F)$. Then for given > 0, There exists n_0 such that $D(u_k, \bar{0}) <$ for all $k \geq n_0$

Now, $D(x_k u_k, \bar{0}) \leq D(x_k, \bar{0}) D(u_k, \bar{0})$
which implies

$D(x_k u_k, \bar{0})^{1/k} \leq D(x_k, \bar{0})^{1/k} D(u_k, \bar{0})^{1/k}$

Hence $(x_k u_k) \in \Lambda(F)$. Therefore $(x_k) \in c^\Lambda(F)$. Thus $\Lambda(F) \subset c^\Lambda(F)$. . On the other hand, Let $(x_k) \subset c^\Lambda(F)$. Suppose if $(x_k) \notin \Lambda(F)$ Construct a sequence $u = (u_k)$ as $u_k = (0, 0,.. , 1, 0, 0,...)$ with 1 in the k^{th} place and 0 elsewhere. Then $(u_k) \in c_0 F)$ Since$(x_k) \notin \Lambda(F)$,the sequence $(x_k u_k) \notin \Lambda(F)$ which is a contradiction to the fact that $(x_k) \in c_0$ (F). Thus $(x_k) \in \Lambda(F)$ and hence c_0 (F) = $\Lambda(F)$. Therefore c_0 (F) = $\Lambda(F)$. Similar proof can be given for $c_o(F) = N(F)$. Similar Proof can be given for c(F) and $l_\infty(F)$ Now we prove that $\Gamma^\Lambda(F) = \Lambda(F)$. Since $\Gamma(F) \subset \Lambda$ (F) , we have from the proposition 3.2 $\Lambda(F) = l_\infty^\Lambda(F) \subset \Gamma^\Lambda(F)$. Thus $\Lambda(F) \subset \Gamma^\Lambda(F)$. The other inclusion arises as in the case of $c_0(F)$.

Theorem 3.5. The Γ -dual of $c_0(F)$, $c(F)$, $l_{\infty(F)}$ and $\Gamma(F)$ is $\Lambda(F)$.

Proof. The proof is similar to theorem 3.4

Theorem 3.6 The Λ -and Γ -duals of a sequence space X(F) of fuzzy numbers are normal.

Proof We give the proof for Λ -dual. Similar proof follows for Γ -dual. Let (x_k), (y_k) are two sequences of fuzzy numbers such that

$$D(y_k, \bar{0}) \leq D(x_k, \bar{0}) \text{ f or all } k \in N \tag{1}$$

Let $(x_k) \in X^\Lambda(F)$. Then there exists M > 0 such that $D(x_k y_k, 0)^{1/k} < M$ for all $(x_k) \in X(F)$ and for all k. From (2), $D(y_k, \bar{0}) \leq D(u_k x_k, \bar{0})$ implies $D(u_k y_k, \bar{0})^{1/k} \leq D(u_k x_k, \bar{0})^{1/k} < M$ for all $(u_k) \in X(F)$ and for all k. Thus $(y_k) \in X^\Lambda(F)$.

Theorem 3.7 If $l(F) \subset X(F) \subset \Lambda(F)$ then $X^\Gamma(F) = \Gamma(F)$.

Proof. From proposition 3.2, $l(F) \subset X((F) \subset \Lambda(F)$ implies $l^\Gamma(F) \supset X^\Gamma(F) \supset \Lambda^\Gamma(F)$
Therefore $\Gamma(F) \supset X^\Gamma(F) \supset \Gamma(F)$. Thus $X^\Gamma(F) = \Gamma(F)$.

Theorem 3.8 Let X(F) be a sequence of fuzzy numbers such that $\Gamma(F) \subset X(F) \subset$

$\Lambda(F)$. Then $X^{\Lambda}(F) = \Lambda(F)$.

Proof. We prove the theorem in three steps. In the first step we prove that $\Gamma^{\Lambda}(F) = \Lambda(F)$. Let $(x_k) \in \Lambda(F)$. Then sup Then sup $D(x_k, 0) < M < \infty$. For any $(u_k) \in \Gamma(F)$, $D(u_k x_k, 0)^{1/k} < M$. Thus $\Lambda(F) \subset \Gamma^{\Lambda}(F)$. On the other hand suppose if $x_k \notin \Lambda(F)$, then there exists an increasing sequence $n_1 < n_2 < n_3 <\ldots$ such that

$$D(x_{nk},0)^{1/k} > p^{nk} \text{ where } p>1$$

is an integer construct a sequence $u = (u_k)$ as

$$u_n(t) = \begin{cases} n = n_k,\ 0 < t < 1; \\ 0 \quad \text{otherwise} \end{cases}$$

Then $u \in \Gamma(F)$. But $D(x_{nk}\, u_{nk}, \bar{0}) \leq D(x_{nk}, \bar{0})^{1/nk} D(u_{nk}, \bar{0})^{1/nk} > p^{n_k \frac{k}{p^{n_k}} t}$

So that $\{x_{nk}\, u_{nk}\} \notin \Lambda(F)$. In the second step we prove that $\Lambda^{\Lambda}(F) = \Lambda(F)$. Let $(x_k) \in \Lambda^{\Lambda}(F)$ Then

$$\sup D(x_k u_k, \bar{0})1/k < \infty \text{ f or all } (u_k) \in \Lambda(F)$$

Hence there exists M > 0 such that

$$\sup D(x_k u_k, \bar{0})^{1/k} < \infty \text{ f or all } (u_k) \in \Lambda(F)$$

Since $D(x_k u_k, \bar{0})^{1/k}$ is true for $(u_k) \in \Lambda(F)$, it is true for $(u_k) = I \in \Lambda(F)$ also. Thus we have $D(x_k u_k, \bar{0})^{1/k} < M$. Hence $\Lambda^{\Lambda}(F) \subset \Lambda(F)$, similarly we can prove that $\Lambda(F) \subset \Lambda^{\Lambda}(F)$. Therefore $\Lambda^{\Lambda}(F) = \Lambda(F)$. In the third step we prove that $X^{\Lambda}(F) = \Lambda(F)$. From the given hypothesis, $\Gamma(F) \subset X(F)\Lambda(F)$. Also from *proposition 3.2*,

$$\Gamma(F) \subset X(F) \text{ implies } X^{\Lambda}(F) \subset \Gamma^{\Lambda}(F)$$

By the first step we have

$$X^{\Lambda}(F) \subset \Lambda(F) \qquad (2)$$

Again by proposition 3.2,

$$X(F) \subset \Lambda(F) \text{ implies } \Lambda^{\Lambda}(F) \subset X^{\Lambda}(F)$$

By the second step we have

$$\Lambda(F) \subset X^{\Lambda}(F) \qquad (3)$$

From (2) and (3), we have $X^{\Lambda}(F) = \Lambda(F)$

Corollary 3.9. The only Λ perfect space $X(F)$ with $\Gamma(F) \subset X(F) \subset \Lambda(F)$ is $\Lambda(F)$

Proof. Since $\Gamma(F)$ $X(F)$, we have from proposition 3.2

$X^{\Lambda}(F)$ $\Gamma^{\Lambda}(F) = X(F)$ Again by applying proposition 3.2

$X^{\Lambda}(F) = X(F)$. Also $X(F)$ satisfies $\Gamma(F) \subset X(F) \subset \Lambda(F)$.

Thus $\Lambda(F) = X(F)$.

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