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Stationarity and Separability of Spatiotemporal Covariance Functions

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Abstract

The modeling of spatiotemporal processes involving in both space and time is crucial in many fields such as epidemiology, climate prediction and meteorology, environmental sciences, biology and other disciplines. Separability and stationarity of spatiotemporal covariance function is a sole property of complex spatiotemporal dependencies and spatiotemporal modeling because separability process overawed this complexity problem. The main advantage of separability spatiotemporal process is the covariance matrix can be expressed as the Kronecker product of spatial and temporal covariance functions. For the test of separability and stationarity, we used spectral methods the mechanics of the test can be reduced to a simple two-factor analysis of variance (ANOVA) procedure. We apply the statistical methods proposed here to test for separability and stationarity of spatiotemporal leishmaniasis disease infection rate using data from University of Gondar leishmaniasis Research and Treatment Center.

AMS subject classification:
Keywords: Separability, Spectral, Covariance; Spatiotemporal, Stationarity, Leishmaniasis, Periodiogram.

1. Introduction

Many applications of spatiotemporal statistics for instance in epidemiology, oceanography, atmosphere, ecology and other disciplines involves spatiotemporal process and dependencies and the difficult challenges in modeling spatiotemporal arrangement can be overcome by using separability process. A random process $Y(S,t)$ is said to have a separable separable spatiotemporal covariance function if for all $S, X \in \mathrm{D}_s \subset \mathrm{R}^m, t, r \in \mathrm{D}_t \subset \mathrm{R}$

$$Cov(Y(S;t), Y(X;r)) = C^{(S)}(S,X).C^{(t)}(t,r) \tag{1}$$

Where $C^{(S)}$ and $C^{(t)}$ are spatial and temporal covariance functions respectively. This class of separability posses immense computational benefits because the covariance matrix can be expressed as the Kronecker product of two matrixes coming from spatial and temporal processes explicitly. Thus, separability is a desired property of the spatiotemporal process. However; these separable models do not always exist and appropriate, no formal test is available for the separability of the Spatiotemporal process. Undoubtedly, the most relevant work done by [4] he used the spectral methods to test separability of spatiotemporal covariance functions, the mechanics of their test can be reduced to a simple two-factor analysis of variance (ANOVA) procedure. The study by [6] they used the Wald test to test a doubly-geometric process against a more general unilateral autoregressive process in a time series context. [1] Presented and studied a new methodology and accompanying theory to test for separability of spatiotemporal functional data. They focused on testing for the separation of space and time in spatiotemporal data, their methods can be applied to any area where separability is convenient, including biomedical imaging. They presented three tests, one being a functional extension of the Monte Carlo likelihood method of [8] and [9], while the other two are based on quadratic forms. Its tests are based on asymptotic distributions of maximum likelihood estimators and do not require Monte Carlo or bootstrap replications. [5] They discussed separable approximations of nonseparable space-time covariance matrices. Specifically, they described the nearest Kronecker product approximation, in the Frobenius norm, of a space-time covariance matrix. In this work, we consider also the problem of testing a given spatiotemporal process for separability. The approach we propose here is based on a spectral representation of the process, and the proposed method consists essentially in studying if the coherence function of the process [4], is constant across frequencies. The approach we propose is based on only one realization of the spatial-temporal process and it is a nonparametric test similar to [4]. The study by [10], They used a likelihood ratio test based on multivariate repeated measures to the spatiotemporal context. The study by [3], they proposed a testing technique for detecting separability in the spatiotemporal dependence structure. Its approach is based on the representation of the log-periodogram as the response variable in a regression model. [2] and [7] derived a new approach that allows one to obtain many classes of nonseparable, spatiotemporal covariance functions.

2. Spectral interpretation of separability

2.1. Separable stationary processes

We consider $Y(S : t) : S \in \mathrm{D}_s \subset \mathrm{R}^d, t \in \mathrm{D}_t \subset \mathrm{R}$ be a spatiotemporal mean process observed at N spatial coordinates $(s_1, t_1), (s_2, t_2), \ldots (s_N, t_N)$. By assuming the spatiotemporal covariance functions is stationary in space and time,

$$Cov(Y(s_1, t_1), Y(s_2, t_2) = C(s_1 - s_2; t_1 - t_2), s_1, s_2 \in \mathrm{D}_s \subset \mathrm{R}^d, t_1, t_2 \in \mathrm{D}_t \subset \mathrm{R} \quad (2)$$

$$Cov(Y(s_1, t_1), Y(s_2, t_2) = C(A; B)$$

where; $A = s_1 - s_2$ and $B = t_1 - t_2$, By Bochner's theorem, we can write the covariance C interms of the spectral density g of spatiotemporal process Y.

$$\mathrm{C}(A, B) = \int\int exp(iA^T\varpi + iB\tau)\mathrm{g}(\varpi, \tau)d\varpi d\tau \quad (3)$$

If C is integrable then, we have

$$\mathrm{g}(\varpi, \tau) = (2\pi)^{-d-1}\sum_{B=-\infty}^{B=\infty}\int exp(-iA^T\varpi - iB\tau)\mathrm{C}(A, B)dA \quad (4)$$

$$\mathrm{g}(\varpi, \tau) = (2\pi)^{-d}\int exp(-iA^T\varpi)f(A, \tau)dA \quad (5)$$

For any fixed A, $f(A, \tau)$ is the cross-spectral density function time process $X_1(t) = Y(s; t)$ and $X_2(t) = Y(s + A; t)$ and we have

$$f(A; \tau) = (2\pi)^{-1}\sum_{B=-\infty}^{B=\infty} exp(-iB\tau)\mathrm{C}(A, B) \quad (6)$$

If C is a separable spatiotemporal covariance function, then we can write, $C(A, B) = C_1(A)C_2(B)$, where, C_1 is a positive definite function in R^d and C_2 is a positive definite function in R. Thus, $f(A; \tau)$ is the product function of A and a function of τ

$$f(A; \tau) = (2\pi)^{-1}\sum_{B=-\infty}^{B=\infty} exp(-iB\tau)\mathrm{C}(A, B) \quad (7)$$

$$f(A; \tau) = (2\pi)^{-1}\sum_{B=-\infty}^{B=\infty} exp(-iB\tau)\mathrm{C}_1(A)\mathrm{C}_2(B) \quad (8)$$

$$f(A;\tau) = \mathrm{C}_1(A)[(2\pi)^{-1} \sum_{B=-\infty}^{B=\infty} exp(-iB\tau)\mathrm{C}_2(B)] \tag{9}$$

$$= \mathrm{C}_1(A)\mathrm{m}(\tau) \tag{10}$$

[4] use this spectral representation, where m is an integrable and positive function and C_1 for each fixed τ is a covariance function of A and an integrable function of A. We can obtain a non-separable spatiotemporal covariance function by making C_1 depend on τ, Thus we get;

$$\mathrm{C}(A,B) = \int exp(-iB\tau)\mathrm{C}_1(A,\tau)\mathrm{m}(\tau)d\tau$$

[2] and [4] use this spectral representation to generate a parametric model of the non-separable spatiotemporal stationary covariance function. If Y is separable,

$$f(A;\tau) = \mathrm{C}_1(A)\mathrm{m}(\tau) \tag{11}$$

Thus, f could be a complex function, but when Y is separable and stationary f is real.

2.2. Nonstationarity in space

We assume Y is not second-order stationary [4], the cross product function $f(.;t)$ between the two time series process $\mathrm{W}_1(t) = Y(p;t)$ and $\mathrm{W}_2(t) = Y(q;t)$ is not just a function of the spatial distance between two points p and q in D but it depends also the location of p and q. Then, we write the cross-spectral function as $f_{p,q}(\tau)$ for p and q in D, and $\tau \in [0,\infty)$

$$f_{p,q}(\tau) = (2\pi)^{-1} \sum_{B=-\infty}^{B=\infty} exp(-iB\tau)Cov(Y(p;B+t),Y(q,t)) \tag{12}$$

We define the coherency between site p and site q

$$\mathrm{R}_{p,q}(\tau) = \frac{|f_{pq}(\tau)|}{[f_{pp}(\tau)f_{qq}(\tau)]^{\frac{1}{2}}} \tag{13}$$

$|\mathrm{R}_{pq}(\tau)|^2$ is the frequency analog of the coefficient of correlation between the two time process $\mathrm{W}_1(t) = Y(p;t)$ and $\mathrm{W}_2(t) = Y(q;t)$.If the process has separable covariance nonstationary in space (stationary in time), we have

$$Cov(Y(p;B+t),Y(q,t)) = \mathrm{C}_s(p,q)\mathrm{C}_2(B)$$

where C_s is the spatial covariance, and then;

$$f_{p,q}(\tau) = \mathrm{C}_s(p,q)m(\tau)$$

$$f_{pp}(\tau) = \mathrm{C}_s(p, p)m(\tau)$$
$$f_{qq}(\tau) = \mathrm{C}_s(q, q)m(\tau)$$
$$\mathrm{R}_{p,q}(\tau) = \frac{\left|f_{pq}(\tau)\right|}{[f_{pp}(\tau)f_{qq}(\tau)]^{\frac{1}{2}}} = \frac{\mathrm{C}_s(p, q)}{[\mathrm{C}_s(p, p)\mathrm{C}_s(q, q)]^{\frac{1}{2}}}$$

Thus if the process is separable, the coherency $\mathrm{R}_{p,q}(\tau)$, does not depend on the frequency τ.If the process is stationary in space, the covariance function C_s is stationary and then $\mathrm{R}_{p,q}(\tau)$ depend on the vector distance between the two locations.We test for separability by studying if $\mathrm{R}_{p,q}(\tau)$ depends on the frequency τ [4]. A stationary test is a side effect test, we test for stationarity by studying if, $\mathrm{R}_{pi,qi}(\tau)$ depends on locations. When R evaluated (estimated) at pairs (p_i, q_i) all separated the same distance A.

3. Spectral estimates

3.1. Estimating the spectral density

We use a nonparametric estimate of the cross-spectral density density $f_{p,q}(\varpi)$ and we study its asymptotic properties as $\mathrm{T} \rightarrow \infty$ the estimate, $\hat{f}_{p,q}(\varpi)$ is the spatial and temporal smoothed version of the cross periodogram of two time process $Y(p, t)$ and $Y(q, t)$, we use two filters a spectral filter V and spatial filter u_ρ, to obtain the weight average of the cross periodogram value concentrating weight in a neighbouring of ϖ (filter V) and in a spatial neighbourhood of p and q (filter u_ρ).

$$J_p(\varpi) = \sum_{t=0}^{T-1} K\left(\frac{t}{T}\right) Y(p, t)exp(-it\varpi)$$

$$J_q(\varpi) = \sum_{t=0}^{T-1} K\left(\frac{t}{T}\right) Y(q, t)exp(-it\varpi)$$

where, $T = Nt$

$$I_{pq}(\varpi) = \left\{2\pi \sum_{t=0}^{T-1} K\left(\frac{t}{T}\right)^2\right\}^{-1} J_p(\varpi)J_q^c(\varpi)$$

Where K is a tapering function, the entries of $I_{pq}(\varpi)$ is second-order periodogram of tapered values $K\left(\frac{t}{T}\right) Y(p, t)$. We assume the consistent estimate of f_{pq} by taking the weight average of the statistic $I_{pq}(\varpi)$ concentrating weight in the neighbourhood of ϖ having width $O(b)$ where b is a band-width parameter tend to 0 as $T \longrightarrow \infty$, we assume $\mathrm{V}(\alpha)$, $-\infty < \alpha < \infty$

$$\int_{-\infty}^{\infty} \mathrm{V}(\alpha)d\alpha = 1$$

Let us assume $b, T = 1, 2, \ldots$ be a bounded sequence of non negative scale parameter. We consider

$$I^*_{pq}(\varpi) = 2\pi/T \sum_{t=0}^{T} V^{(T)}(\varpi - 2\pi t/T) I_{pq}(2\pi t/T) \tag{14}$$

where, $V^{(T)}(\alpha) = b^{-1} \sum_{j=-\infty}^{\infty} W(b^{-1}[\alpha + 2\pi j])$ weights a periodogram value heavily a frequency within $O(b)$ of ϖ. This suggests that $b \longrightarrow 0$ as $T \longrightarrow \infty$. We estimate by smoothing the value of over neighboring values of p and q. Let u_ρ be a weight function or window depending on the bandwidth parameter ρ, which satisfies,

1. $u_\rho(s) \geq 0 \forall s, \rho$
2. $u_\rho(s)$ decays to zero as $|s| \longrightarrow \infty, \forall \rho$
3. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_\rho(s) ds = 1, \forall \rho$
4. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [u_\rho(s)]^2 ds < \infty, \forall \rho$

Thus; we can estimate

$$f_{pq}(\varpi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_\rho(s - p) u_\rho(s - q) I^*_{p+s,q+s}(\varpi) ds \tag{15}$$

Filtering:- We consider $[u_\rho(s)]$ for $S = (s_1, s_2)$ to be a multiplicative filter (the tensor product of 1- diaminational filter), $u_\rho(s) = u_1(s_1) u_2(s_2)$ where $u_1($ is the form

$$u_1(s_1) = \begin{cases} \frac{1}{\rho} & \frac{-1}{2}\rho \leq S \leq \frac{1}{2}\rho \\ 0 & Otherwise \end{cases} \tag{16}$$

In the application, we consider also the following weight function similar to [1]

$$V_{pq}(T) \left(\frac{2\pi S}{T} \right) = \frac{T}{2\pi} (2m + 1)^{-1} \tag{17}$$

Where,$m = bT$ and $S \leq m$;This is a constant function, it is the weight of $2bT + 1$ periodogram ordinates where frequency fall in the interval $(\lambda - 2\pi b, \lambda + 2\pi b)$.

Asymptotic distribution of the estimated spectrum

We define $\hat{f}_{D_s}(\varpi)$ the matrix with entries $\hat{f}_{p,q}(\varpi)$ for all pair (p, q) in the spatial domain D_s

1. The weight function, $V(\beta), -\infty < \beta < \infty$ is real-valued, even and bounded variation.
$$\int_{-\infty}^{\infty} V(\beta)d\beta = 1$$
and
$$\int_{-\infty}^{\infty} |V(\beta)|\, d\beta < \infty$$
2. the temporal covariance is summable, then for each A
$$\sum_{B=-\infty}^{B=\infty})\,|C(A,B)| < \infty$$
and also,
$$\sum_{u=-\infty}^{u=\infty})\,|B|\,|C(A,B)| < \infty$$
3. b is a bandwidth paramater, $T \longrightarrow \infty, bT \longrightarrow \infty, b \longrightarrow 0$

Theorem 3.1. Consider a Gaussian spatiotemporal process, $Y(S,t)$, $S \in \mathrm{D}_s \subset \mathrm{R}^d, t \in \mathrm{D}_t \subset \mathrm{R}$ observed at N spatiotemporal cordinates $(s_1,t_1),\dots,(S_N,t_N)$ and with the covariance function $Cov(Y(s_1,t_1), Y(s_2,t_2) = C(s_1 - s_2; t_1 - t_2)$ We define for $\varpi \in [-\pi,\pi]$ a second order periodogram function, $\hat{f}_{p,q}(\varpi)$ for all pairs of (p,q) in D_s as an estimate of the cross spectral function $f_{p,q}(\varpi)$ defined (12), we consider filter $u_\rho(s)$ under above assumption (1-4) and $V(\beta)$ under above assumption (1-3),we get the expeted value of the estimated spectrum,

$$E(\hat{f}_{p,q}(\varpi)) = \frac{2\pi}{T}\sum_{s=1}^{T-1} V^T\left(\varpi - \frac{2\pi s}{T}\right)\hat{f}_{p,q}\left(\frac{2\pi s}{T}\right)O(T-1)$$
$$= \int_{-\infty}^{\infty} V(\alpha)\hat{f}_{p,q}(\varpi - b\alpha)d\alpha + O(b^{-1}T^{-1})$$

where the error term is uniform in ϖ ,and

$$\hat{f}_{p,q}(\varpi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_\rho(p-s)u_\rho(q-s)f_{p+s,q+s}(\varpi)ds$$

for $-\infty < \varpi < \infty$ The asymptotic variance of $\hat{f}_{p,q}$ is

$$Cov(\hat{f}_{p1,q1}(\lambda), \hat{f}_{p2,q2}(\varpi)) = \frac{2\pi}{T}\left[\int_{-\pi}^{\pi} V^{(T)}(\lambda-\alpha)V^{(T)}(\varpi-\alpha)[\hat{f}_{p1,p2}(\alpha) * \hat{f}_{q1,q2}(-\alpha)]d\alpha\right.$$

$$\left. + \int_{-\pi}^{\pi} V^{(T)}(\lambda-\alpha)V^{(T)}(\varpi+\alpha)[\hat{f}_{p1,q2}(\alpha) * \hat{f}_{q1,p2}(-\alpha)]d\alpha\right]$$

$$+O(b^{-2}T^(T-2))+O\left(T^{T-1}\right)$$

where;

$$\hat{f}_{p_i,q_i}(\lambda)*\hat{f}_{p_j,q_j}(\varpi)=\int_{\mathrm{R}^2}\int_{\mathrm{R}^2} f_{p_i+s_i,q_i++s_i}(\lambda)f_{p_j++s_j,q_j+s_j}(\varpi)$$

$$\times\left[u_\rho(p_i-s_i)u_\rho(p_j-s_j)u_\rho(q_i-s_i)u_\rho(q_j-s_j)\right]ds_id_j$$

The spatial periodogram $\hat{f}_{p_1,q_2}(\lambda)$ and $\hat{f}_{p_2,q_2}(\varpi)$ value for are asymptotically uncorrelated for $\varpi\neq\lambda\neq 0$

$$\lim_{T\longrightarrow\infty} bTCov\left(\hat{f}_{p_1,q_1}(\lambda),\hat{f}_{p_2,q_2}(\varpi)\right)=2\pi\left(\int V(\alpha)^2d\alpha\right)$$

$$\times\left(\eta(\lambda-\varpi)\left[\hat{f}_{p_1,p_2}(\lambda)*\hat{f}_{q_1,q_2}(-\lambda)\right]+\eta(\lambda+\varpi)\left[\hat{f}_{p_1,q_2}(\lambda)*\hat{f}_{p_2,q_1}(-\lambda)\right]\right)$$

where;

$$\eta(\lambda)=\begin{cases}1 & if\lambda\equiv 0\\ 0 & Otherwise\end{cases}$$

$\hat{f}_{D_s}(\varpi_1),\ldots,\hat{f}_{D_s}(\varpi_j)$ is asymptotically normally distributed with the covariance structure given by ;

$$Cov(\hat{f}_{p_1,q_1}(\lambda),\hat{f}_{p_2,q_2}(\varpi))=\frac{2\pi}{T}\left[\int_{-\pi}^{\pi}V^{(T)}(\lambda-\alpha)V^{(T)}(\varpi-\alpha)[\hat{f}_{p_1,p_2}(\alpha)*\hat{f}_{q_1,q_2}(-\alpha)]d\alpha\right.$$

$$\left.+\int_{-\pi}^{\pi}V^{(T)}(\lambda-\alpha)V^{(T)}(\varpi+\alpha)[\hat{f}_{p_1,q_2}(\alpha)*\hat{f}_{q_1,p_2}(-\alpha)]d\alpha\right]+O(b^{-2}T^(T-2))$$

$$+O\left(T^{T-1}\right).$$

3.2. Estimating the coherency

The coherency is usually estimated by replacing f with the second order periodogram using [4].

$$\hat{\mathrm{R}}_{p,q}(\tau)=\frac{\left|\hat{f}_{pq}(\tau)\right|}{[\hat{f}_{pp}(\tau)\hat{f}_{qq}(\tau)]^{\frac{1}{2}}} \tag{18}$$

$$f_{pq}(\varpi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_\rho(s-p)u_\rho(s-q)I^*_{p+s,q+s}(\varpi)ds \tag{19}$$

where $\hat{f}$ is the proposed estimate (16) of the cross temporal spectrum. Thus $\hat{\mathrm{R}}_{D_s}$ is the matrix with entires $\hat{\mathrm{R}}_{pq}(\tau)$ for all pairs of (p,q) in the spatial domain D_s

Asymptotic distribution of coherence

1. $u_\rho(s) \geq 0 \forall s, \rho$
2. $u_\rho(s)$ decays to zero as $|s| \longrightarrow \infty, \forall \rho$
3. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_\rho(s)ds = 1, \forall \rho$
4. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [u_\rho(s)]^2 ds < \infty, \forall \rho$

1. The weight function, $V(\beta), -\infty < \beta < \infty$ is real-valued, even and bounded variation.

$$\int_{-\infty}^{\infty} V(\beta)d\beta = 1$$

and

$$\int_{-\infty}^{\infty} |V(\beta)|\, d\beta < \infty$$

2. the temporal covariance is summable, then for each A

$$\sum_{B=-\infty}^{B=\infty})\,|C(A,B)| < \infty$$

and also,

$$\sum_{u=-\infty}^{u=\infty})\,|B|\,|C(A,B)| < \infty$$

3. b is a bandwidth paramater, $T \longrightarrow \infty, bT \longrightarrow \infty, b \longrightarrow 0$

Under the above assumptions (1-4) and (1-3), the variates $\hat{\mathrm{R}}_{D_s}(\tau_i)$ are asymptotically jointly normal for $j = 1, 2, \ldots J$

$$E\left(\left|\hat{\mathrm{R}}_{pq}(\tau)\right|^2\right) = \left|\hat{\mathrm{R}}_{pq}(\tau)\right|^2 + O(b) + O(b^{-1}T^{-1})$$

Where;

$$\hat{\mathrm{R}}_{p,q}(\tau) = \frac{\left|\hat{f}_{pq}(\tau)\right|}{[\hat{f}_{pp}(\tau)\hat{f}_{qq}(\tau)]^{\frac{1}{2}}}$$

$$Cov\left[\left|\hat{\mathrm{R}}_{pq}(\tau)\right|^2, \left|\hat{\mathrm{R}}_{pq}(\lambda)\right|^2\right] = [\eta(\tau-\lambda)+\eta(\tau+\lambda)]\left|\hat{\mathrm{R}}_{p,q}(\tau)\right|^2 \times\left[1-\left|\hat{\mathrm{R}}_{p,q}(\tau)\right|^2\right]4\pi\int V(\alpha)^2 d\alpha b^{-1}T^{-1} + O\left(b^{-2}T^{-2}\right)$$

[4]; suggested that the inverse hyperbolic tangent for $\hat{R}$ as a variance stabilizing transformation. Thus we consider a standardized asymptotic distribution of the variance stabilizing transformation of R. Thus;

$$\Phi_{pq}(\tau) = tanA^{-1}\left(\mathrm{R}_{pq}(\tau)\right)$$

and thus we can estimate;

$$\hat{\Phi}_{pq}(\tau) = tanA^{-1}\left(\hat{R}_{pq}(\tau)\right)$$

[7] and [4] suggested that the transformed variate may be more nearly normal than the untransformed one. This is the straightforward consequences of the Delta method [4] and [9]. We write $\hat{\Phi}_{D_s}(\tau)$ to denote the matrix with entries $\hat{\Phi}_{pq}(\tau)$ for all pairs (p,q). We study the asymptotic properties and distribution of $\hat{\Phi}_{D_s}(\tau_j)$.

3.3. Testing Separablity

We used a test of separability of the spatiotemporal process proposed by [4]. The main advantages of using this method are that the mechanics of the test can be reduced to those of the simple two-way Analysis of Variance (ANOVA) procedure. We test the separability by study if $\hat{\Phi}_(p,q)(\tau)$ is a function of τ. We evaluate $\hat{\Phi}_(p_i,q_i)(\tau)$ at k pairs of locations $[(p_i,q_i)]_{i=1}^{k}$ and a set of frequencies $\tau_1,\ldots,\tau_n$ that cover the domain. We can write;

$$\hat{\Phi}_(p_i,q_j)\left(\tau_j\right) = \Phi(p_i,q_i)(\tau_j) + \epsilon((p_i,q_i),\tau_i)$$

Asymptotically; $E\left[\epsilon((p_i,q_i),\tau_i)\right] = 0$ and $Var\left[\epsilon((p_i,q_i),\tau_i)\right] = \sigma^2$ Where σ^2 is independent of (p_i,q_i) and ϖ_j. Assuming the (p_i,q_i) and τ_j are sufficiently spaced, then $\epsilon((p_i,q_i),\tau_i)$ will be approximately uncorrelated, this is based on asymptotic properties of $\hat{f}_(p_i,q_j)\left(\tau_j\right)$, Thus we have;

$$U_{ij} = \hat{\Phi}_(p_i,q_j)\left(\tau_j\right)$$

$$M_{ij} = \Phi_(p_i,q_j)\left(\tau_j\right)$$

and

$$\epsilon_{ij} = \epsilon((p_i,q_i),\tau_j)$$

Thus we have the model;

$$U_{ij} = M_{ij} + \epsilon_{ij}$$

This can be the usual two-factor analysis of variance model can be written as;

$$H_1 : U_{ij} = \mu + \alpha_i + \beta_j + \epsilon_{ij}$$

for $i = 1, 2 \ldots, k$ and $j = 1, 2 \ldots, n$. The parameter (α_i) and (β_j) represents the main effect of space and frequency factor. Then we test separability by using the standard technique to test the model $(\beta_j = 0)$

$$H_0 : U_{ij} = \mu + \alpha_i + \epsilon_{ij}$$

vs

$$H_1 : U_{ij} = \mu + \alpha_i + \beta_j + \epsilon_{ij}$$

3.4. Testing for stationarity

The methodology in geostatistics is based on the assumptions of second-order stationary random fields [4]. We test spatial stationary by studying if the coherence R is a function of space when R is evaluated at pairs of locations, $[(p_i, q_i)]_{i=1}^{k}$ separated by the same distance A. $i.e$, $p_i - q_i = A, \forall i = 1, 2, \ldots, k$.If the process is stationary in space, at each fixed time the covariance of the process between two locations p and q separated by a distance A, is only a function of A. Thus the spectral function $f_{pq}(\tau)$ is only a function of τ and A.We denote it,

$$f(A;\tau) = (2\pi)^{-1} \sum_{B=-\infty}^{B=\infty} exp(-iB\tau)\mathrm{C}(A, B)$$

Thus, the coherency is just a function of vector distance between pairs of locations.

$$R(A;\tau) = \frac{|f(A;\tau)|}{|f(0;\tau)|}$$

Under the assumptions of separability below;

$$f(A;\tau) = \mathrm{C}_1(A)[(2\pi)^{-1} \sum_{B=-\infty}^{B=\infty} exp(-iB\tau)\mathrm{C}_2(B)]$$

We have;

$$R(A;\tau) = \frac{|f(A;\tau)|}{|f(0;\tau)|} = \frac{C_1(A)}{C_1(0)}$$

We define R_{ij} as

$$U_{ij} = \hat{\Phi}_{(}p_i, q_j)\left(\tau_j\right)$$

$$M_{ij} = \Phi_{(}p_i, q_j)\left(\tau_j\right)$$

and

$$\epsilon_{ij} = \epsilon((p_i, q_i), \tau_j)$$

$$\mathrm{R}_{ij} = U_{ij} + \epsilon_{ij} = \mu + \alpha_i + \beta_j + \epsilon_{ij}$$

but we evaluate it only at pairs, $[(p_i, q_i)]_{i=1}^{k}$ such that, $p_i - q_i = A, \forall i = 1, 2, \ldots, k$. We test the stationary in space by using standard Analysis of Variance (ANOVA) techniques to test the model ($\alpha_i = 0$).

$$H_0 : R_{ij} = \mu + \beta_j + \epsilon_{ij}$$

vs

$$H_1 : R_{ij} = \mu + \alpha_i + \beta_j + \epsilon_{ij}$$

$\forall i = 1, 2, \ldots, k$ and $\forall j = 1, 2, \ldots, n$. Where, the parameter ($\alpha_i$) and ($\beta_j$) represents the main effect of space and frequency factor.

4. Application

We use spatiotemporal data of the infected rate of leishmaniasis in Amhara regional state of Ethiopia. We have a daily infected rate for 2 months from January 1, 2015, to February 28, 2015, at 5 locations a total of 295 spatiotemporal observations and 59 observations over time.

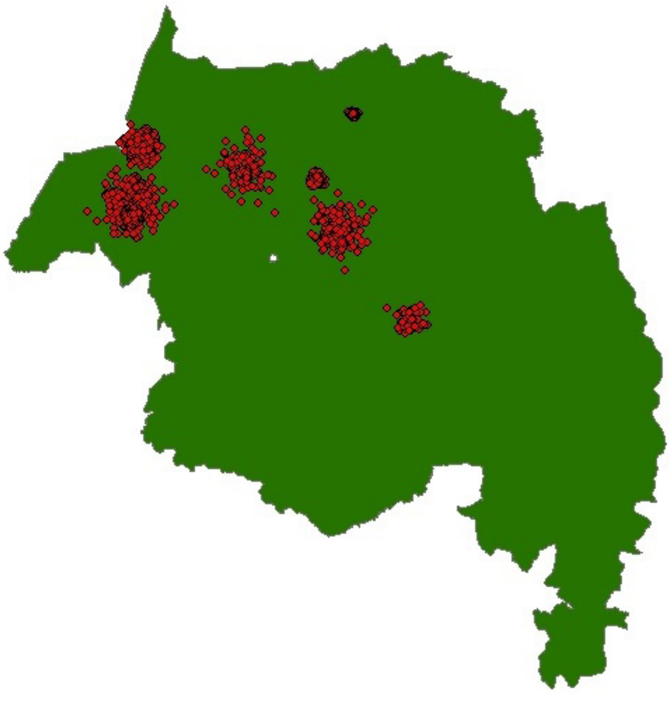

Figure 1: This map shows the spatial points of the infected rate of leishmaniasis in Amhara region, Ethiopia for 5 locations from January 1, 2015, to February 28, 2015.

Figure 1 shows the locations of p_i for $i = 1, 2, \ldots, 5$ corresponding to 5 location points in Amhara regional State Ethiopia, Chilga, Dabat, Metema , Quara and West

Armachiho and the locations of q_i for $i = 1, 2, \ldots, 5$ are other five sites with the same latitude but 59, 121, 178, 89 and 210 Km east respectively.

Before applying our test for stationarity and separability we need to remove spatial and temporal trends. In order to remove the spatial trend, we calculate at each location of the infected rate of leishmaniasis anomaly, that is the corresponding infected rate of leishmaniasis minus the mean over time (using 59 observation over time at each location).

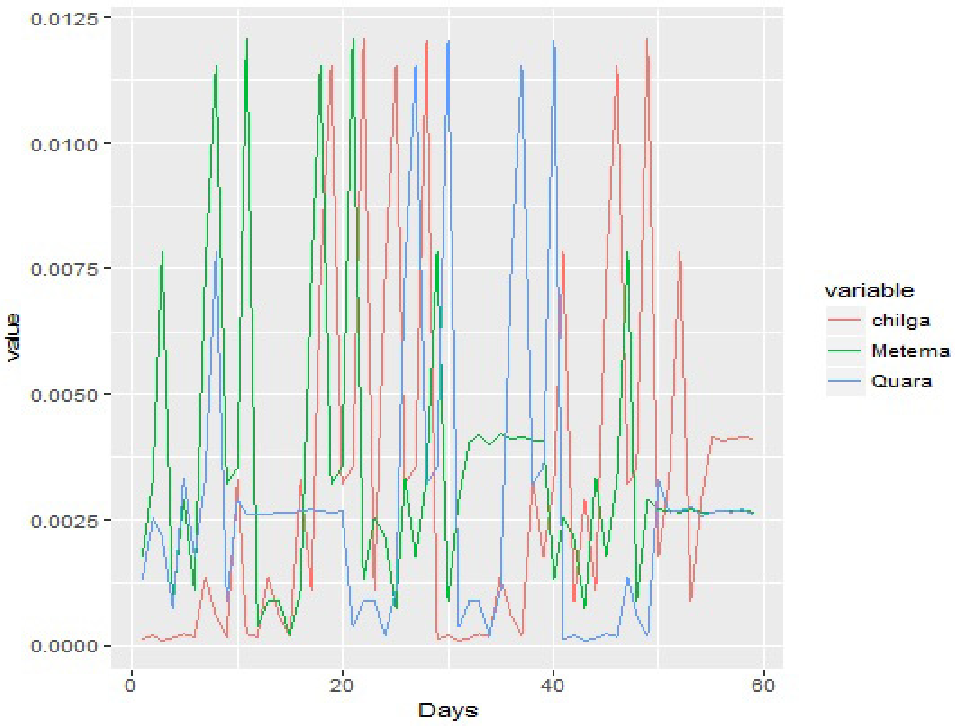

Figure 2: The time series in this graph shows the daily infected rate of leishmaniasis in Amhara Regional state, Ethiopia, from January 1, 2015, to February 28, 2015, at three locations with its corresponding latitude and longitude. The Y-axis indicates the daily infected rate of leishmaniasis.

The estimate of $\hat{f}_{p,q}(\varpi)$ were obtained using equation (15) in which $V(\alpha)$is obtained by equation (16) with the bandwidth $2\pi b$ with $b = \frac{1}{10}$ and $u_\rho(A)$ is of the form(17) with $\rho = 5units$ (1 unit=42 Km), $m = bT$,Thus in order to obtain approximately uncorrelated estimate, the frequency ϖ_j and pairs $[(p_i, q_i)]_{i=1}^{k}$ should be selected. So that the spacing between the ϖ_j are at least $\frac{\pi}{5}$ and the distance between any pairs (p_i, q_i) and (p_j, q_j) is at least 5 grid cells (210 Km). The ϖ_j were selected as follows;$\varpi_j = \frac{\pi j}{15}$ with $j = 1, 2 \ldots 5$ $\varpi_1 = \frac{\pi}{15}$,$\varpi_2 = \frac{2\pi}{15}$,$\varpi_3 = \frac{3\pi}{15}$, $\varpi_4 = \frac{5\pi}{15}$ and $\varpi_5 = \frac{12\pi}{15}$ we consider 5 pairs $[(p_i, q_i)]_{i=1}^{5}$ Table 1 shows that the result of the test for separability using five pairs of location points (sites). The between spatial location effects is significant (P-value < 0.05) confirming that there is a clear evidence of lack of separability (spatiotemporal covariance function is not separable), this result is in line to [4] and coefficient of determination is 0.721. In table 2 we study separability in a smaller region (using site Metema and Quara) by considering pairs of locations (p_3, q_3) and (p_5, q_5). In table 2 the between spatial location and the between frequencies is not significant, suggesting that there is no evidence of lack of stationarity in two locations Metema and Quara using pairs of locations (p_3, q_3) and (p_5, q_5). Similarly, the between frequencies effect is also insignificant confirming that the spatiotemporal process is separable, with the coefficient of determination 0.816, in the smaller subregions the assumption of separability is satisfied. Since the distance between the two components in both pairs is the same, thus we can use this pairs to test stationarity.

Table 1: Analysis of variance for all five locations (sites)

Source	DF	Sum of squares	F values	Pr(F)
Between Spatial location (points)	4	8.106	4.94	0.00025
Between frequencies	4	11.184	6.82	0.0048
Residuals	16	6.577		

Table 2: Analysis of variance between Metema and Quara locations (sites)

Source	DF	Sum of squares	F values	Pr(F)
Between Spatial location (points)	1	1.0623	0.9355	0.05235
Between frequencies	4	5.7414	1.27154	0.0048
Residuals	4	0.23300		

Figure 3 shows that QQ plot for the infected rate of leishmaniasis it seems the data comes from approximately normally distributed. Table 3 shows that $\left|\hat{R}_{p,q}(\varpi)\right|^2$ values at each ϖ_i for $i = 1, 2 \ldots 5$ frequency for the pair of (p_3, q_3) and (p_5, q_5). $|R|^2$ indicates that the coefficient of correlation between two pairs as it closer to 1 it indicates a high correlation between two-time series both pairs have very similar coherency functions which support the assumption of stationarity. However, the coherency is changed with frequency but it seems that spatially to assume stationary for the infected rate of leishmaniasis to be satisfied even for a small geographical area of the study.

5. Discussion

The test of spatial stationarity based on the coherency between pairs of locations. The test of separability based on spectral methods using a nonparametric estimate of the spectral

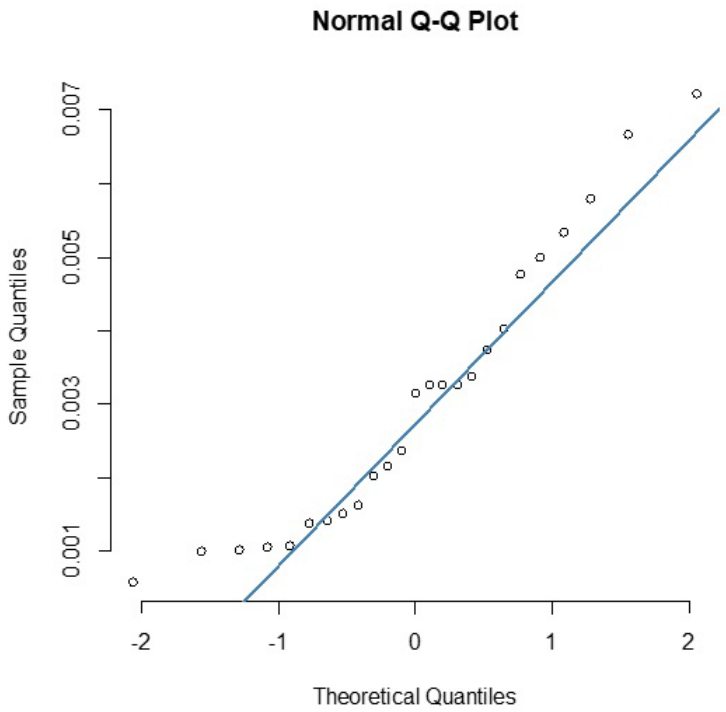

Figure 3: This graph shows that the quantile plot of the infected rate of leishmaniasis

Table 3: Coherence pair between (p_3,q_3) and (p_5,q_5) at five Frequencies

Pairs	ϖ_1	ϖ_2	ϖ_3	ϖ_4	ϖ_5
(p_3,q_3)	0.621	0.916	0.729	0.924	0.921
(p_5,q_5)	0.827	0.803	0.916	0.912	0.916

density of the process. The spectral estimate $\hat{f}_{p,q}(\varpi)$ this spectral estimate is a function of two filter $u_\rho(S)$ and $V(\varpi)$. These two filter used to pledge the independence of the estimated spectrum at frequency and locations that apart with its respective bandwidth so that we can apply two âŁ“factor Analysis of variance (ANOVA).

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