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Construction of Projected Tilings from Crystallographic Tilings By Cut and Project method

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Abstract

Known quasi-crystallographic tilings like the Penrose tiling can be obtained by projecting a subset of a point lattice onto a plane. We will describe a more general construction of cut-and-project tilings from an arbitrary given crystallographic tiling T, and not only lattices and given cut-and-project data (that is, projection subspace and window), in detail.
As a first step, points must be chosen in each prototile to obtain a Delone set. The points in one prototile should be invariant under the isometry group of the prototile, so it does not matter which isometry is applied on the prototile to obtain an actual tile in the tiling; we always choose the same points in the tile.
Then, the cut-and-project Delone set can be constructed using the cut-and-project data, and from this set, we can construct the Voronoi-cell tiling. One has to show that, the projected point set is also a Delone set, and that the associated Voronoi-cell tiling is simple.

1. INTRODUCTION

Take T to be a crystallographic tiling of $\mathbb{E}^n$, and construct a Delone set X out of it. To this purpose, choose finite sets of points X_i in each prototile t_i fixed by the symmetry group of the prototile.

2. DELONE SETS FROM CRYSTALLOGRAPHIC TILINGS

An obvious set of points to choose would be the vertices of the prototiles since vertices must be mapped to vertices by isometries. On the other hand, vertices are not the only choice of the set of points. There are some cases where one can determine the symmetry centres of the prototiles.

Example 2.1 Take the standard lattice tiling in any dimension, there is no difference whether we choose vertices or symmetry centres, since the symmetry centres look like the shifted points of the vertices, as shown in Figure caption.1.

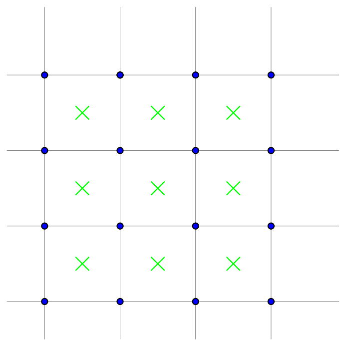

Figure 1: Standard lattice tiling with vertices $\circ$ and symmetry centers $\times$.

Definition 2.2 A *point set data* $\{(X_i, t_i)_i\}$ of a tiling T consists of a finite set of points X_i for each prototile t_i that is invariant under the isometry group of t_i.

Proposition 2.3 *Given a point set data* $\{(X_i, t_i)_i\}$ *the point set*

$$X_T = \bigcup_{t \in T, \gamma(t_i)=t} \gamma(X_i) \ ; \ \gamma \in Isom(\mathbb{E}^n)$$

is a Delone set.

PROOF Note that the union runs over all tiles $t \in T$, and for each t, we choose an isometry $\gamma \in Isom(\mathbb{E}^n)$ mapping the prototile t_i behind t to $t = \gamma(t_i)$. Now, $\bigcup_{t \in T} t = \mathbb{E}^n$ and $\exists R : t \subset B_R(x)$ for all points $x \in t$, where R only depends on the prototile behind t. Since we only have a finite number of prototiles, there exists an R working for all t at once. This means that $\bigcup_{x \in X_T} B_R(x) = \mathbb{E}^n$, because for each t, $x \in t \cap X_T$. Hence, the covering radius of X_T is less than or equal to R, in particular the covering radius of X_T is finite.
If the packing radius of X_T is r, then open balls of radius r centered at the points of X_T

will be disjoint from each other, and each open ball centered at one of the points of X_T with radius $2r$ will be disjoint from the rest of X_T.

Now, for a given R and all choices of $y \in \mathbb{E}^n$, there is only a finite number of patches $[T]_{B_R(y)}$ up to isometries. This was already used in the proof of compactness of hull. This means that there are only a finite number of intersection sets $B_R(y) \cap X_T$ up to isometries. Furthermore, for $y \in \mathbb{E}^n$, the set

$$\{d(x, x') : x \neq x' \in B_R(y) \cap X_T\}$$

is finite, as $B_R(y) \cap X_T$ only intersects a finite number of tiles and each tile intersects X_T in a finite set of points. Since,

$$\{d(x, x') : x \neq x' \in B_R(y) \cap X_T\}$$

is invariant under isometries, we conclude that

$$r := \frac{1}{2} \inf\{d(x, x') : x \neq x' \in B_R(y) \cap X_T, y \in \mathbb{E}^n\} > 0.$$

Hence, from all what we have discussed, we have shown that X_T is a Delone set.

□

Remark 2.4 The important thing about choosing points in prototiles that are fixed under the symmetry group of t_i is that, for the definition of X_T, we need to get the same points in tile t independent of the isometry used to get from t_i to t. In Example 2.1, X_i is the symmetry centres of the prototiles t_i in the standard lattice tiling T where we can get the tile t_1 from the tile t_1 by a 90° rotation. It is clear that Example 2.1 satisfies Proposition 2.3.

Proposition 2.5 *For a point set X_T associated to a tiling T as above, the Voronoi-cell tiling T_{X_T} is $id_{\mathbb{E}^n}$-i-LD from T.*

PROOF T_{X_T} is $id_{\mathbb{E}^n}$-i-LD from T if there exists a radius R such that, for $x \in \mathbb{E}^n$ and $\phi \in Isom(\mathbb{E}^n)$, we have:

$$[T]_{B_R(x)} = [\phi(T)]_{B_R(x)} \Longrightarrow [T_{X_T}]_{\{x\}} = [\phi(T_{X_T})]_{\{x\}}.$$

Now,

$$\begin{aligned} [T]_{B_R(x)} = [\phi(T)]_{B_R(x)} &\Longrightarrow [X_T]_{B_R(x)} = [\phi(X_T)]_{B_R(x)} \\ &\Longrightarrow [T_{X_T}]_{\{x\}} = [\phi(T_{X_T})]_{\{x\}}; \end{aligned} \tag{1}$$

for large enough R independent of x, the Voronoi-cell around a point $x \in X_T$ only depends on points of X_T up to a distance of x that is independent of x. □

There are many counterexamples of the other direction of Proposition 2.5. That is, T is not LD from T_{X_T}. If T is a crystallographic tiling this is equivalent to $Aut(T) \subsetneqq Aut(T_{X_T})$. We will also check this condition in the following examples.

Example 2.6 Choose the symmetry centres as the set of points of the prototiles. The Voronoi-cell tiling we gain is just the shifted standard lattice tiling (see Figure caption.2). So, T is LD from T_{X_T}.

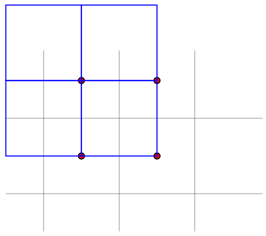

Figure 2: Standard lattice tiling (——) and Voronoi-cell tiling of its vertices (——).

Example 2.7 For the slanted lattice tiling, if we choose the point set X_T as the vertices of T, this set coincides with the vertices of the standard lattice tiling; if we choose the point set data as the vertices or the symmetry centres, both cases will give us Delone set as a standard lattice tiling, which has more automorphisms than the slanted tiling. Therefore, $Aut(T) \subsetneqq Aut(X_T) = Aut(T_{X_T})$, and T is not LD from T_{X_T}.
If we take the standard lattice tiling and take points in the tiles that are close to all the vertices and invariant under the automorphism group of the square then from Figure caption.3 it is clear that $Aut(T_{X_T})$ contains horizontal and vertical translations by $\frac{1}{2}$. On the other hand, if we look at X_T, the horizontal translations by $\frac{1}{2}$ are not contained in $Aut(X_T)$. Hence, $Aut(T) = Aut(X_T) \subsetneqq Aut(T_{X_T})$, and T is not LD from T_{X_T}.

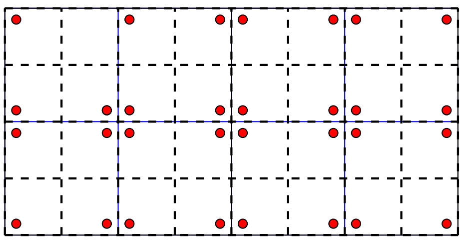

Figure 3: Standard lattice tiling, point set data and Voronoi-cell tiling

Corollary 2.8

$$Aut(T) \subset Aut(X_T) \subset Aut(T_{X_T}).$$

PROOF First of all, we will show that $Aut(T) \subset Aut(X_T)$. Let $\phi \in Aut(T)$, such that $\forall t \in T : \phi(t) \in T$. Now, by construction, if $\gamma(t_i) = t$ for the prototile t_i and the isometry γ, we have:

$$\phi(t \cap X_T) = \phi\big(\gamma(X_i)\big);$$

since by construction $t \cap X_T = \gamma(X_i)$. This implies that:

$$\phi(X_T) = \bigcup_{t\in T;\gamma(t_i)=t} \phi\big(\gamma(X_i)\big) = \bigcup_{t\in T;\gamma(t_i)=t} \phi(t) \cap X_T = \bigcup_{t\in T;\gamma'(t_i)=t} \gamma'(X_i) = X_T,$$

since ϕ is an automorphism of T, hence, $\phi(T)$ runs over all tiles of T if t does, and $\gamma' = \phi\gamma \in Aut(T)$. Therefore, $\phi \in Aut(X_T)$.

The second step now is to prove that $Aut(X_T) \subset Aut(T_{X_T})$. Assume $\phi \in Aut(X_T)$ and let $t \in T_{X_T}$. Notice that $t = t_x$ for some $x \in X_T$, where

$$t_x = \{y \in \mathbb{E}^n : d(y,x) \leq d(y,x')\forall x' \in X_T\} \qquad (2)$$

This implies $d\big(\phi(y), \phi(x)\big) \leq d\big(\phi(y), \phi(x')\big)$ for all $x' \in X_T$, and since $\phi(x')$ runs through all points of X_T if x' does, we have, $\phi(tx) = t\phi(x)$which means that $\phi \in Aut(T_{X_T})$. □

3. GENERAL CUT-AND-PROJECT CONSTRUCTION

We will first define cut-and-project data for the Euclidean space $\mathbb{E}^n$.

Let $E \subset \mathbb{E}^n$ be an m-dimensional hyperplane, and $E^\perp \subset \mathbb{E}^n$ be an $(n{-}m)$-dimensional hyperplane orthogonal to E. Let Π be the orthogonal projector onto E, and $\Pi^\perp$ the orthogonal projector onto $E^\perp$, that is $\Pi : \mathbb{E}^n \to E$ and $\Pi^\perp : \mathbb{E}^n \to E^\perp$.

Then, we fix a compact subset $K \subset E^\perp$ such that $K^\circ \neq \emptyset$ and $\overline{K^\circ} = K$. K will be called the *window* for the projection, E the *projection hyperplane*, $K \times E$ the *cylinder*, (see Figure caption.4), and (K, E) *cut-and-project data* for $\mathbb{E}^n$.

Construction in steps:

Given a crystallographic tiling T of $\mathbb{E}^n$, cut-and-project data (K, E) for $\mathbb{E}^n$ can be used to construct a new tiling T' of E through the following steps:

(i) Choose point-set data $\{(X_i, t_i)_i\}$ of T as in section 2, where $t_1, t_2, ..., t_k$ are the prototiles of T.

(ii) Construct the Delone set $X_T = \bigcup_{t \in T; \gamma(t_i)=t} \gamma(X_i)$ from the point-set data, as in section 2.

(iii) Cut and project: Set $X_{T'} = \Pi\big(X_T \cap (K \times E)\big)$, where the cylinder $K \times E$ is defined with respect to the unique intersection point in $E^{\perp} \cap E$ as the origin. This step requires that $X_T \cap (K \times E)$ is not empty, which will be the case under the assumption on the window K discussed later.

(iv) T' is the Voronoi-cell tiling $VT(X_{T'})$ associated to $X_{T'}$.

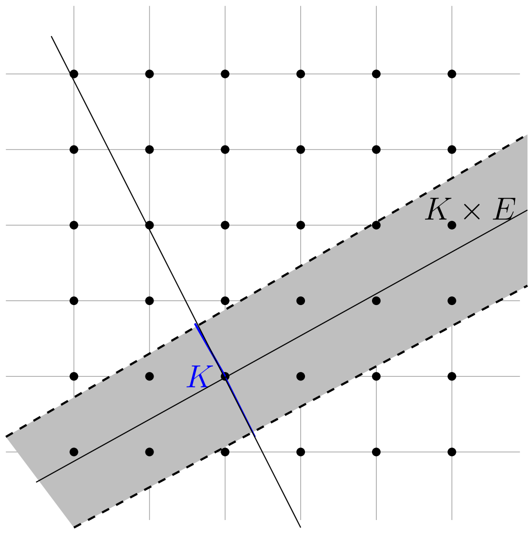

Figure 4: Cut-and-project method with projection subspace E and window K.

Remark 3.1 The intersection $X_T \cap (K \times E)$ could be empty as in Figure caption.5). Here the projection subspace E has slope 1 with respect to the standard lattice, so never passes through one of the lattice points, and the minimal distance to the lattice points will even be positive. Therefore, if we choose a small enough $K \subset E^{\perp}$ window, the cylinder $K \times E$ will not contain any of the lattice points.

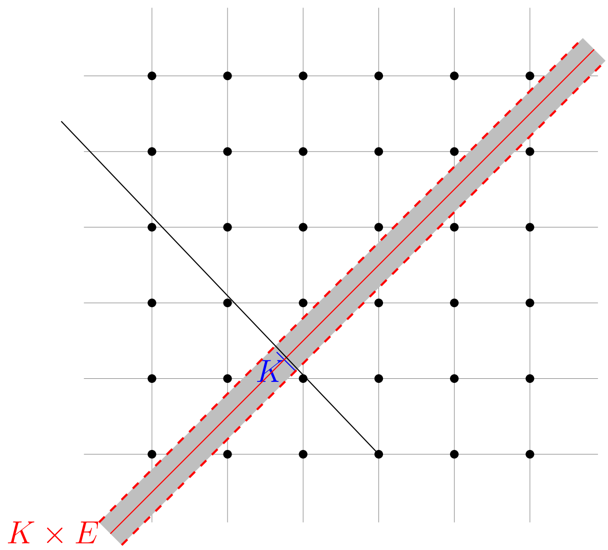

Figure 5: The projection result is empty.

Assumption on window K:

$$(*) \qquad K^\circ \cap \Pi^\perp(X_T) \neq \emptyset \ .$$

Remark 3.2 Notice that:

(i) If the interior of a window K does not intersect $\Pi^\perp(X_T)$, we can move K with an isometry ρ of $E^\perp$ to a window $\rho(K)$ with non-empty intersection with $\Pi^\perp(X_T)$.

(ii) If only the boundary of K intersects $\Pi^\perp(X_T)$, there are lots of cases to distinguish, depending on whether components of the boundary contain enough image points of X_T.

Under this assumption with regard to the window K, T' displays the following properties:

Theorem 3.3 *$X_{T'}$ is a Delone set.*

Theorem 3.4 *T' is a simple tiling.*

The proof for these facts will occupy the rest of the section.

The main tool for the proof of Theorem 3.3 and Theorem 3.4 is Kronecker's Approximation Theorem (in several dimensions).

Theorem 3.5 [2222, First form of Kronecker's Approximation Theorem]
If $\alpha_1, ..., \alpha_n$ are arbitrary real numbers, if $\theta_1, ..., \theta_n$ are $\mathbb{Z}$-linearly independent real numbers, and if $\epsilon > 0$ is arbitrary, then there exists a real number $t > 0$ and integers $h_1, ..., h_n$, such that:

$$|t\theta_i - h_i - \alpha_i| < \epsilon \textit{ for } i = 1, 2, ..., n.$$

Remark 3.6 Kronecker's Approximation Theorem as in 2222 only states that t is a real number, but the proof goes through if one stays restricted to $t > 0$; this is what we need later.

Under additional assumptions on the α_i Kronecker's Approximation Theorem holds for arbitrary real numbers $\theta_1, ..., \theta_N$, irrespective of whether they are $\mathbb{Z}$-linearly independent or not.

Corollary 3.7 If $\theta_1, ..., \theta_N$ are real numbers and $\alpha_1, ..., \alpha_N$ are real numbers satisfying the same $\mathbb{Z}$-linear relations as $\theta_1, ..., \theta_N$, then for every $\epsilon > 0$, there exists a real number $t > 0$ and integers $h_1, ..., h_N$, such that:

$$|t\theta_i - h_i - \alpha_i| < \epsilon \text{ for } i = 1, 2, ..., N.$$

PROOF By reordering, we can achieve that $\theta_1, ..., \theta_k$ are $\mathbb{Z}$-linearly independent, $1 \leq k \leq N$, and for each θ_i; $i = k+1, ..., N$, there is a $\mathbb{Z}$-linear relation:

$$n_1^{(i)}\theta_1 + ... + n_k^{(i)}\theta_k - n_i\theta_i = 0; \quad n_1^{(i)}, ..., n_k^{(i)}, n_i \in \mathbb{Z}.$$

By multiplying with $\frac{\Pi_{j=k+1}^{N} n_j}{n_i}$ we can achieve $n_{k+1} = ... = n_n = n > 0$. By Kronecker's Approximation Theorem, for any $\epsilon' > 0$, there is a real number $t' > 0$, and there are integers $h'_1, ..., h'_k$, such that:

$$|t'\theta_1 - h'_1 - \frac{\alpha_1}{n}| < \epsilon', ..., |t'\theta_k - h'_k - \frac{\alpha_k}{n}| < \epsilon'.$$

This implies that:

$$\Longleftrightarrow |t'n\theta_i - h'_i - n\alpha_i| < |n_1^{(i)} + ... + n_k^{(i)}|\epsilon'; \text{ with } h'_i = n_1^{(i)}h'_1 + ... + n_k^{(i)}h'_k \in \mathbb{Z}.$$

Now, if we multiply $|t'\theta_i - h_i - \frac{\alpha_i}{n}| < \epsilon'$ for $i = 1, ..., k$ with n, we get:

$$|t'n\theta_i - nh_i - \alpha_i| < |n|\epsilon'.$$

Hence, $\epsilon' := \min\{\frac{\epsilon}{n}, \frac{\epsilon}{|n_1^{(i)}+...+n_k^{(i)}|}\}$, $t = t'n > 0$, $h_i := nh'_i$ for $i = 1, ..., k$, and $h_i = h'_i$ for $i = k+1, ..., N$ are the choices required for the claim. □

The next theorem shows the relative denseness of $X_{T'}$ in a special case, where $dim(E) = 1$.

Theorem 3.8 *The set* $X_{T'} = \Pi\big(X_T \cap (K \times E)\big)$ *is relatively dense in the case that* $dim(E) = 1$ *and* $dim(E^{\perp}) = N - 1$.

PROOF $Aut(T)$ is crystallographic, which means that there exists a lattice of full rank $\Lambda \subset Aut(T)$. Since $\Lambda \subset Aut(T) \subset Aut(X_T)$, the orbit $\Lambda \cdot x$ of a point $x \in X_T$ is contained in X_T. Choose $x \in X_T$ such that $\Pi^{\perp}(x) \in K^{\circ}$ (possible by Assumption (*) on the window K). Also, choose a basis of Λ and let x be the origin of the coordinate system on $\mathbb{E}^n$ given by this basis.

In terms of this basis, $E = \mathbb{R} \cdot (\theta_1, ..., \theta_N)$. Permuting the coordinates, we can arrive at the conclusion that $\theta_1, ..., \theta_K$ are $\mathbb{Z}$-linearly independent and $\theta_{K+1}, ..., \theta_N \in \mathbb{Q}\theta_1 + ... + \mathbb{Q}\theta_K$. In particular, there is a $(N-K) \times K$-matrix M with integer entries such that:

$$M' \cdot \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_N \end{pmatrix} = \left(\begin{array}{c|ccc} M & d_{K+1} & & 0 \\ & & \ddots & \\ & 0 & & d_N \end{array}\right) \cdot \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_K \\ \theta_{K+1} \\ \vdots \\ \theta_N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Set $H = \left\{ (x_1, ..., x_N) \ : \ \left(\begin{array}{c|ccc} M & d_{K+1} & & 0 \\ & & \ddots & \\ & 0 & & d_N \end{array}\right) \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$. Then, $E \subset H$ and $dim\ H = K$.

Claim 1: $H \cap \Lambda$ is a lattice Λ_H of full rank K.

Proof of claim 1: $H \cap \Lambda = ker\ \phi_{M'}$, where $\phi_{M'} : \mathbb{Z}^N \to \mathbb{Z}^{N-K}$ is defined by $\begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} \to M' \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix}$. M' has rank $N-K$ because of the $(N-K) \times (N-K)$ diagonal matrix on the right. Hence, the $\mathbb{Q}$-linear map $\phi_{M'} \otimes_{\mathbb{Z}} \mathbb{Q}$ has rank $N-K$, therefore, $dim\ \big(ker(\phi_{M'} \otimes_{\mathbb{Z}} \mathbb{Q})\big) = K$ and $dim\ \big(ker(\phi_{M'} \otimes_{\mathbb{Z}} \mathbb{Q})\big)$ is the rank of the torsion-free part of the finitely generated abelian group $ker\ \phi_{M'}$. Since $ker\ \phi_{M'}$ is torsion-free as a subgroup of $\mathbb{Z}^N$, we have $ker\ \phi_{M'} \cong \mathbb{Z}^K$.

□

Now, choose a $\mathbb{Z}$-basis of Λ_H. This is also an $\mathbb{R}$-basis of H. In terms of this, write $E = \mathbb{R} \cdot (\theta_1^H, ..., \theta_K^H)$.

Claim 2: $\theta_1^H, ..., \theta_K^H \in \mathbb{R}$ are $\mathbb{Z}$-linearly independent.
Proof of claim 2: It is enough to show that $\theta_1^H, ..., \theta_K^H$ are $\mathbb{Q}$-linearly independent. Let $\tau_i = (t_{1i}^H, ..., t_{Ni}^H) \in \mathbb{Z}^N$, $i = 1, ..., K$ be the chosen $\mathbb{Z}$-basis vectors of $\Lambda_H \subset \mathbb{Z}^N$. Then,

$$T \cdot \begin{pmatrix} \theta_1^H \\ \theta_2^H \\ \vdots \\ \theta_K^H \end{pmatrix} = \begin{pmatrix} t_{11}^H & \dots & t_{1K}^H \\ \vdots & & \vdots \\ t_{N1}^H & \dots & t_{NK}^H \end{pmatrix} \cdot \begin{pmatrix} \theta_1^H \\ \theta_2^H \\ \vdots \\ \theta_K^H \end{pmatrix} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_N \end{pmatrix};$$

$\theta_1, ..., \theta_K$ are $\mathbb{Z}$-linearly independent, hence, $\mathbb{Q}$-linearly independent. Therefore, the homomorphism $\theta : \mathbb{Q}^K \to (\theta_1, ..., \theta_K)\mathbb{R}$ given by $e_1 \mapsto \theta_1, ..., e_K \mapsto \theta_K$ is injective.
θ factorises by the map $\mathbb{Q}^K \xrightarrow{T_H} \mathbb{Q}^K$ given by the matrix $T_H = \begin{pmatrix} t_{11}^H & \dots & t_{1K}^H \\ \vdots & & \vdots \\ t_{K1}^H & \dots & t_{KK}^H \end{pmatrix}$
through $\theta_H : \mathbb{Q}^K \xrightarrow{(\theta_1^H, ..., \theta_K^H)} \mathbb{R}$ given by $\theta_1^H, ..., \theta_K^H$:
The matrix T_H is of full rank since its rows are the first K rows of the rank K matrix T, and the last $N - K$ rows of T are linear combinations of the first rows of T because the columns of T are in the kernel of the linear map described by the matrix $\left(\begin{array}{c|ccc} & d_{K+1} & & 0 \\ M & & \ddots & \\ & 0 & & d_N \end{array} \right)$. Hence, $\mathbb{Q}^K \xrightarrow{T_H} \mathbb{Q}^K$ is an isomorphism; therefore, $\mathbb{Q}^K \xrightarrow{(\theta_1^H, ..., \theta_K^H)} \mathbb{R}$ is injective.

□

For the theorem, it is enough to show that:

$$\Pi\big(\Lambda_H \cdot x \cap ((K \cap H) \times E)\big)$$

is relatively dense on E because $\Lambda_H \cdot x \subset \Lambda \cdot x \subset X_T$.
Setting $H := \mathbb{R}^N, E^\perp := E^\perp \cap H$ and $\Lambda \cdot x := \Lambda_H \cdot x$. We can reduce to the situation where $(\theta_1, ..., \theta_N)$ consists of $\mathbb{Z}$-linearly independent coordinates. Choose basis vectors $\sigma_1, ..., \sigma_{N-1}$ of $E^\perp$. The vectors $\theta, \sigma_1, ..., \sigma_{N-1}$ are also a basis of $\mathbb{R}^N$. Therefore, we can use two basis of $\mathbb{R}^N$ to get two maximum norms on $\mathbb{R}^N$, denoted by $\|\cdot\|_\Lambda$ and $\|\cdot\|_{E,E^\perp}$. Since $\mathbb{R}^N$ is finite-dimensional, these two norms are comparable. Henceforth, we will use $\|\cdot\|_{E,E^\perp}$.
Claim 3:
$\forall \epsilon > 0 \; \exists (n_1, ..., n_N) \in \mathbb{Z}^N$ and $\exists t > 0 : \|t \cdot (\theta_1, .., \theta_N) - (n_1, ..., n_N)\|_{E,E^\perp} < \epsilon$
and $\Pi^\perp(n_1, ..., n_N) = \sum_{i=1}^{N-1} s_i \underline{\sigma_i}$ with $s_i \geq 0$.

Proof of claim 3: $\theta_1, ..., \theta_N$ are $\mathbb{Z}$-linearly independent. As the metric is comparable, we can achieve claim 3 by applying Theorem 3.5 (first form of Kronecker's Approximation Theorem). Take $x' := x + \sum_{i=1}^{N-1} \frac{\epsilon}{2}\sigma_i$.

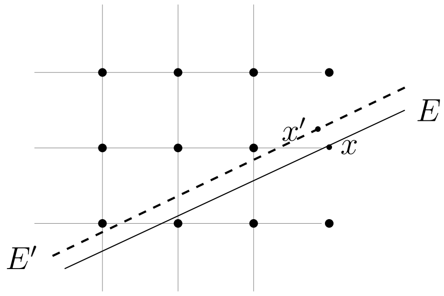

Figure 6: Construction of E' through x'.

From the first form of Kronecker's Approximation Theorem there exists $t > 0$ and $(n_1, ..., n_N) \in \mathbb{Z}^N$, such that:

$$\|t \cdot (\theta_1, ..., \theta_N) + \sum_{i=1}^{N-1} \frac{\epsilon}{2}\sigma_i - (n_1, ..., n_N)\|_{E,E^\perp} \leq \frac{\epsilon}{2}.$$

Then,

$$\begin{aligned} \|t \cdot (\theta_1, ..., \theta_N) - (n_1, ..., n_N)\|_{E,E^\perp} &\leq \|t \cdot (\theta_1, ..., \theta_N) + \sum_{i=1}^{N-1} \frac{\epsilon}{2}\sigma_i - (n_1, ..., n_N)\|_{E,E^\perp} \\ &\quad + \|\frac{\epsilon}{2} \sum_{i=1}^{N-1} \sigma_i\|_{E,E^\perp} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned} \tag{3}$$

Furthermore,

$$\begin{aligned} \|\Pi^\perp\big(\sum_{i=1}^{N-1} \frac{\epsilon}{2}\sigma_i - (n_1, ..., n_N)\big)\|_{E,E^\perp} &= \|\Pi^\perp\big(t \cdot (\theta_1, ..., \theta_N) + \sum_{i=1}^{N-1} \frac{\epsilon}{2}\sigma_i - (n_1, ..., n_N)\big)\|_{E,E^\perp} \\ &\leq \|t \cdot (\theta_1, ..., \theta_N) + \sum_{i=1}^{N-1} \frac{\epsilon}{2}\sigma_i - (n_1, ..., n_N)\|_{E,E^\perp} \\ &\leq \frac{\epsilon}{2}. \end{aligned} \tag{4}$$

This means that the coefficients s_i of $\Pi^\perp(n_1, ..., n_N) = \sum_{i=1}^{N-1} s_i\sigma_i$ deviate at most by $\frac{\epsilon}{2}$ from $\frac{\epsilon}{2}$, so that $s_i \geq 0$. □

Now, we can use claim 3 to find a radius $R > 0$, such that

$$\forall y \in E, \quad B_R(y) \cap \Pi(\Lambda \cdot x) \cap (K^\circ \times E) \text{ is not empty .}$$

For each $\epsilon > 0$, claim 3 gives 2^{N-1} points $x_i = (n_1, ..., n_N) \in \mathbb{Z}^N, i = 1, ..., 2^{N-1}$ such that for each x_i there exists $t_i > 0$ with

$$x_i = (n_1, ..., n_N) = t_i(\theta_1, ..., \theta_N) + ...$$

and the s_{ji} run through all combinations of being positive and negative when i runs from 1 to 2^{N-1}.
The negative signs can be brought in by changing the relevant basis vectors σ_j to $-\sigma_j$. If we choose a small enough $\epsilon > 0$, all the points x_i have an orthogonal projection $\Pi^\perp(x_i) = \sum s_{ji}\sigma_j \in K^\circ$.
Claim 4: For every $k \gg 0$, there exists $k_1, ..., k_{2^{N-1}} \geq 0$, such that $\sum_{i=1}^{2^{N-1}} k_i \geq k$ and $\sum_{i=1}^{2^{N-1}} k_i x_i \in K^\circ \times E$.
Proof of claim 4: Assume that

$$\|\Pi^\perp(\sum_{i=1}^{2^{N-1}} k_i x_i)\|_{E,E^\perp} = \|\sum_{i=1}^{2^{N-1}} \sum_{j=1}^{N-1} k_i s_{ji}\sigma_j)\|_{E,E^\perp} < \epsilon.$$

Since all s_{ji} have absolute value less than ϵ, adding $\sum_{j=1}^{N-1} s_{ji}\sigma_j$ to $\Pi^\perp(\sum_{i=1}^{2^{N-1}} k_i x_i)$ will not increase the norm of this vector if the s_{ji} has the correct sign. Since all combinations of signs are achieved when i runs from 1 to 2^{N-1}, this shows that there exists an $i_\circ$ such that

$$\|\Pi^\perp\big(\sum_{i=1, i\neq i_\circ}^{2^{N-1}} k_i x_i + (k_{i_\circ} + 1)x_{i_\circ}\big)\|_{E,E^\perp} < \epsilon.$$

Repeating this argument will make $\sum_{i=1}^{2^{N-1}} k_i$ arbitrarily large.

□

The argument above also shows that, the points in $\Pi(\Lambda \cdot x) \cap (K^\circ \times E)$ are separated at most by $R = \max_{i=1,...,2^{N-1}} \|\Pi(x_i)\|_{E,E^\perp}$. Since $\|\Pi(\sum_{i=1}^{2^{N-1}} k_i x_i)\|_{E,E^\perp} = \sum_{i=1}^{2^{N-1}} k_i \|\Pi(x_i)\|_{E,E^\perp}$, claim 4 shows that the points in $\Pi(\Lambda \cdot x) \cap (K^\circ \times E)$ occur arbitrarily far away from x. Consequently,

$$B_R(y) \cap \Pi(\Lambda \cdot x) \cap (K^\circ \times E) \text{ always contains a point.}$$

□

The following more general theorem is a consequence of the proof of Theorem 3.8 above.

Theorem 3.9 *If the dimension of the hyperplane E is n, then the set*

$$X_{T'} = \Pi\big(X_T \cap (K \times E)\big)$$

is relatively dense (if the window K satisfies the assumption (*)).

PROOF Choose lines $E_1, ..., E_n \subset E$ through a point $x \in X_T$, whose spanning vectors e_i are linearly independent. Then, construct points in X_T arbitrarily close to lines E_i, as in claims 3 and 4 in the proof of Theorem 3.8 (using $E_i^\perp$ instead of $E^\perp$ and the preimage of the window K in $E_i^\perp$, under the orthogonal projection $E_i^\perp \to E^\perp$). For $p \in E$, split up $p - x = \sum p_i e_i$. Choose points $x_i \in X_T$ approximating E_i closest to $x + p_i e_i$. Then, the distance of $\sum_{i=1}^n x_i$ to p is bounded independently of p. □

Theorem 3.10 *The set $X_{T'} = \Pi\big(X_T \cap (K \times E)\big)$ is uniformly discrete.*

PROOF $Aut(X_T)$ is crystallographic, that is, a subgroup of a product of a lattice Λ of translations of full rank and a finite point group, of finite index. For a fixed R, consider the following intersections:

$$B_R(y) \cap (K \times E) \cap X_T; \quad \forall y \in X_T \cap (K \times E) \;.$$

Claim: There are only a finite number of these bounded point sets, up to translations.
Proof of the claim: Take a fundamental domain $D \subset \mathbb{R}^N$ of Λ. Notice that D is compact as Λ is a lattice of full rank. Therefore, $D \cap X_T$ is finite, i.e., $D \cap X_T = \{x_1, ..., x_s\}$. Now, for all $x \in X_T$, there exists $\tau \in \Lambda$ such that $\tau(x) \in D$ and $\tau(x) = x_i$; therefore, $\bigcup_{i=1}^s \Lambda \cdot x_i = X_T$.
In particular, if $y = \tau(x_i)$ with $\tau \in \Lambda$, then, for any radius $R > 0$:

$$B_R(y) \cap X_T = \tau\big(B_R(x_i) \cap X_T\big),$$

as τ is an isometry in $Aut(X_T)$. Therefore, there are only finitely many point sets $B_R(y) \cap X_T$, up to translations in $Aut(X_T)$.
If $B_R(y) \cap X_T$ and $B_R(y') \cap X_T$ are mapped to each other by a translation, then $B_R(y) \cap X_T \cap (K \times E)$ and $B_R(y') \cap X_T \cap (K \times E)$ may not be mapped to each other by this translation because $B_R(y') \cap X_T \cap (K \times E)$ and $B_R(y') \cap X_T \cap \tau(K \times E)$ are different. On the other hand, there are only finitely many different point sets $B_R(y') \cap X_T \cap \tau(K \times E)$ for all $\tau \in Aut(X_T)$, because the number of points in

$B_R(y') \cap X_T$ is finite. Hence, the claim follows. □

$X_{T'} \subset E$ is relatively dense, that is, $\exists R > 0$, such that $B_R(y) \cap X_{T'} \neq \emptyset$ for all $y \in E$. Choose points $\{y_i\}_{i\in I}$ such that $\bigcup B_R(y_i) = E$. For each y_i, choose $x_i \in B_R(y_i) \cap X_{T'}$ such that if we take the radius $2R$, then $B_R(y_i) \subset B_{2R}(x_i)$. This implies that $\bigcup_{i\in I} B_{2R}(x_i) = E$, $\bigcup_{i\in I} B_{2R}(x_i) = \Pi\big(\bigcup_{i\in I} B_{2R}(\overline{x}_i)\big)$ with $\overline{x}_i \in X_T \cap (K \times E)$ and $\Pi(\overline{x}_i) = x_i$.
From the claim, the number of distances of points in the sets

$$\Pi\big(B_{2R}(\overline{x}_i) \cap X_T \cap (K \times E)\big) = B_{2R}(x_i) \cap X_{T'}\ ; \forall x_i, i \in I$$

is finite, because projected translated points have the same distance as the projected points themselves.
Now, choose $0 < r < 2R$ as the minimum of these distances:
For $y, y' \in X_{T'}$, there is x_i, such that $y \in B_{2R}(x_i)$. If $y' \in B_{2R}(x_i)$, then $d(y, y') \geq r$ (by the choice of r). On the other hand, if $y' \notin B_{2R}(x_i)$, then $d(y, y') \geq 2R > r$, also by the choice of r. Hence, in both cases $d(y, y') > r$, so $X_{T'}$ is uniformly discrete.

□

Theorem 3.11 *$X_{T'}$ is a Delone set.*

PROOF Straightforward from Theorem 3.9 and Theorem 3.10. □

Theorem 3.12 *The Voronoi-cell tiling $VT(X_{T'})$ associated to the Delone set $X_{T'}$ is a simple tiling.*

PROOF The claim made by Theorem 3.10 implies that there are only finitely many types of projections:

$$\Pi\big(B_R(y) \cap X_T \cap (K \times E)\big),\ \text{ up to translation in } E\ ,\ \text{for all } y \in X_T.$$

This is the case since translating, and then projecting to E is the same as projecting first and then translating inside E.
If we decompose the translation τ as $\tau = \tau_E \oplus \tau_{E^\perp}$, then, $\Pi \circ \tau = \tau_E \circ \Pi$. Since projections of balls to E are balls of the same radius in E, we have:

$$\Pi\big(B_R(y) \cap X_T \cap (K \times E)\big) = X_{T'} \cap B_R\big(\Pi(y)\big).$$

Consequently, there are only finitely many point sets of type $X_{T'} \cap B_R\big(\Pi(y)\big)$, up to translations in E.
We know that for large enough $R \gg 0$, $X_{T'} \cap B_R\big(\Pi(y)\big)$ determines the Voronoi-cell of $\Pi(y)$ in $VT(X_{T'})$. Hence, there is only a finite number of tile types in $VT(X_{T'})$; up to translations in E, and so, $VT(X_{T'})$ is a simple tiling. □

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